Jet Propulsion Laboratory, California Institute of Technology

A new concept of stability in orbit propagation

Quantifying numerical errors

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Main results

Topological stability

How well is the structure of KS space preserved during the integration?

Quantifying numerical errors

Monitoring topological stability $\longrightarrow$ direct estimation of global numerical errors

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KS transformation

Cartesian space ($\mathbf{x}$ in 3D) $\rightarrow$ KS space ($\mathbf{u}$ in 4D)

$$\mathbf{x} = \mathcal{K}(\mathbf{u})\,\mathbf{u}$$

- $\mathbf{x} = [x, y, z, 0]^\top$.
- $\mathcal{K}(\mathbf{u})$ is the KS matrix:

$$\mathcal{K}(\mathbf{u}) = \begin{bmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{bmatrix}$$

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KS transformation II

KS + Sundman time transformation $\dfrac{\mathrm{d}t}{\mathrm{d}s} = g(\mathbf{x}, \dot{\mathbf{x}}) \equiv r$

Two-body problem

$$\mathbf{u}'' + \frac{h}{2}\mathbf{u} = \frac{u^2}{2}\mathcal{K}^\top \mathbf{a}_p$$

N-body problem

Canonical transformation:

$$\tilde{\mathcal{H}} = g(\mathcal{H} - \mathcal{E}_0)$$

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Why four dimensions?

A classical problem

$$(x_1^2 + \ldots + x_r^2)(y_1^2 + \ldots + y_s^2) = z_1^2 + \ldots + z_n^2$$

Can we write z_k as bilinear functions of x_i and y_j for any (r, s, n)?

- Simplest solution: (1,1,1) $\longrightarrow \mathbb{R}$
- Diophantus of Alexandria: (2,2,2) $\longrightarrow \mathbb{C}$
- Euler: (4,4,4) $\longrightarrow \mathbb{H}$
- Hurwitz (8,8,8) $\longrightarrow \mathbb{O}$

There are no transformation of this kind in 3D!

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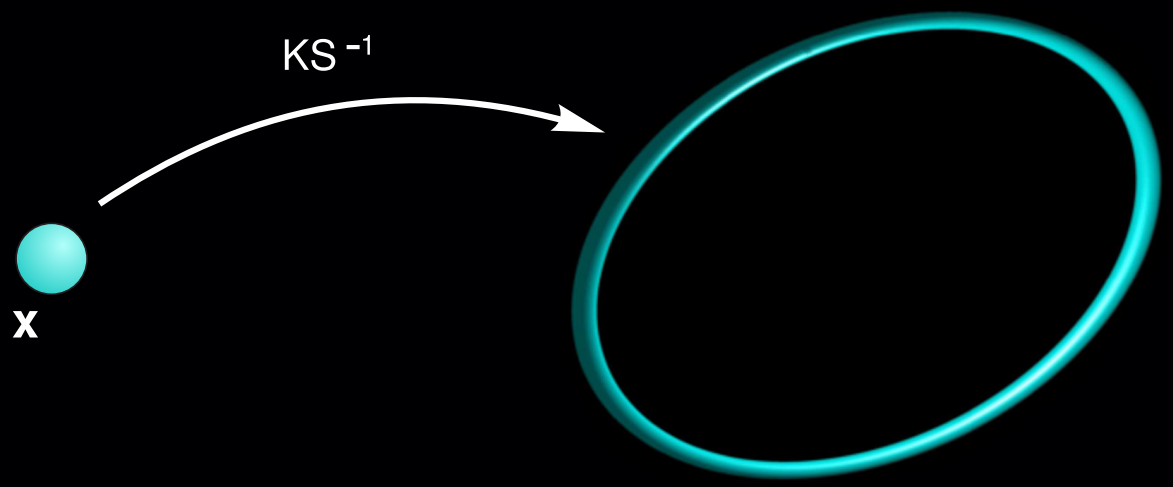
The Hopf fibration

Extra dimension $\Longrightarrow$ Degree of freedom

Point $\mathbb{R}^3$ Hopf fiber $\mathbb{U}^4$



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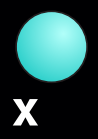
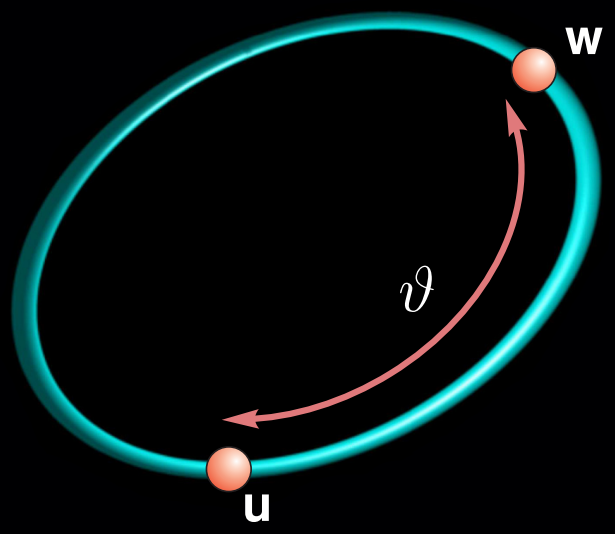
The Hopf fibration

Extra dimension $\Longrightarrow$ Degree of freedom

Point $\mathbb{R}^3$



Hopf fiber $\mathbb{U}^4$



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The Hopf fibration

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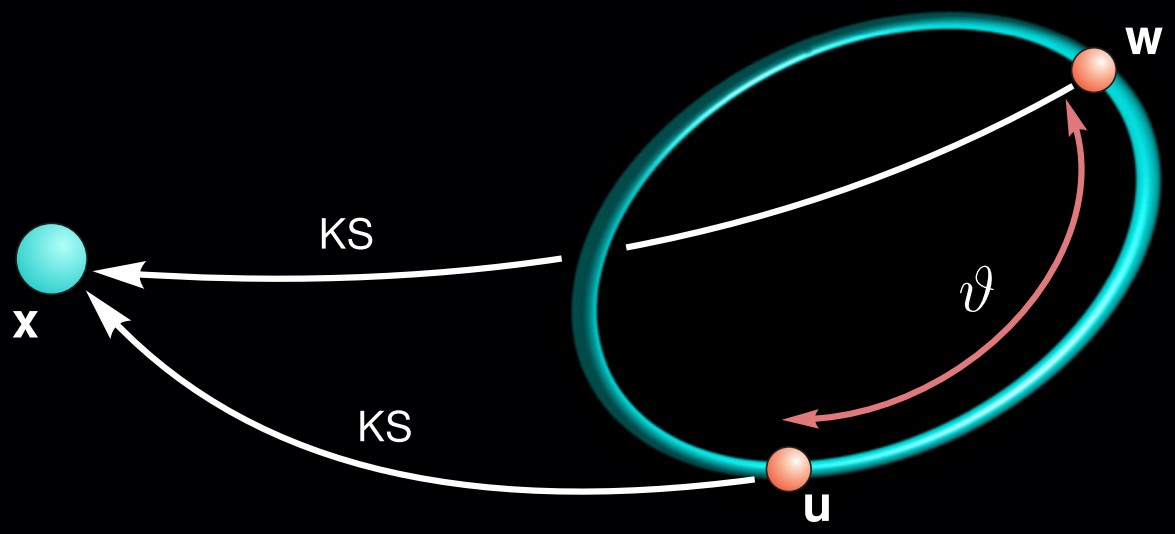

$$\mathbf{x} = \mathcal{K}(\mathbf{u})\,\mathbf{u} = \mathcal{K}(\mathbf{w})\,\mathbf{w}$$

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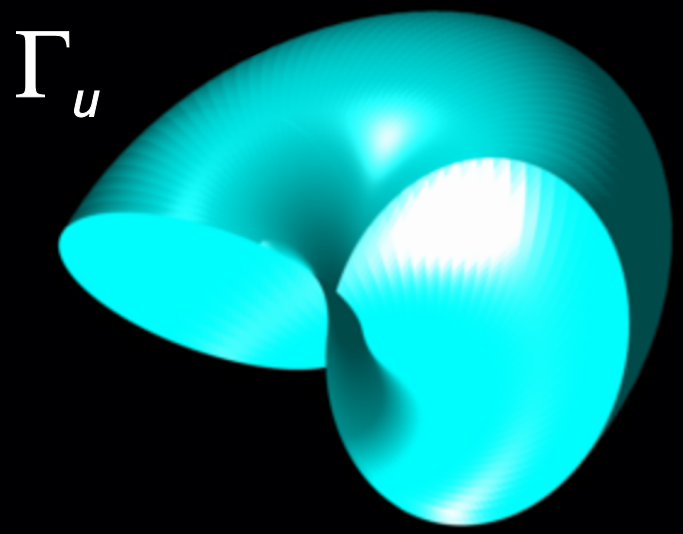
The Hopf fibration II



The role of the degree of freedom

- We can choose any point on the initial fiber: same solution in $\mathbb{R}^3$.
- But $\mathbf{u}_0$ and $\mathbf{w}_0$ yield different trajectories in KS space!
- The angular separation ϑ remains constant.

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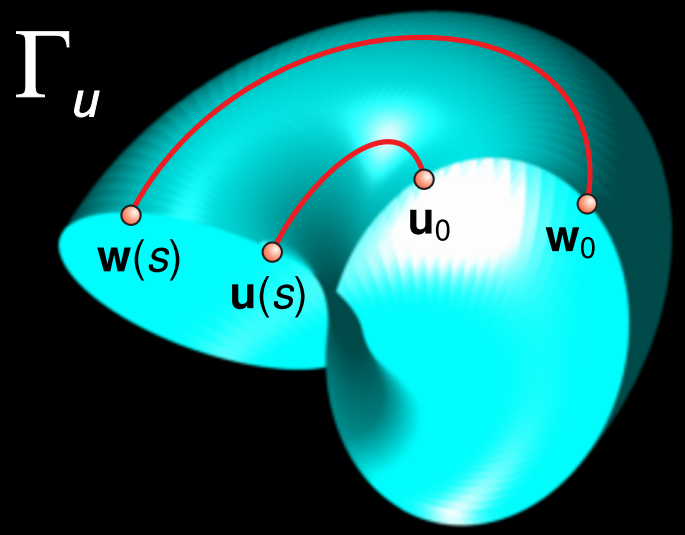
The Hopf fibration II



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Topological stability

Topological constraint

All solutions in KS space that represent the same 3D solution relate by the rotation:

$$\mathbf{w} = \mathcal{R}(\vartheta)\,\mathbf{u}$$

- Due to numerical errors, this constraint might not be satisfied.
- $\mathcal{R}(\vartheta)\,\mathbf{u}$ provides an estimate of the integrated solution $\mathbf{w}^*$.

$\mathscr{K}$-separation

$$d(s) = ||\mathbf{w}^*(s) - \mathcal{R}(\vartheta)\,\mathbf{u}(s)||$$

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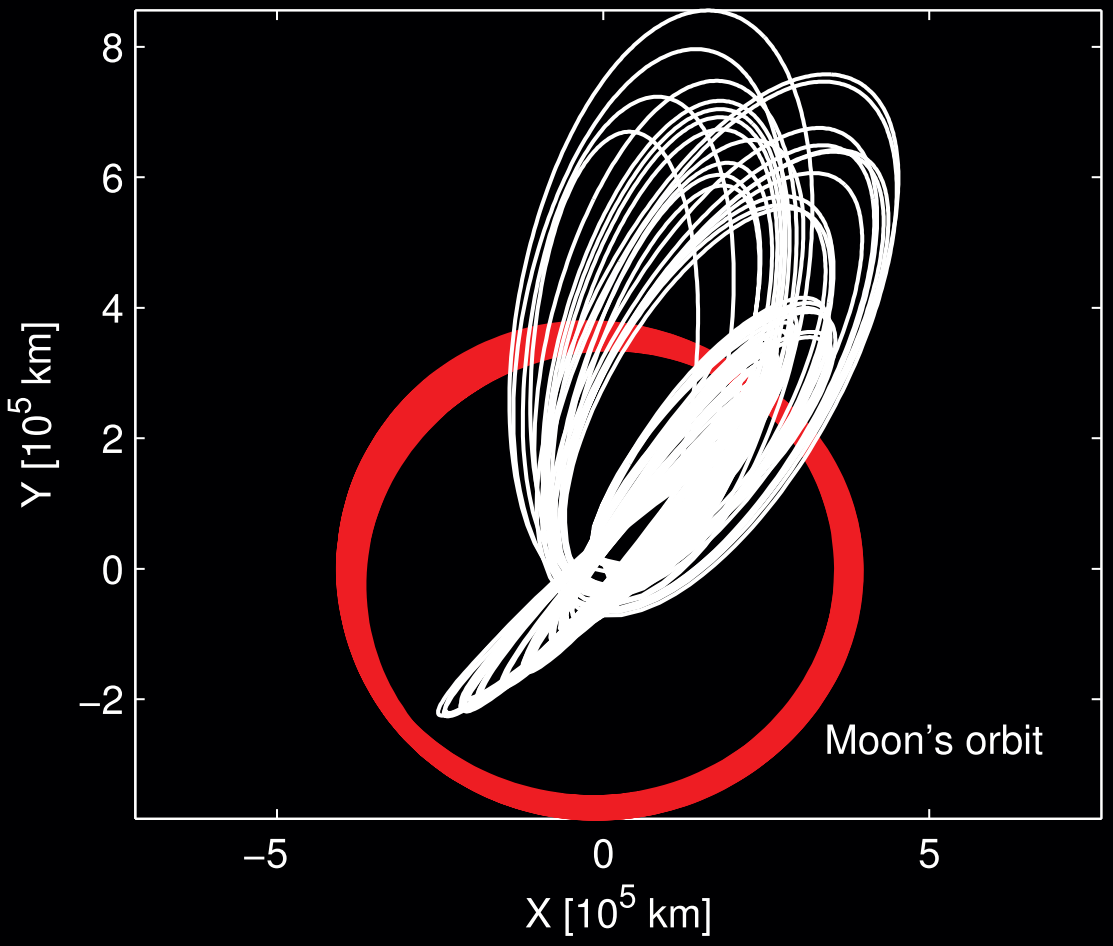
Resonant encounters



Initial orbit $a = 340000$ km, $e = 0.94$, 4-year propagation

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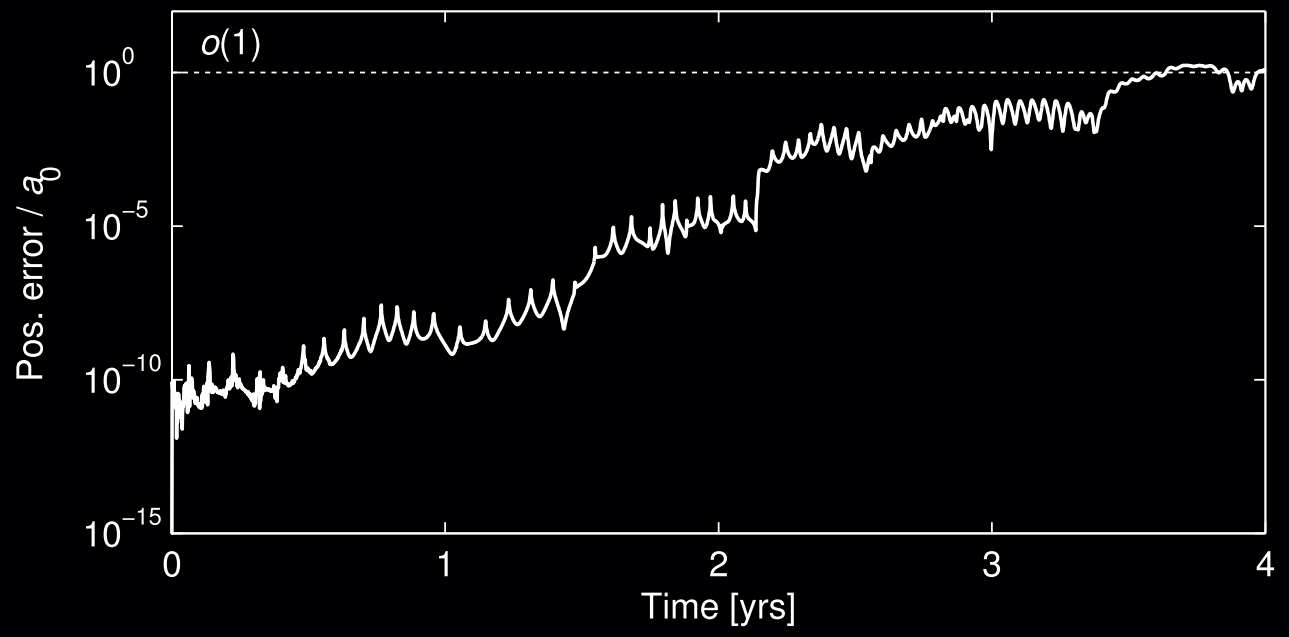
Resonant encounters II



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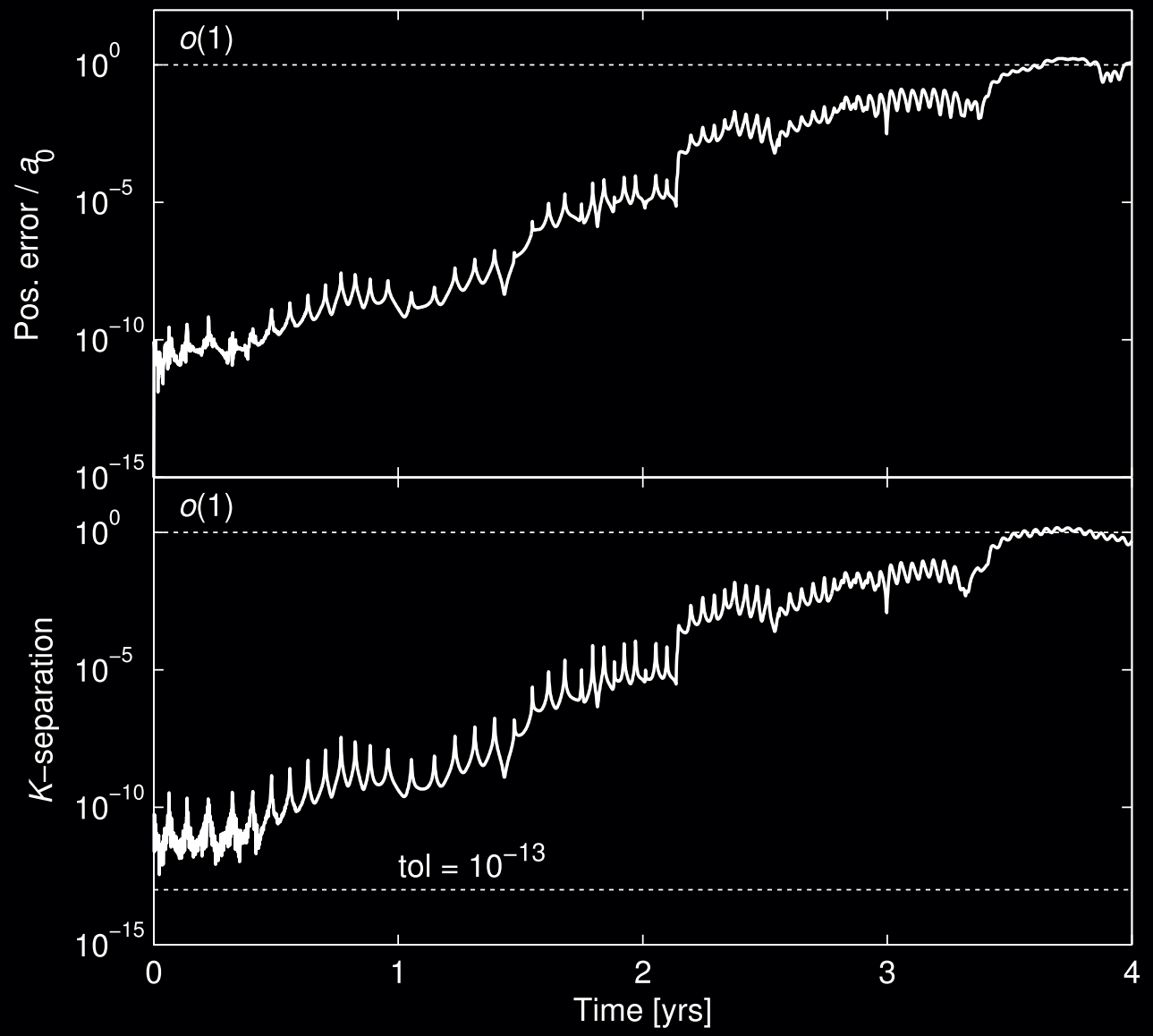
Resonant encounters II



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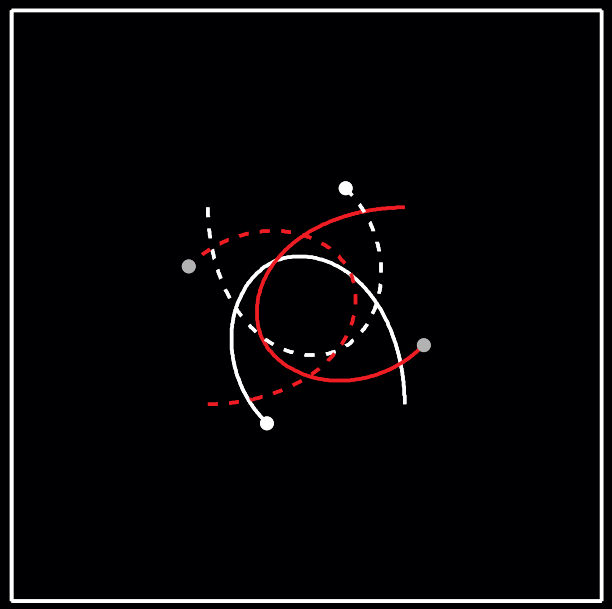
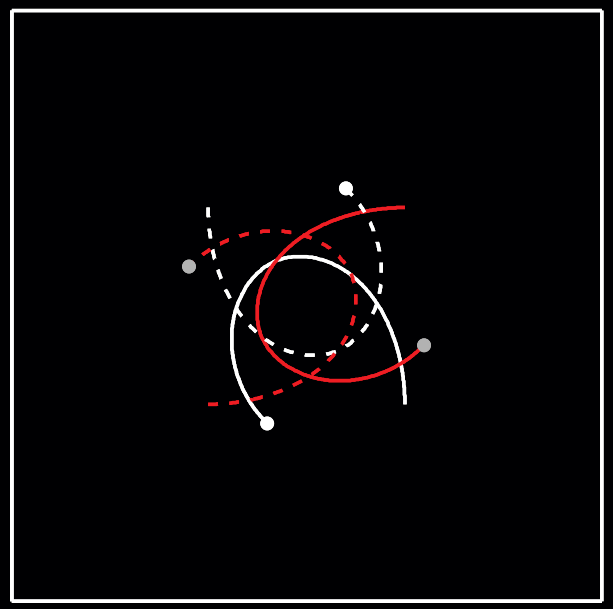
8-BODY PROBLEM

Two different solutions in KS space ($\vartheta = 270°$):

Time = 2



Time = 2



Errors dominate the solution after $t \sim 15$

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8-BODY PROBLEM

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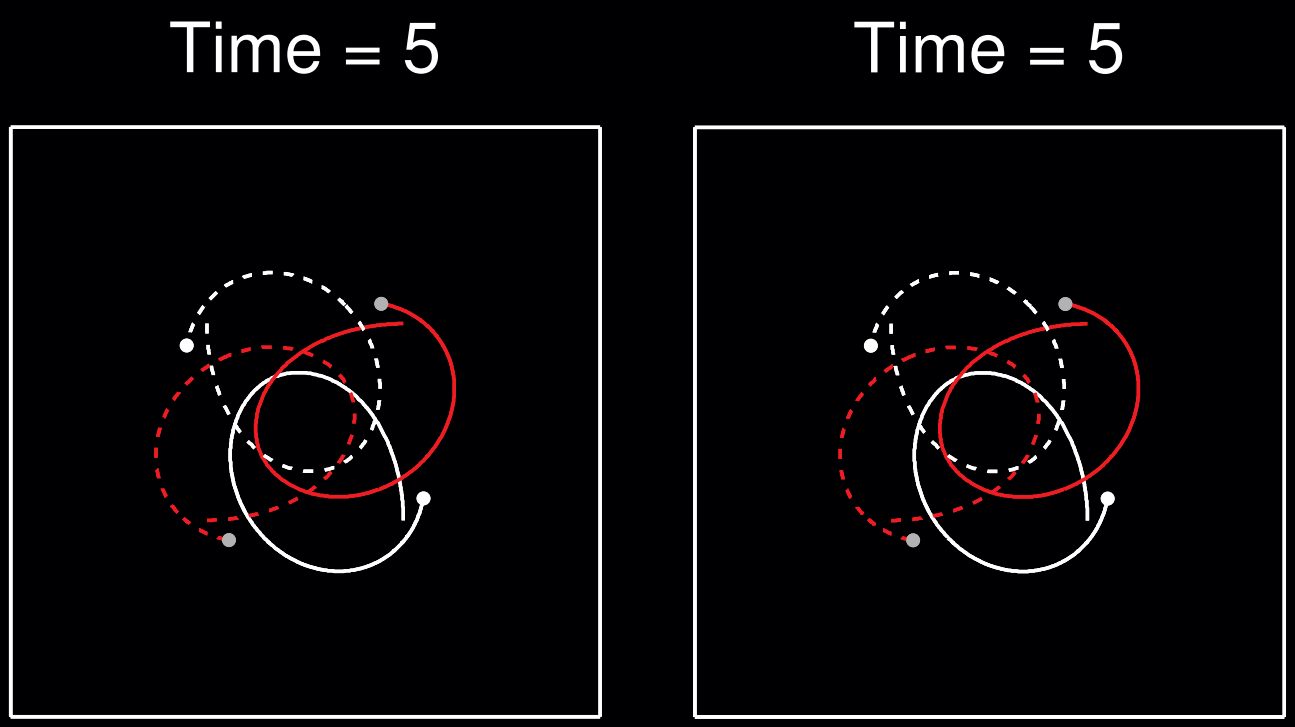

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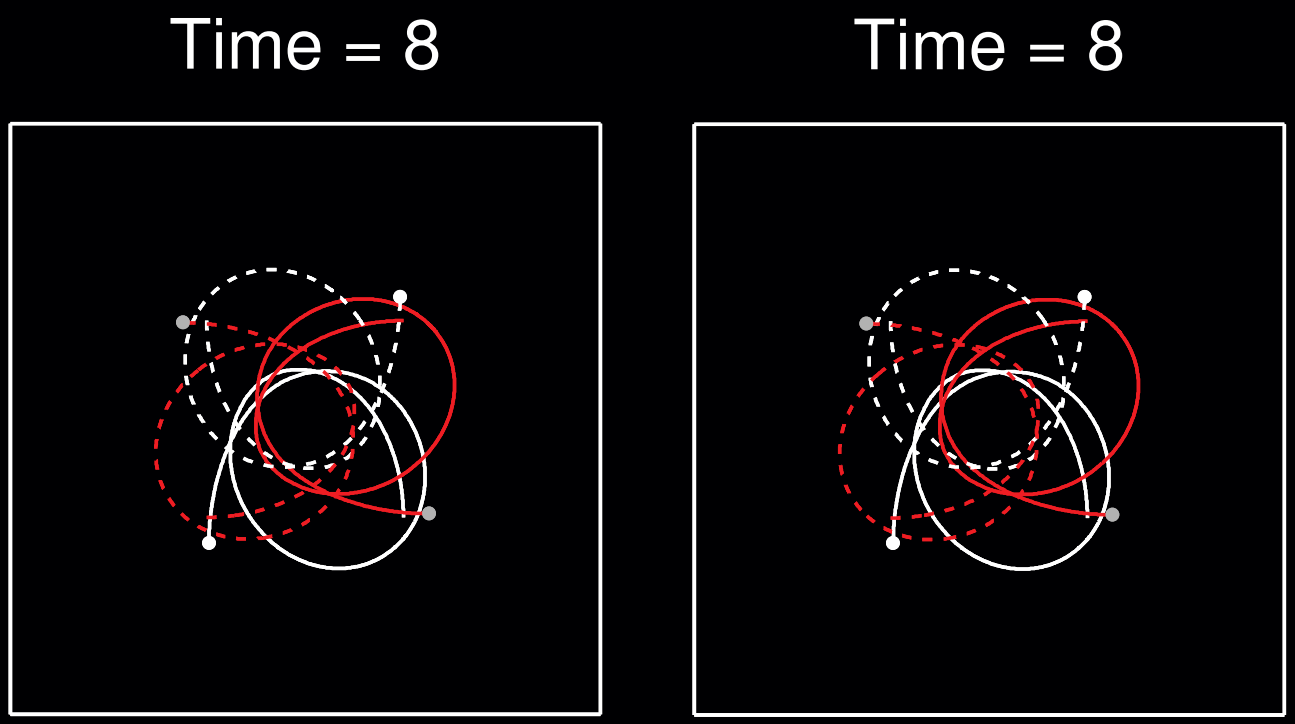

Errors dominate the solution after $t \sim 15$

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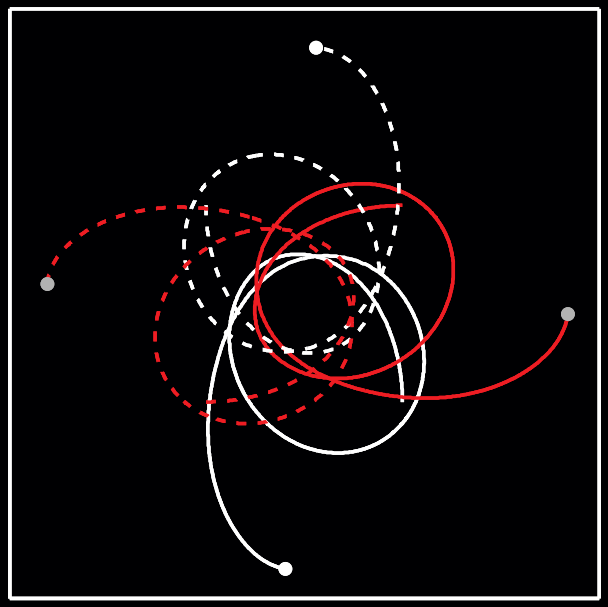
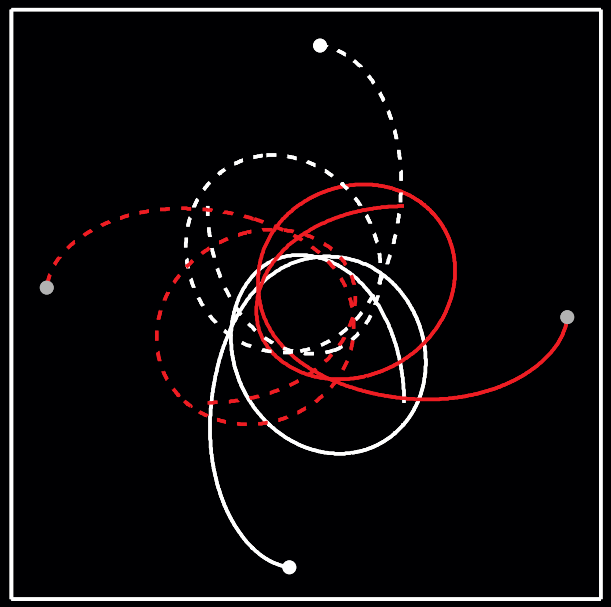
8-BODY PROBLEM

Two different solutions in KS space ($\vartheta = 270°$):

Time = 11



Time = 11



Errors dominate the solution after $t \sim 15$

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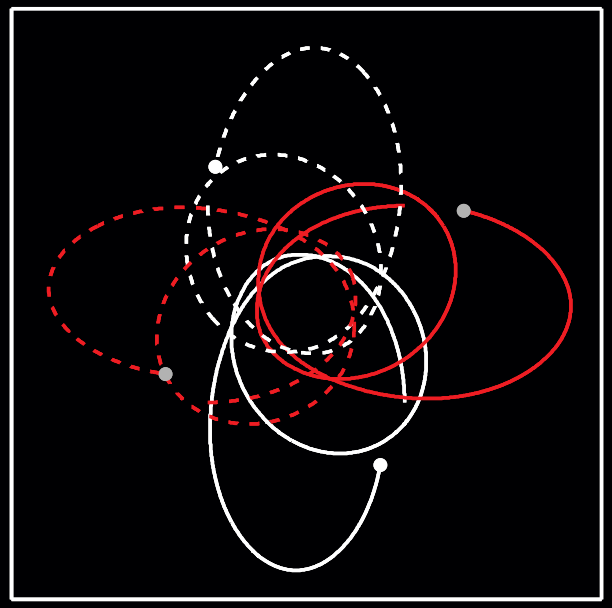
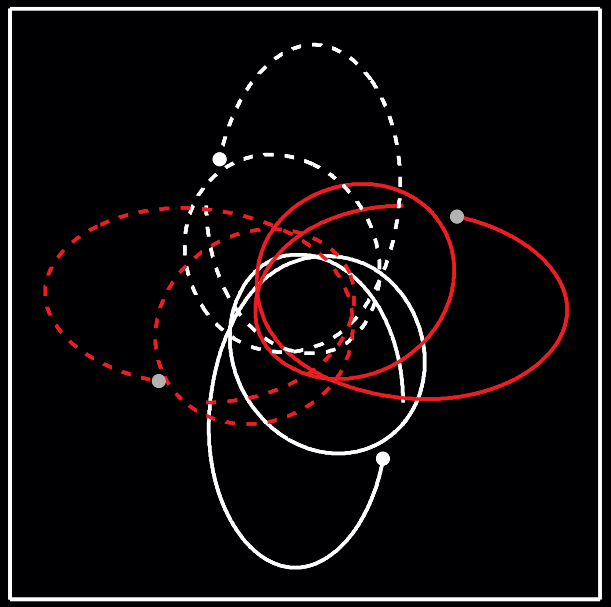
8-BODY PROBLEM

Two different solutions in KS space ($\vartheta = 270°$):

Time = 14



Time = 14



Errors dominate the solution after $t \sim 15$

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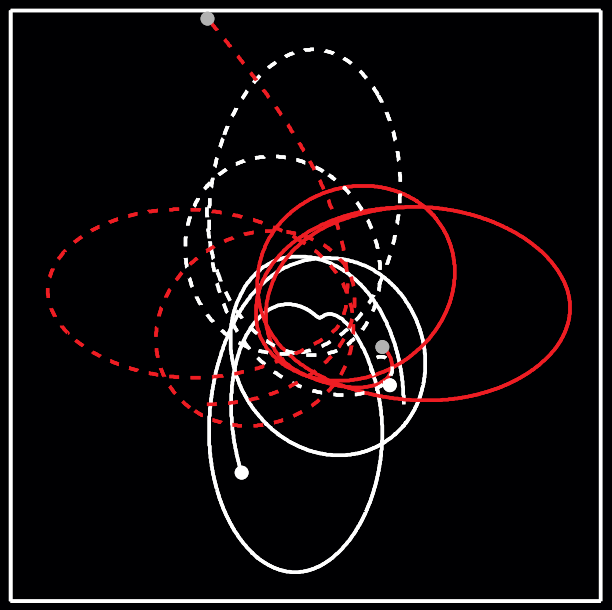
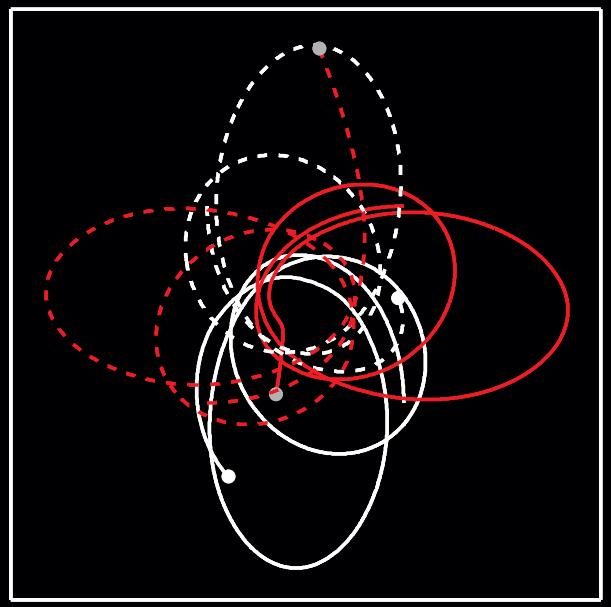
8-BODY PROBLEM

Two different solutions in KS space ($\vartheta = 270°$):

Time = 17



Time = 17



Errors dominate the solution after $t \sim 15$

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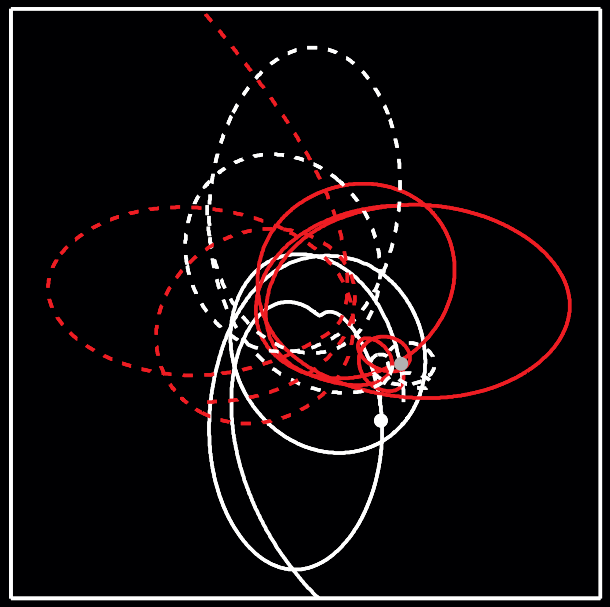
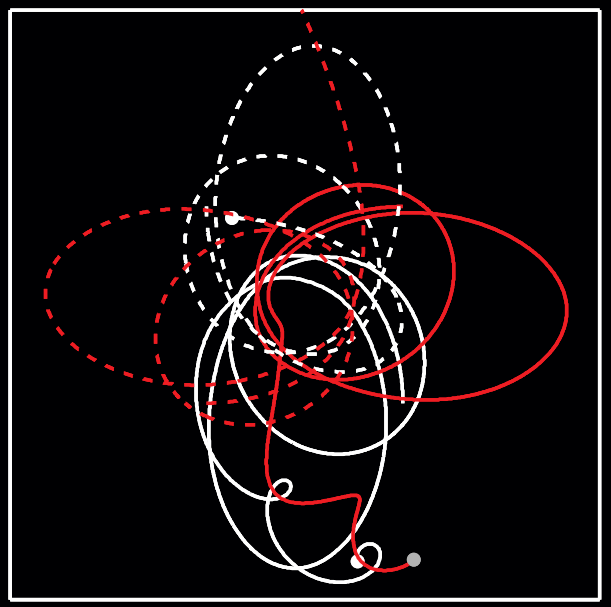
8-BODY PROBLEM

Two different solutions in KS space ($\vartheta = 270°$):

Time = 20



Time = 20



Errors dominate the solution after $t \sim 15$

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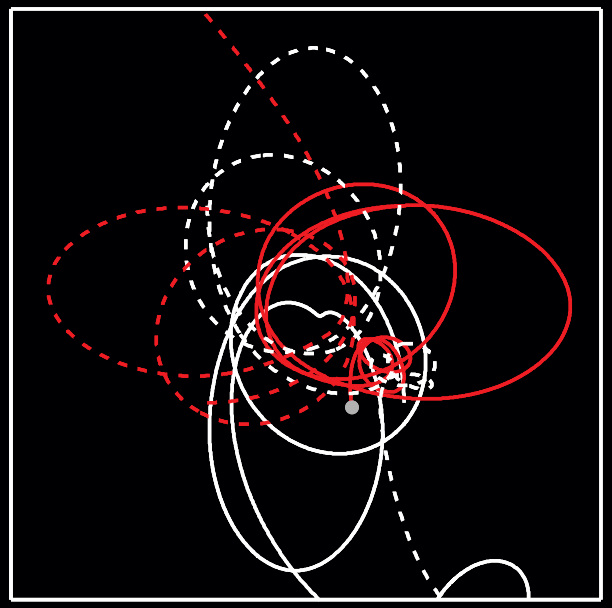
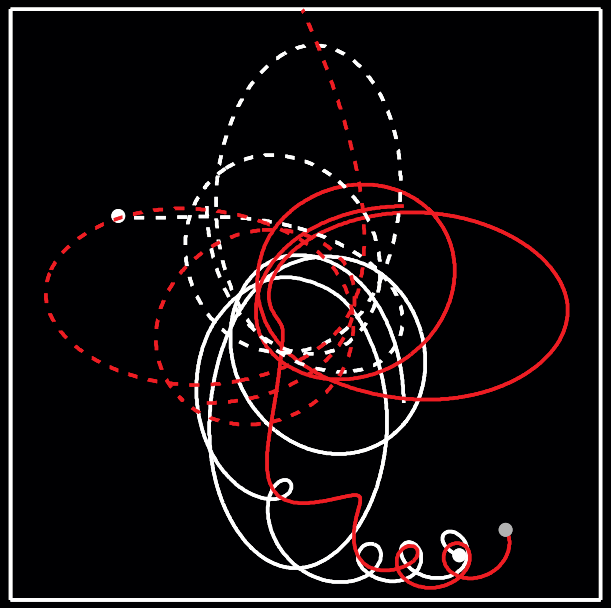
8-BODY PROBLEM

Two different solutions in KS space ($\vartheta = 270°$):

Time = 23



Time = 23



Errors dominate the solution after $t \sim 15$

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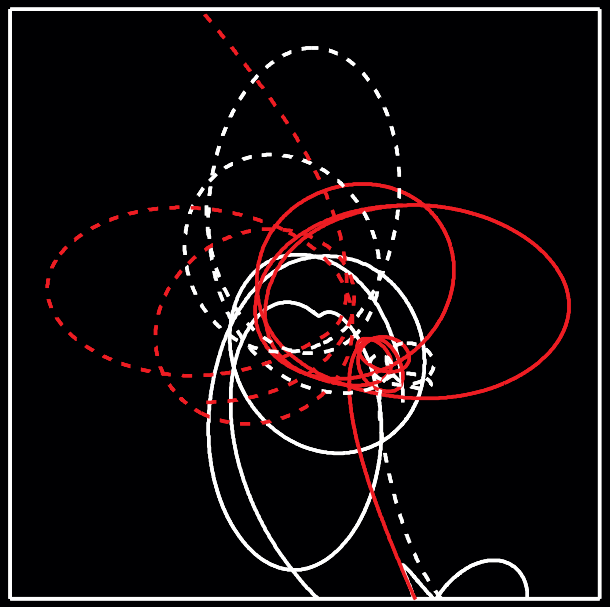
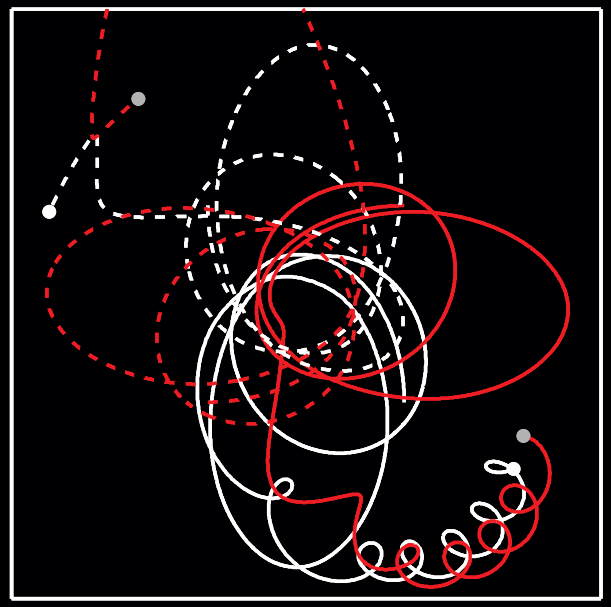
8-BODY PROBLEM

Two different solutions in KS space ($\vartheta = 270°$):

Time = 26



Time = 26



Errors dominate the solution after $t \sim 15$

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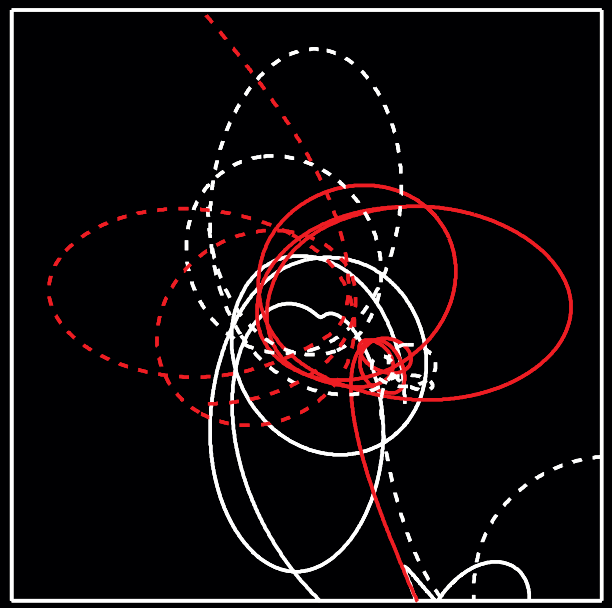
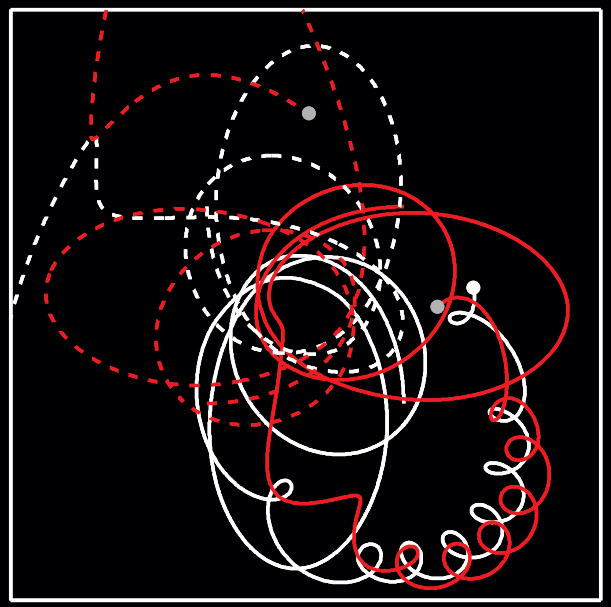
8-BODY PROBLEM

Two different solutions in KS space ($\vartheta = 270°$):

Time = 30



Time = 30



Errors dominate the solution after $t \sim 15$

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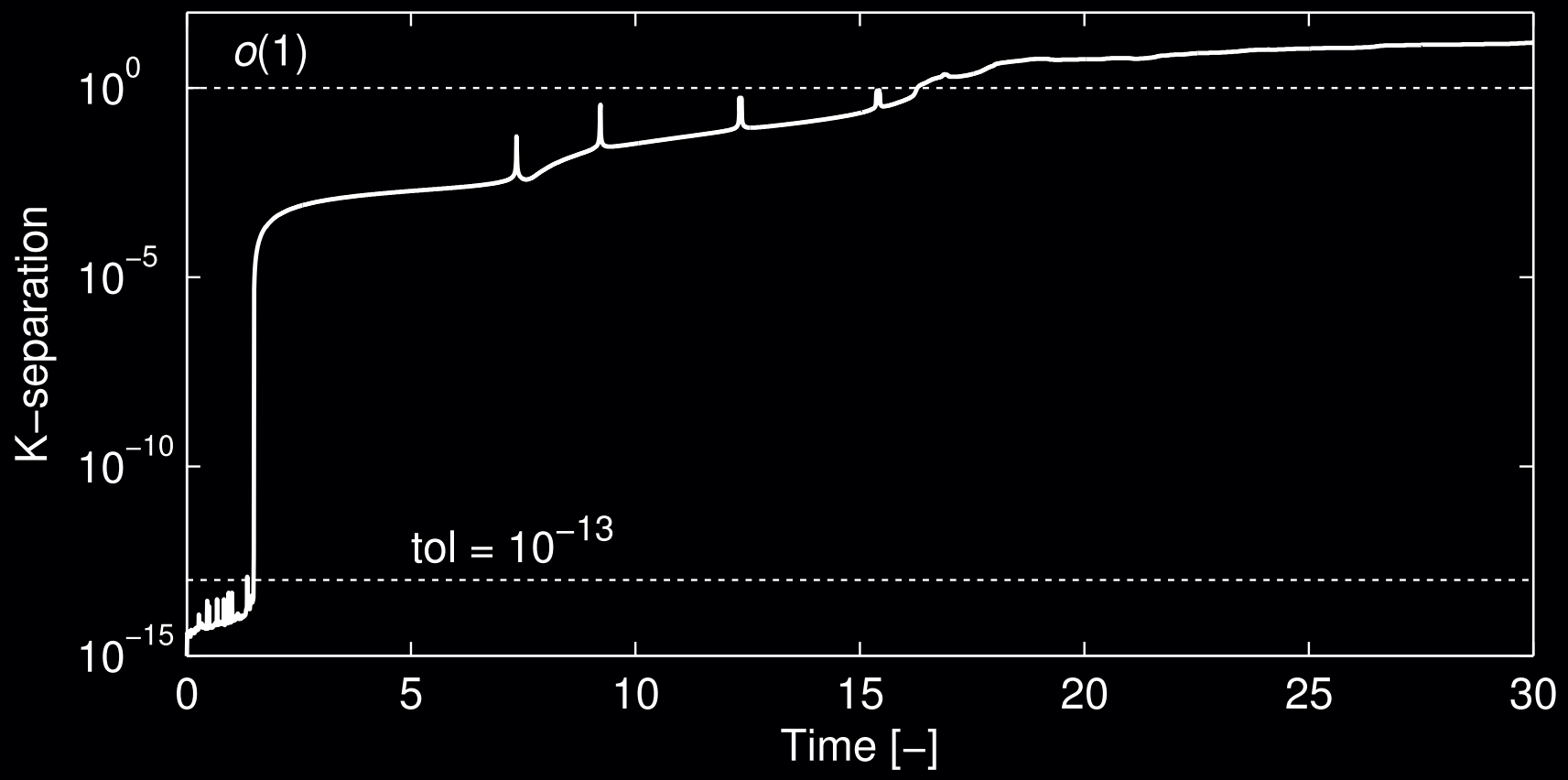
8-body problem II



Errors dominate the solution after $t \sim 15$

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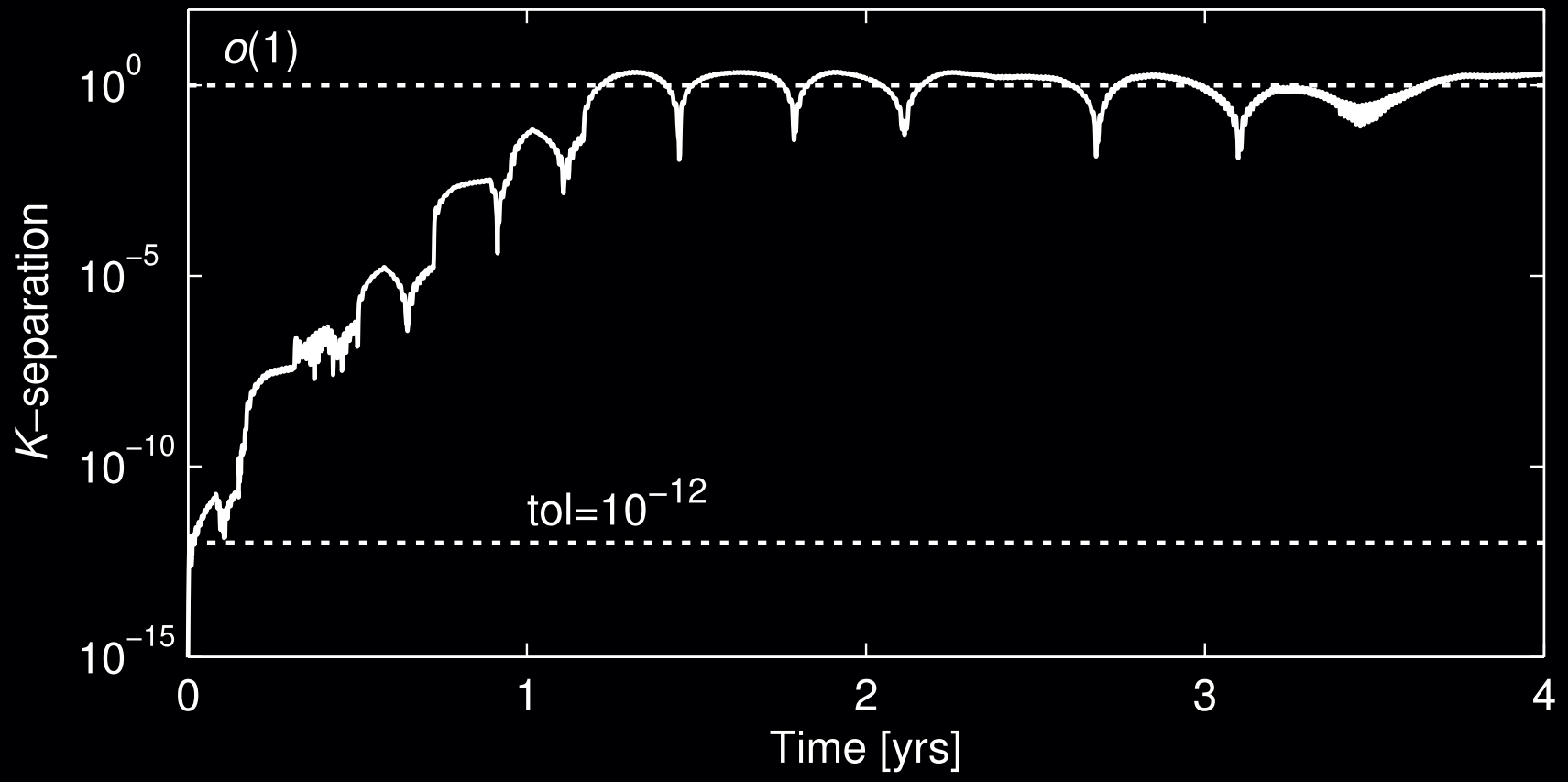
Jovian system



Errors dominate the solution after $t \sim 1.25$ years

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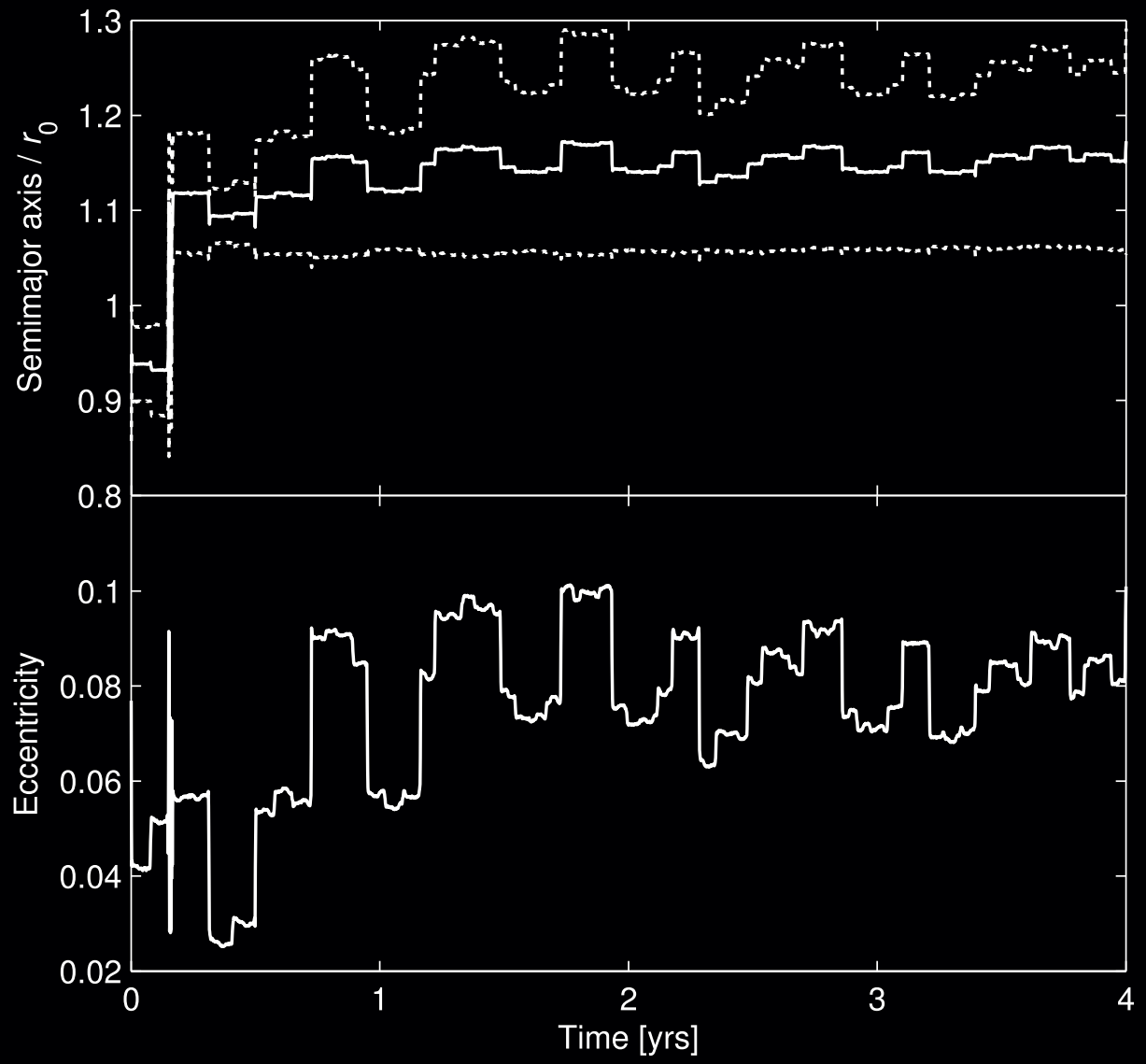
Jovian system II



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Conclusions

Topological stability

- There are infinitely many trajectories in KS space that represent the same orbit.
- Connected by the structure of the Hopf fibration.
- Constraint on the solutions $\longrightarrow$ Topological stability.

Quantifying numerical errors

- Numerical errors might destroy topological stability.
- $\mathscr{K}$-separation as a measure of the numerical error.

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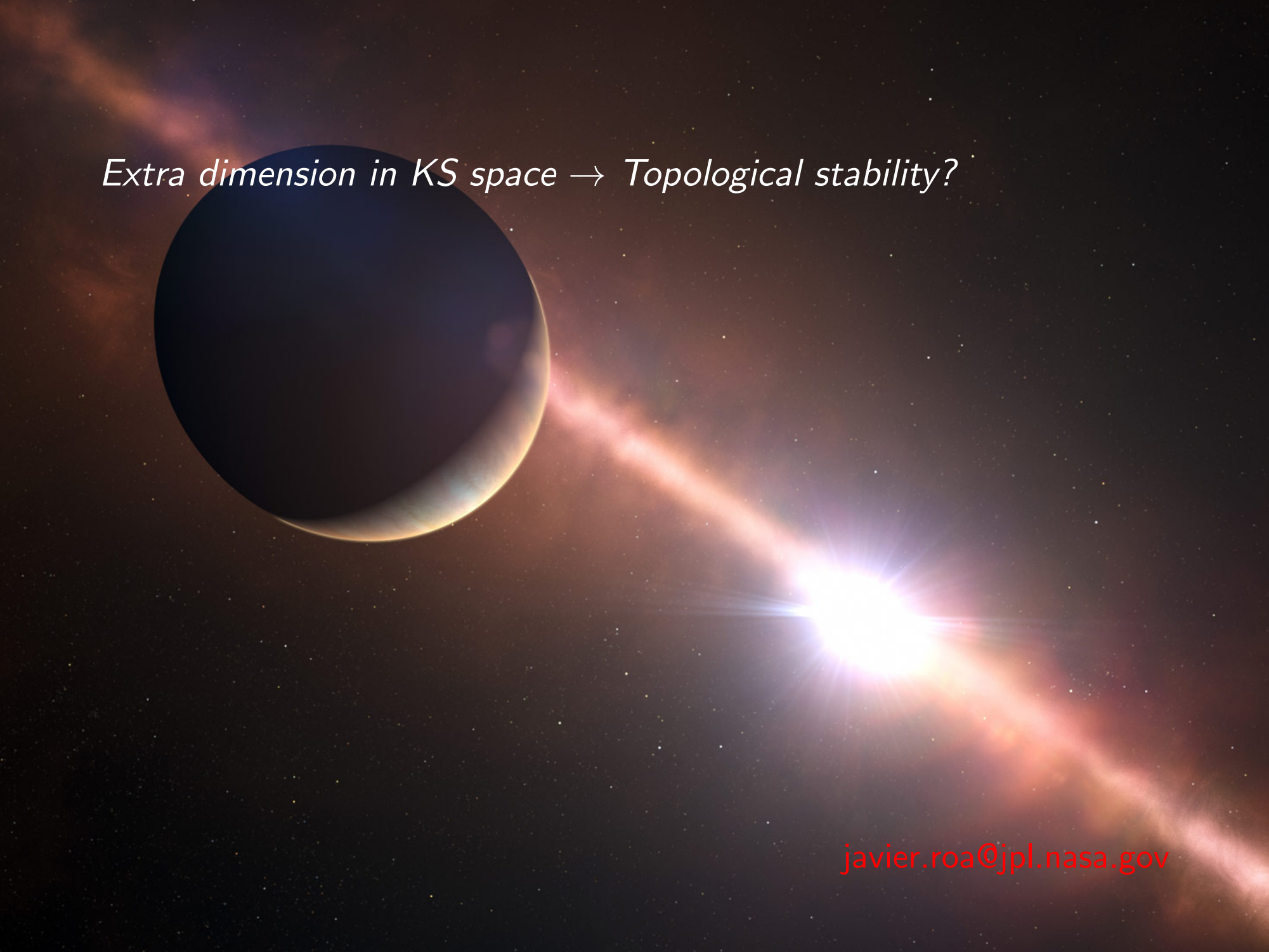
Extra dimension in KS space $\rightarrow$ *Topological stability?*



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