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Applications of Ergodic Theory to Coverage Analysis

Martin W. Lo[i]

Abstract

The study of differential equations, or dynamical systems in general, has two fundamentally different approaches. We are most familiar with the construction of solutions to differential equations. Another approach is to study the statistical behavior of the solutions. Ergodic Theory is one of the most developed methods to study the statistical behavior of the solutions of differential equations. See Sinai (Ref. 1) and Arnold (Ref. 2) for r eferences and an introduction to this field. In the theory of satellite orbits, the statistical behavior of the orbits is used to produce "Coverage Analysis" or how often a spacecraft is in view of a site on the groud. In this paper, we consider the use of Ergodic Theory for Coverage Analysis. This allows us to greatly simplify the computation of quantities such as the total time for which a ground station can see a satellite without ever integrating the trajectory, see Lo (Ref. 1, 2). More over, for any quantity which is a function of the ground track, its average may be computed similarly without the integration of the trajectory. For example, the data rate for a simple telecom system is a function of the distance between the satellite and the ground station. We show that such a function may be averaged using the Ergodic Theorem.

Extended Abstract

1. Review of The Coverage Analysis Problem and Ergodic Theory

The Coverage Analysis Problem, at its simplest, is the study of the visibility properties of a satellite in orbit around the Earth from a point on Earth. In Figure 1, we depict the satellite ground track of a circular orbit and the circular region of visibility from a point P (at the center of the circle) on the Equator in the Pacific Ocean. Geometrically, whenever the satellite ground track enters this circle, it is in view from the station on the Equator. We define the following variables for this discussion:

D	=	Circular region of visibility of a spacecraft from a ground station centered on the Equator in the Pacific Ocean.
A	=	Annulus region defined by the ground tracks of the spacecraft.
$\aleph(\ )$	=	Area function, i.e. provides area of region D on the sphere.

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Simplistically, one may think that the percentage of time T spent by the satellite in the circle D would be well approximated by the area of the intersection between the circle and the annulus defined by the ground track, divided by the area of the annulus defined by the ground track, i.e. T would be equal to the expression:

$$\aleph(D \cap A) / \aleph(A)$$

In fact, this is a very bad approximation. A clue as to why this is a bad approximation is given by the density of the ground tracks which depends on the latitude and is quite uneven. Moreover, the speed of the nadir of the satellite along the ground track is not constant because the Earth is rotating. Also, the orbital plane is precessing due to the J2 gravity harmonic. But all is not lost.

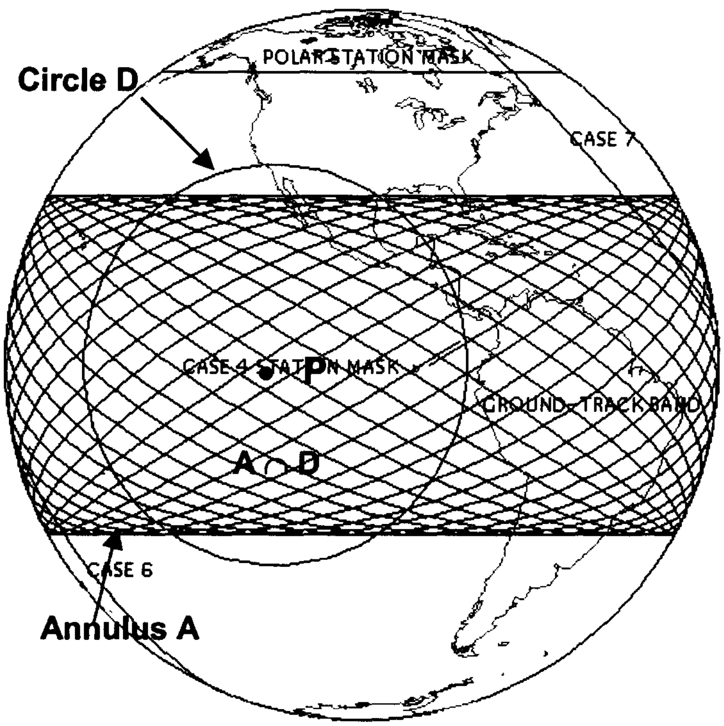

Figure 1 The ground tracks of a circular satellite forming the annulus A and the circular coverage region D of a point P on the Equator in the Pacific Ocean.

The heuristics of this reasonsing is intuitively correct. But, instead of the ratio of the geometric areas of the two regions mentioned earlier, we need to weigh the area depending on the ground velocity and some how account for the expansion and contraction of the ground tracks. This new weighted area function is technically called an "invariant measure" usually denoted by "μ". As a weighted area element following the satellite nadir along the ground track, the area of the element is preserved. Hence the weighted area element is invariant under the motion of the satellite ground track. When such a measure of the area is available, then indeed the percentage of time spent by the satellite in the circular region is given by the measure of the intersection of the circular region with the annulus, divided by the measure of the annulus, i.e.

$$T = \mu(A \cap D) / \mu(A).$$

Such a measure was constructed in Ref. 3, and 4. However, for this to work, it is necessary that the ground tracks not be periodic. But, in Ref. 1 it is shown that even when the ground track is periodic, provided the repeat cycle is not too small, this approximation is fairly good. This means that instead of finding the view periods from a propagated trajectory to compute the amount of time a satellite is in view of a ground station, also called the "time average", we can replace this by a simple area integral with a weighted area. This weigted average is called the "space average". This in essence is the Ergodic Theorm that we can replace time averages by space average. Typically, time averages are more difficult to compute since it requires the solution of differential equations. Where as the space average is much easier to solve as it requires only a single area integral. Note, for the space average, J2 is only used for the verification that the orbit ground track is not periodic. It never enters into the area integral.

We should note here that we are assuming that the satellite orbit is being maintained so that effects of the Earth gravity's higher harmonics, the luni-solar perturbation, solar radiation pressure, and drag are being compensated. The maneuvers will perturb the orbit node, but the orbital elements such as semimajor axis, eccentricity are essentially preserved. Assuming the maneuvers are radom without a bias that would cause the ground tracks to become periodic in some fashion, this theory applies to the coverage problem.

2. A Simpler Way of Computing the Data Volume

But, the Ergodic Theorem is much more general. It actually says that the time average of any well behaved function of the trajectory is equal to the space average of the trajectory. A simple example of how this can be useful is the computation of the total data volume which can be transmitted by the satellite to a point on the ground. Instead of finding the time average using a propagated trajectory, this quantity can be easily computed with an area integral using the weighted area measure. This greatly simplifies the calculation.

References

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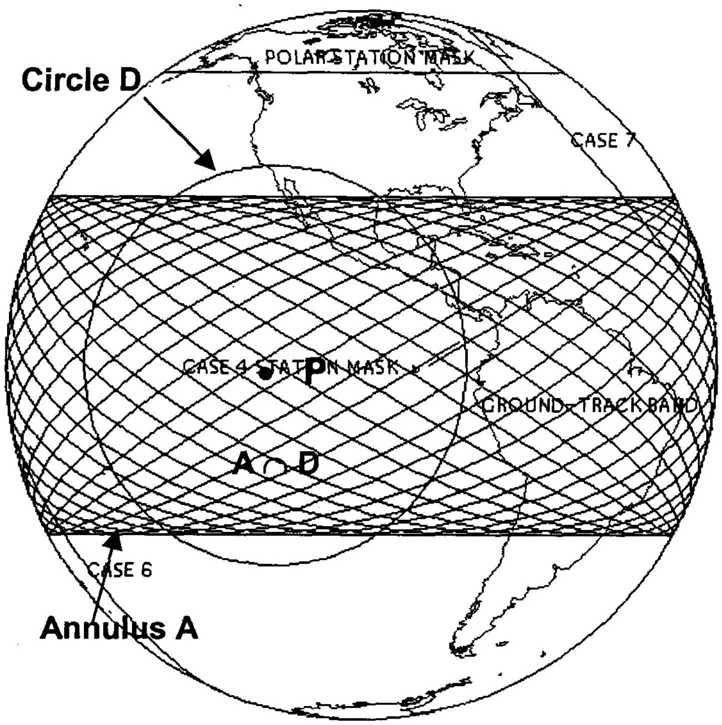

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