High Aperture Efficiency Ka-Band Horn Using Hard Boundary Walls

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Overview

- Introduction
- Rationale
- Mathematical Development
- Analytical Approach
- Preliminary Results
- Summary

Introduction

- 'Hard' Boundary Definition [1]

$$Z_z << Z_{transverse}$$

  - Dielectric-filled longitudinal corrugations force $E_z = 0$ while allowing $E_{tan} \neq 0$
  - Possible to achieve a near uniform amplitude distribution

- Goal
  - Define a complete waveguide mode set
  - Track the roots of the characteristic equations
  - Develop a tool to design/optimize hard horns utilizing mode-matching technique

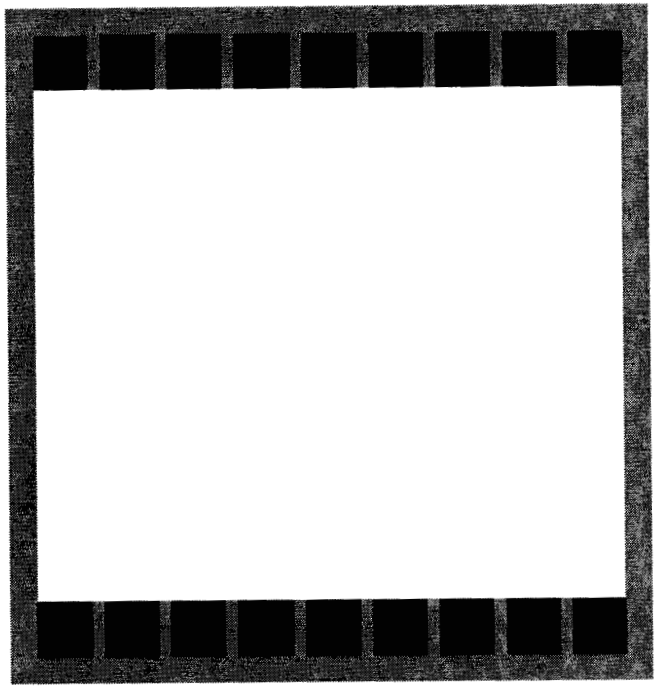


Fig. 1 Longitudinally corrugated horn

Rationale

- Solid-state Ka-Band amplifier
  - Hard horn couples to a microstrip patch antenna array feeding MMIC amplifiers

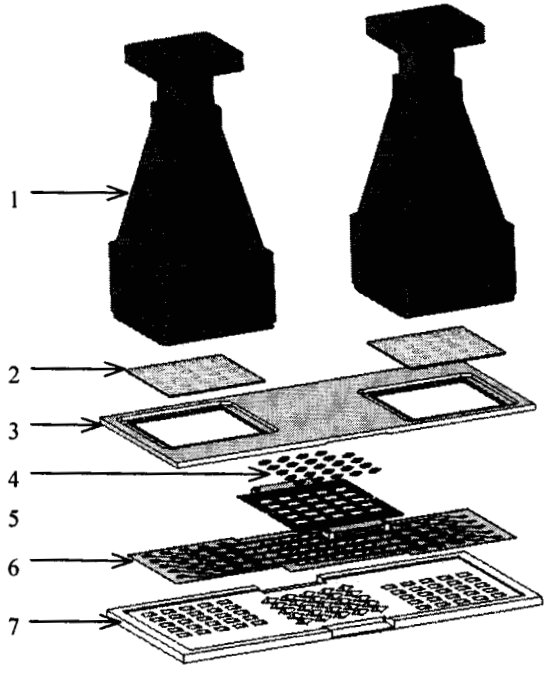


Fig. 2 Solid-State Amplifier (Graphic taken from [2])

- Maximum directivity antenna

Analytical Development – Overview

- Field Relations
- Impedance Relations
- Constraints
- Cosine-Cosine Field Contribution Derivation
- Cosine-Cosine Characteristic Equations
- Standard TM Mode Verification

Field Relations

- E- & H-Field Functional Relations [3]

$$E_x = f_1(x) f_2(y) \tag{1-a}$$

$$E_y = -\frac{\delta \cdot f_1'(x) f_2(y)'}{k_x^2} \tag{1-b}$$

$$E_z = -\frac{\Gamma}{\gamma} \frac{f_1'(x) f_2(y)}{k_x^2} \tag{1-c}$$

$$H_x = -j \frac{\Gamma f_1'(x) f_2(y)'}{\omega\mu \cdot k_x^2} \left(\frac{1}{\gamma} + \delta \right) \tag{1-d}$$

$$H_y = -j \frac{\Gamma f_1(x) f_2(y)}{\omega\mu} \left(1 + \frac{1}{\gamma} \right) \tag{1-e}$$

$$H_z = j \frac{f_1(x) f_2(y)'}{\omega\mu} (\delta - 1) \tag{1-f}$$

Field Relations

- Total Field Is Comprised of Four Components
  - Cosine-Cosine

$$f(x) = A_{cc} k_{xycc} \cos(k_{xcc} x) \quad f(y) = A_{cc} \cos(k_{ycc} y) \tag{1-g}$$

  - Sine-Cosine

$$f(x) = A_{sc} k_{xysc} \cos(k_{xsc} x) \quad f(y) = A_{sc} \cos(k_{ysc} y) \tag{1-h}$$

  - Sine-Sine

$$f(x) = A_{ss} k_{xyss} \cos(k_{xss} x) \quad f(y) = A_{ss} \cos(k_{yss} y) \tag{1-i}$$

  - Cosine-Sine

$$f(x) = A_{cs} k_{xycs} \cos(k_{xcs} x) \quad f(y) = A_{cs} \cos(k_{ycs} y) \tag{1-j}$$

Impedance Relations

- Wall Impedances

$$Z_1 = \pm \frac{E_x}{H_z}\bigg|_{y=\mp\frac{b}{2}} = \mp \frac{j\omega\mu \cdot f_2(y)}{(\delta-1)f_2'(y)}\bigg|_{y=\mp\frac{b}{2}} \qquad (2\text{-a})$$

$$Z_2 = \pm \frac{E_z}{H_x}\bigg|_{y=\mp\frac{b}{2}} = \mp \frac{j\omega\mu \cdot f_2(y)}{(1+\gamma\delta)f_2'(y)}\bigg|_{y=\mp\frac{b}{2}} \qquad (2\text{-b})$$

$$Z_3 = \pm \frac{E_z}{H_y}\bigg|_{x=\mp\frac{a}{2}} = \mp \frac{j\omega\mu \cdot f_1'(x)}{k_x^2(1+\gamma)f_1(x)}\bigg|_{x=\mp\frac{a}{2}} \qquad (2\text{-c})$$

$$Z_4 = \mp \frac{E_y}{H_z}\bigg|_{x=\mp\frac{a}{2}} = \mp \frac{j\omega\mu \cdot f_1'(x)\delta}{k_x^2(\delta-1)f_1(x)}\bigg|_{y=\mp\frac{a}{2}} \qquad (2\text{-d})$$

- Impedance 'Compatibility' Relation

$$Z_1Z_3 - Z_2Z_3 + Z_2Z_4 = 0 \qquad (3)$$

Constraints

- Isotropic, source free medium
  - Modified separation equation

$$\nabla \times \overline{E} = 0 \rightarrow -k_x^2 - k_y^2 + k_o^2 + \Gamma^2 \rightarrow 1 + \delta\left(\frac{k_y}{k_x}\right)^2 + \left(\frac{\Gamma}{k_x}\right)^2 \Big/ \gamma = 0 \qquad (4)$$

- Structure must be symmetric

‘TM’ Cosine-Cosine Field Components

- ‘TM’ Mode

$$E_x = f_1(x) f_2(y) = A_{cc} k_x \cos(k_x x) \cos(k_y y) \tag{5-a}$$

$$E_y = \frac{-\delta \cdot f_1'(x) f_2'(y)}{k_x^2} = -A_{cc} k_y \sin(k_x x) \sin(k_y y) \tag{5-b}$$

$$E_z = \frac{-\Gamma}{\gamma} \frac{f_1'(x) f_2(y)}{k_x^2} = \frac{-A_{cc}\left(k_x^2 + k_y^2\right)}{\Gamma} \sin(k_x x) \cos(k_y y) \tag{5-c}$$

$$H_x = \frac{-j\Gamma \cdot f_1'(x) f_2'(y)}{\omega \mu k_x^2} \left(\frac{1}{\gamma} + \delta \right) = \frac{j A_{cc} \Gamma k_y \omega \varepsilon}{\omega \mu} \sin(k_x x) \sin(k_y y) \tag{5-d}$$

$$H_y = \frac{-j\Gamma \cdot f_1(x) f_2(y)}{\omega \mu} \left(1 + \frac{1}{\gamma} \right) = \frac{j A_{cc} k_x \omega \varepsilon}{\Gamma} \cos(k_x x) \cos(k_y y) \tag{5-e}$$

$$H_z = 0 \tag{5-f}$$

‘TE’ Cosine-Cosine Field Components

- ‘TE’ Mode

$$E_x = f_1(x) f_2(y) = A_{cc} k_y \cos(k_x x) \cos(k_y y) \tag{6-a}$$

$$E_y = \frac{-\delta \cdot f_1'(x) f_2'(y)}{k_x^2} = A_{cc} k_x \sin(k_x x) \sin(k_y y) \tag{6-b}$$

$$E_z = 0 \tag{6-c}$$

$$H_x = \frac{-j\delta \cdot \Gamma f_1'(x) f_2'(y)}{\omega \mu k_x^2} = \frac{j A_{cc} \Gamma k_x}{\omega \mu} \sin(k_x x) \sin(k_y y) \tag{6-d}$$

$$H_y = \frac{-j\Gamma \cdot f_1(x) f_2(y)}{\omega \mu} = \frac{-j A_{cc} \Gamma k_y}{\omega \mu} \cos(k_x x) \cos(k_y y) \tag{6-e}$$

$$H_z = \frac{j \cdot f_1(x) f_2'(y)}{\omega \mu} (\delta - 1) = \frac{j A_{cc} \left(k_x^2 + k_y^2\right)}{\omega \mu} \cos(k_x x) \sin(k_y y) \tag{6-f}$$

Cosine-Cosine Characteristic Equations

- Horizontal Walls Impedances

$$Z_1 = \pm \frac{E_x}{H_z}\bigg|_{y=\mp\frac{b}{2}} = \frac{\pm j\omega\mu \cot(k_y y)}{(\delta-1)k_y}\bigg|_{y=\mp\frac{b}{2}} = \frac{\pm j\omega\mu \cot(k_y y' - k_y b/2)}{(\delta-1)k_y}\bigg|_{y'=0,b} = Z_1 \Rightarrow \frac{\cot(k_y b/2)}{(\delta-1)} = \frac{Z_1 k_y}{-j\omega\mu}$$

$$\delta = 1 \Rightarrow \cot(k_y b/2) = 0 \rightarrow k_y = \frac{(2n-1)\pi}{b} \qquad \text{(7-a)}$$

$$\gamma = \infty \Rightarrow j\omega\mu k_y \cot(k_y b/2) = Z_1\left(k_x^2 + k_y^2\right) \qquad \text{(7-b)}$$

$$Z_2 = \mp \frac{E_z}{H_x}\bigg|_{y=\mp\frac{b}{2}} = \frac{\mp j\omega\mu \cot(k_y y)}{(1+\delta\gamma)k_y}\bigg|_{y=\mp\frac{b}{2}} = \frac{\mp j\omega\mu \cot(k_y y' - k_y b/2)}{(1+\delta\gamma)k_y}\bigg|_{y'=0,b} = Z_2 \Rightarrow \frac{\cot(k_y b/2)}{(1+\delta\gamma)} = \frac{Z_2 k_y}{j\omega\mu}$$

$$\delta = 1 \Rightarrow \frac{j\omega\mu}{k_o^2}\left(k_x^2 + k_y^2\right)\cot(k_y b/2) = Z_2 k_y \qquad \text{(7-c)}$$

$$\gamma = \infty \Rightarrow \frac{\cot(k_y b/2)}{(1+\delta\gamma)} \rightarrow 0 \qquad \text{(7-d)}$$

Cosine-Cosine Characteristic Equations

- Vertical Wall Impedances

$$Z_3 = \pm \frac{E_z}{H_y}\bigg|_{x=\mp\frac{a}{2}} = \frac{\pm j\omega\mu \tan(k_x x)}{(1+\gamma)k_x}\bigg|_{x=\mp\frac{a}{2}} = \frac{\pm j\omega\mu \tan(k_x x' - k_x a/2)}{(1+\gamma)k_x}\bigg|_{x'=0,a} \Rightarrow \frac{\tan(k_x a/2)}{(1+\gamma)} = \frac{Z_3 k_x}{-j\omega\mu}$$

$$\delta = 1 \Rightarrow \underline{\underline{\frac{-j\omega\mu}{k_o^2}\left(k_x^2 + k_y^2\right)\tan(k_x a/2) = Z_3 k_x}} \qquad (7\text{-e})$$

$$\gamma = \infty \Rightarrow \frac{\tan(k_x a/2)}{(1+\gamma)} \to 0 \qquad (7\text{-f})$$

$$Z_4 = \mp \frac{E_y}{H_z}\bigg|_{x=\mp\frac{a}{2}} = \frac{\pm j\omega\mu\delta \tan(k_x x)}{(\delta-1)k_x}\bigg|_{x=\mp\frac{a}{2}} = \frac{\pm j\omega\mu\delta \tan(k_x x' - k_x a/2)}{(\delta-1)k_x}\bigg|_{x'=0,a} \Rightarrow \frac{\delta \tan(k_x a/2)}{(\delta-1)} = \frac{Z_4 k_x}{-j\omega\mu}$$

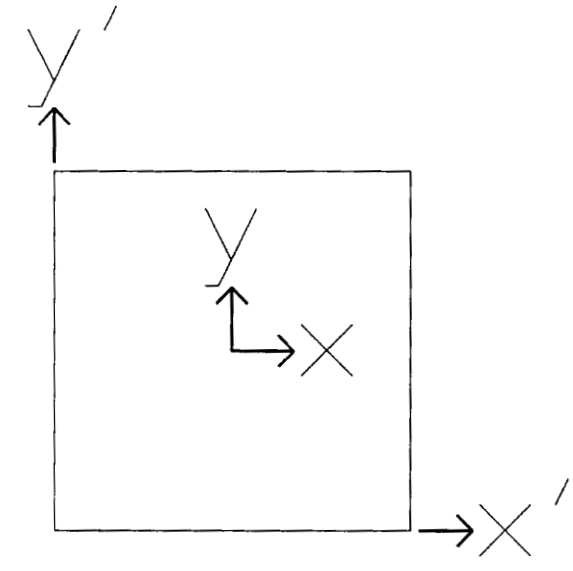
$$\delta = 1 \Rightarrow \tan(k_x a/2) = 0 \to \underline{\underline{k_x = \frac{2m\pi}{a}}} \qquad (7\text{-g})$$

$$\gamma = \infty \Rightarrow \underline{\underline{-j\omega\mu k_x \tan(k_x a/2) = Z_4\left(k_x^2 + k_y^2\right)}} \qquad (7\text{-h})$$

Normalization Constants

- Substitution of variables

$$x = x' - a/2$$
$$y = y' - b/2$$



- Power Normalization

$$\hat{z}\cdot\left(\overline{E}\times\overline{H}^{*}\right)$$
$$= \frac{A_{cc}^{2}\Gamma}{\omega\mu}\left[\begin{array}{l}\left(1+\frac{1}{\gamma}\right)\cos^{2}\left(k_{x}x'-k_{x}\,a/2\right)\cos^{2}\left(k_{y}y'-k_{y}\,b/2\right)\\ +\left(\frac{\delta\cdot k_{y}}{k_{x}}\right)^{2}\left(\frac{1}{\delta\gamma}+1\right)\sin^{2}\left(k_{x}x'-k_{x}\,a/2\right)\sin^{2}\left(k_{y}y'-k_{y}\,b/2\right)\end{array}\right]=1 \qquad (8)$$

Standard TM Mode Verification

- Provided By Larry Epp, JPL
- E_x Summation

$$E_x = A_{cc}k_{xcc}\cos(k_{xcc}x)\cos(k_{ycc}y) + A_{sc}k_{xsc}\sin(k_{xsc}x)\cos(k_{ysc}y) + A_{ss}k_{xss}\sin(k_{xss}x)\sin(k_{yss}y) + A_{cs}k_{xcs}\cos(k_{xcs}x)\sin(k_{ycs}y) \tag{9}$$

- Evaluate Normalization Constants

$$k_x = \frac{m\pi}{a}, \quad k_y = \frac{n\pi}{b}$$

$$A_{cc}^2 = Z_e\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]\frac{\varepsilon_{om}\varepsilon_{on}}{ab}, \quad \varepsilon_{oq} = \begin{cases} 1 & q = 0 \\ 2 & q > 0 \end{cases} \tag{10}$$

$$A_{sc}^2 = Z_e\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]\frac{\varepsilon_{om}\varepsilon_{on}}{ab}, \quad \varepsilon_{oq} = \begin{cases} 1 & q = 0 \\ 2 & q > 0 \end{cases} \tag{11}$$

Standard TM Mode Verification

- Combine CC and SC Terms

$$E_{x(cc+sc)} = A_{cc}\frac{m\pi}{a}\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\Bigg|_{\substack{m=0,2,4\ldots\\ n=1,3,5\ldots}}$$

$$+A_{sc}\frac{m\pi}{a}\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\Bigg|_{\substack{m=1,3,5\ldots\\ n=1,3,5\ldots}}$$

$$=\frac{m\pi}{a}\left[A_{cc}\cos\left(\frac{m\pi}{a}x\right)\Bigg|_{m=2,4,6\ldots}+A_{sc}\sin\left(\frac{m\pi}{a}x\right)\Bigg|_{m=1,3,5\ldots}\right]\cos\left(\frac{n\pi}{b}y\right)\Bigg|_{n=1,3,5} \qquad (12)$$

- It can be shown that

$$\cos\left(\frac{m\pi}{a}x'\right)=\cos\left(\frac{m\pi}{a}x+\frac{m\pi}{2}\right)=\begin{cases}(-1)^{\frac{m+1}{2}}\sin\left(\frac{m\pi}{a}x\right) & m=1,3,5\ldots\\ (-1)^{\frac{m}{2}}\cos\left(\frac{m\pi}{a}x\right) & m=2,4,6\ldots\end{cases} \qquad (13)$$

Standard TM Mode Verification

- Using (13) in (12)

$$E_{x(cc+sc)} = A\frac{m\pi}{a}\cos\left(\frac{m\pi}{a}x+\frac{m\pi}{a}\right)\cos\left(\frac{n\pi}{b}y\right)\Bigg|_{n=1,3,5\ldots} \tag{13}$$

where the signs are absorbed

$$(-1)^{\frac{m+1}{2}} \rightarrow A_{sc}, A_{ss}$$
$$(-1)^{\frac{m}{2}} \rightarrow A_{cc}, A_{cs}$$

- Similarly for the SS and CS Terms We Use

$$\sin\left(\frac{n\pi}{b}y'\right) = \sin\left(\frac{n\pi}{b}y+\frac{n\pi}{2}\right) = \begin{cases} (-1)^{\frac{n-1}{2}}\cos\left(\frac{n\pi}{b}y\right) & n=1,3,5\ldots \\ (-1)^{\frac{n}{2}}\sin\left(\frac{n\pi}{b}y\right) & n=2,4,6\ldots \end{cases} \tag{14}$$

Standard TM Mode Verification

- To Show That

$$E_{x(ss+cs)} = A\frac{m\pi}{a}\cos\left(\frac{m\pi}{a}x+\frac{m\pi}{a}\right)\sin\left(\frac{n\pi}{b}y\right)\Bigg|_{n=2,4,6\ldots} \qquad (15)$$

where the signs are absorbed

$$(-1)^{\frac{n-1}{2}} \rightarrow A_{sc}, A_{cc}$$
$$(-1)^{\frac{n}{2}} \rightarrow A_{ss}, A_{cs}$$

- Adding (15) and (13) Yields

$$E_{x(cc+sc+ss+cs)} = A\frac{m\pi}{a}\cos\left(\frac{m\pi}{a}x+\frac{m\pi}{2}\right)\sin\left(\frac{n\pi}{b}y+\frac{n\pi}{2}\right) \qquad (16)$$

Standard TM Mode

Analytical Approach

- Limit Concern to Hard Vertical Walls Only
  - k_y will be purely real integer multiple of π/b
- Linear Polarization Only
- Find Roots of Eq. (7-h) For All Four Cases
  - Cosine-Cosine

$$\omega\mu k_x \tan(k_x a/2) + X_4\left(k_x^2 + k_y^2\right) = 0 \qquad (17\text{-a})$$

  - Sine-Cosine

$$\omega\mu k_x \cot(k_x a/2) - X_4\left(k_x^2 + k_y^2\right) = 0 \qquad (17\text{-b})$$

  - Sine-Sine

$$\omega\mu k_x \cot(k_x a/2) - X_4\left(k_x^2 + k_y^2\right) = 0 \qquad (17\text{-c})$$

  - Cosine-Sine

$$\omega\mu k_x \tan(k_x a/2) + X_4\left(k_x^2 + k_y^2\right) = 0 \qquad (18\text{-d})$$

Root Finders

- Muller's Method
  - Works for all cases
  - Must be near root with initial guess
  - Gets trapped at trivial solution ($k_x = 0$)
  - Not 'smart'
- Newton-Raphson Method (1-dimension using derivative)
  - Fails for $Z_{wall} > 0$ (complex root case)
- Newton-Raphson for Nonlinear System of Equations (2-dimensions using Jacobian)
  - Fails for $Z_{wall} > 0$ (complex root case)
  - Still in progress

Preliminary Results

- Real Roots
  - Track along real axis from PEC values toward zero
  - Muller fails to converge for $Z_{wall} >> 0$
- Imaginary Roots (Surface Waves)
  - Show up dominant mode for capacitive-reactive wall-impedance cases
  - Appear in higher-order modes for any non-PEC wall impedance
  - Track along imaginary axis toward infinity
- No Mixed Roots Found

Preliminary Results

- Propagation Constant (k_x) vs. Groove Depth

Summary & Conclusions

- Completed Work
  - E- and H-mode sets defined
  - Complex transcendental characteristic equations identified
  - Standard TM mode set meets hard wall BCs
  - Capacitive-reactive wall impedance generates surface modes
- Continuing Work
  - Continue investigating/evaluating root finding methods
  - Implement mode set in a Mode-Matching code for simulation
  - Prototype testing & evaluation

References

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