A Trilateration Scheme for Relative Positioning

Kar-Ming Cheung, Charles Lee
Jet Propulsion Laboratory, California Institute of Technology

March 04-11, 2017

2017 IEEE Aerospace Conference
Big Sky, Montana

© 2016, California Institute of Technology. Government sponsorship acknowledged.

Outline of Talk

Theory and Analysis

- Different Trilateration Approaches
- Major Error Sources in Trilateration

Simulations of Performance

- Two Scenarios of Using Ground Stations to Track Spacecraft in GEO
- Relative Accuracy Performance

Conclusion and Latest Results

- Concluding Remarks
- Latest Results and Ongoing Work

Theory/Analysis: Different Trilateration Schemes (1)

- Trilateration is the process of determining absolute and relative locations of points by measurements of distances

- In this paper, we discuss three trilateration approaches:
  - Newton-Raphson Method (absolute positioning)
  - Montenbruck's Scheme (2002) for relative positioning
  - New relative positioning scheme introduced in this paper

- The Newton-Rapson Method is the current GPS approach for absolute positioning

- Montenbruck's scheme was introduced in 2002 that suggests using GPS to perform relative positioning between two or more spacecraft at LEO
  - Very similar, or the same approach was proposed for X-ray navigation – use X-ray pulsars to estimate the relative position of a spacecraft relative to Earth
  - Montenbruck's scheme has an underlying interferometric approximation in its key equation, which is valid for interstellar case (X-ray pulsars), but not for GEO distance

- The new scheme uses similar approach as Montenbruck's, but is an <u>exact</u> solver

Theory/Analysis: Different Trilateration Schemes (2)
Newton-Raphson Method: Current GPS Approach

$$\sqrt{(x-x_i)^2+(y-y_i)^2+(z-z_i)^2}+c\Delta t=d_i$$

Given sat_i and ρ_i for $i=1,\ldots,n;\quad n\geq 4$

Initialize $\begin{pmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ and $T=0$

LS Iterative Scheme:

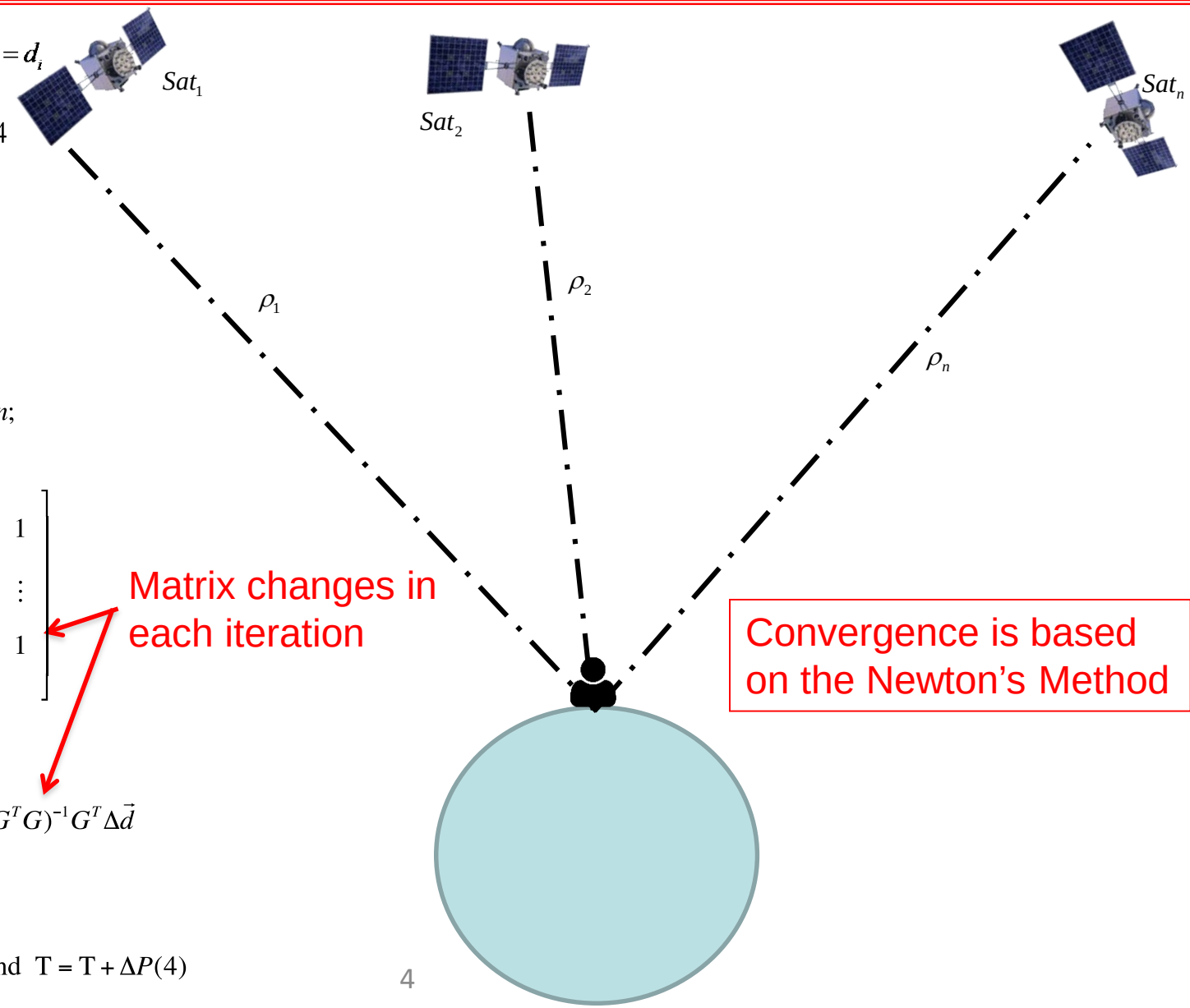
$$r_i=\left\|sat_i-\begin{pmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{pmatrix}\right\| \quad \text{for } i=1,\ldots,n;$$

$$G=\begin{bmatrix}\frac{x_1-\tilde{x}}{r_1} & \frac{y_1-\tilde{y}}{r_1} & \frac{z_1-\tilde{z}}{r_1} & 1\\ \vdots & \vdots & \vdots & \vdots\\ \frac{x_n-\tilde{x}}{r_n} & \frac{y_n-\tilde{y}}{r_n} & \frac{z_n-\tilde{z}}{r_n} & 1\end{bmatrix}$$

Matrix changes in each iteration

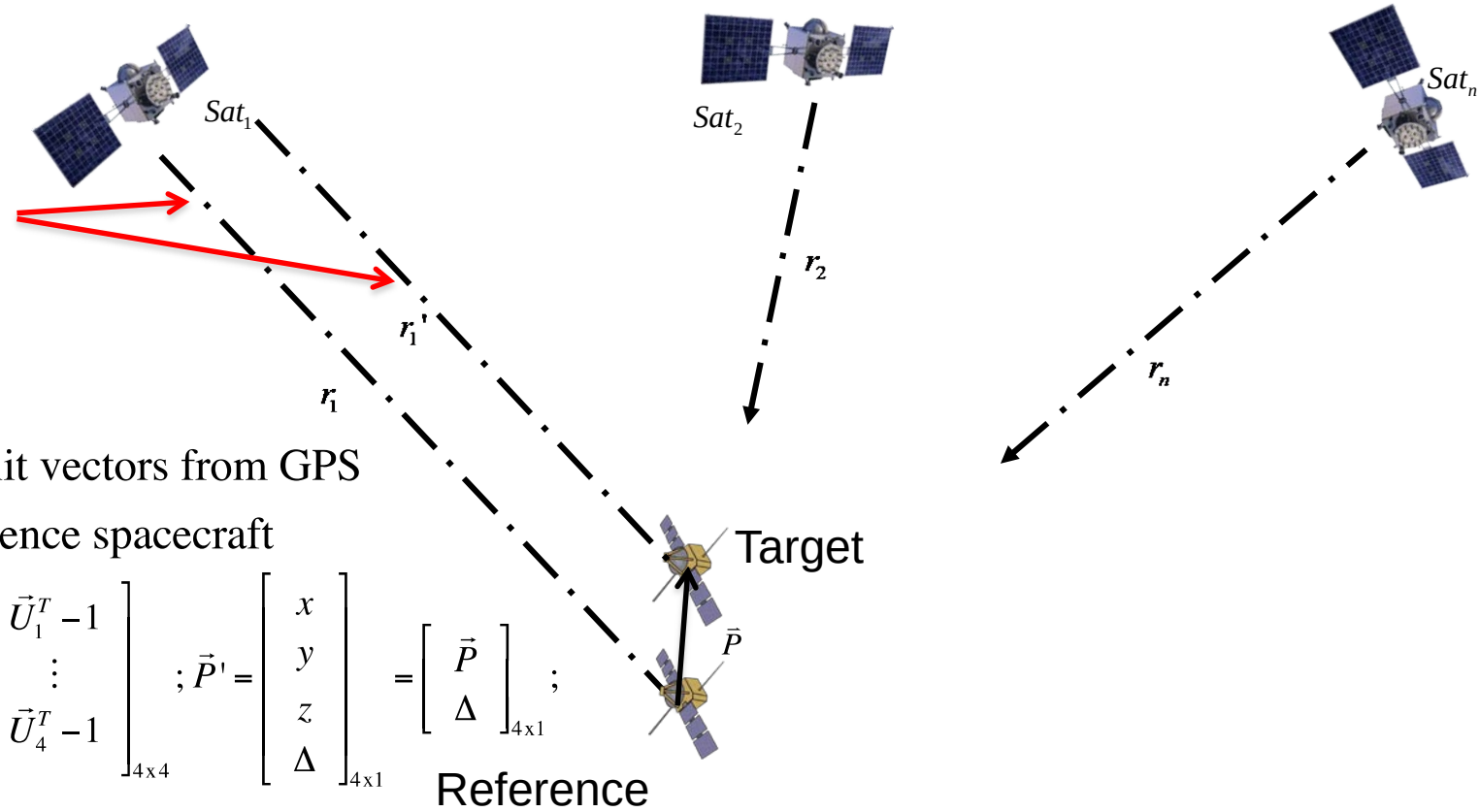
$$\Delta\vec{d}=\begin{bmatrix}\rho_1-r_1+T\\ \vdots\\ \rho_n-r_n+T\end{bmatrix};\quad \Delta\vec{P}=(G^TG)^{-1}G^T\Delta\vec{d}$$

$$\begin{pmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{pmatrix}=\begin{pmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{pmatrix}+\begin{pmatrix}\Delta P(1)\\ \Delta P(2)\\ \Delta P(3)\end{pmatrix} \text{ and } \mathrm{T}=\mathrm{T}+\Delta P(4)$$



Convergence is based on the Newton's Method

Theory/Analysis: Different Trilateration Schemes (3)
Montenbruck's Scheme: interferometric Approximation



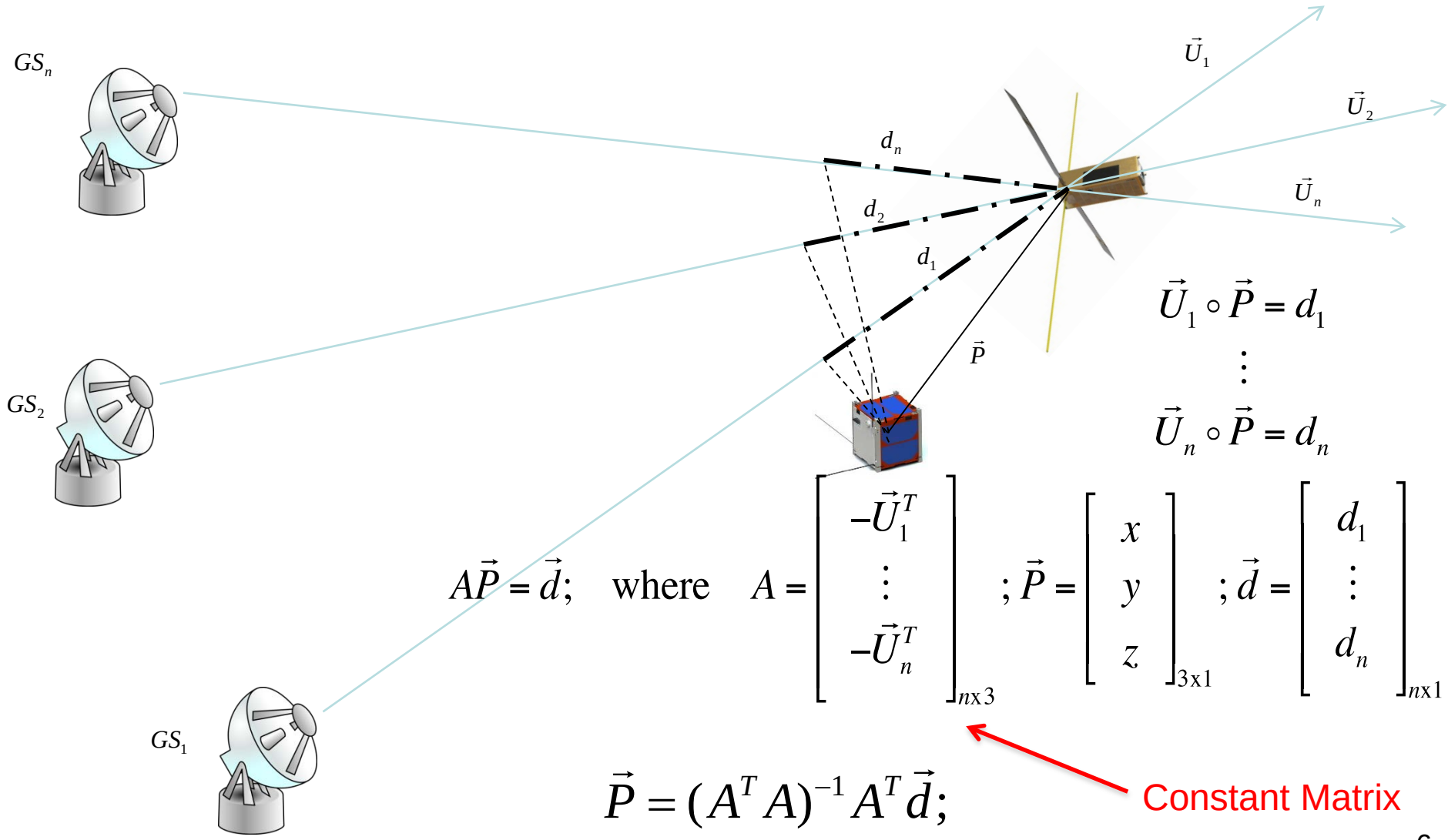
$\vec{U}_1, \vec{U}_2, \vec{U}_3, \vec{U}_4$ are unit vectors from GPS satellites to the reference spacecraft

$$A\vec{P}' = \vec{d}; \quad \text{where} \quad A = \begin{bmatrix} \vec{U}_1^T & -1 \\ \vdots & \\ \vec{U}_4^T & -1 \end{bmatrix}_{4\text{x}4}; \vec{P}' = \begin{bmatrix} x \\ y \\ z \\ \Delta \end{bmatrix}_{4\text{x}1} = \begin{bmatrix} \vec{P} \\ \Delta \end{bmatrix}_{4\text{x}1};$$

$$\vec{d} = \begin{bmatrix} r_1 - r_1' - \Delta \\ \vdots \\ r_4 - r_4' - \Delta \end{bmatrix}_{4\text{x}1}$$

$$\vec{P}' = (A^T A)^{-1} A^T \vec{d};$$

Theory/Analysis: Different Trilateration Schemes (4)
New Scheme: Relative Positioning High-Level Concept



$$\vec{U}_1 \circ \vec{P} = d_1$$
$$\vdots$$
$$\vec{U}_n \circ \vec{P} = d_n$$

$$A\vec{P} = \vec{d}; \quad \text{where} \quad A = \begin{bmatrix} -\vec{U}_1^T \\ \vdots \\ -\vec{U}_n^T \end{bmatrix}_{n\text{x}3}; \vec{P} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\text{x}1}; \vec{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}_{n\text{x}1}$$

$$\vec{P} = (A^T A)^{-1} A^T \vec{d};$$

Constant Matrix

Theory/Analysis: Different Trilateration Schemes (5)
New Scheme: No Clock Bias btw Reference and Target

- Direct measurement from 2-way CDMA ranging signals* (Initialization)

$$d_1 \approx r_1 - r_1' \qquad d_2 \approx r_2 - r_2' \qquad d_3 \approx r_3 - r_3'$$

$$\vec{P} = A^{-1}\vec{d}$$

- Updated estimation of d_1, and re-compute P (Iteration)

$$s_1^2 = \|\vec{P}\|^2 - \|\vec{U}_1 \circ \vec{P}\|^2 \qquad s_2^2 = \|\vec{P}\|^2 - \|\vec{U}_2 \circ \vec{P}\|^2 \qquad s_3^2 = \|\vec{P}\|^2 - \|\vec{U}_3 \circ \vec{P}\|^2$$

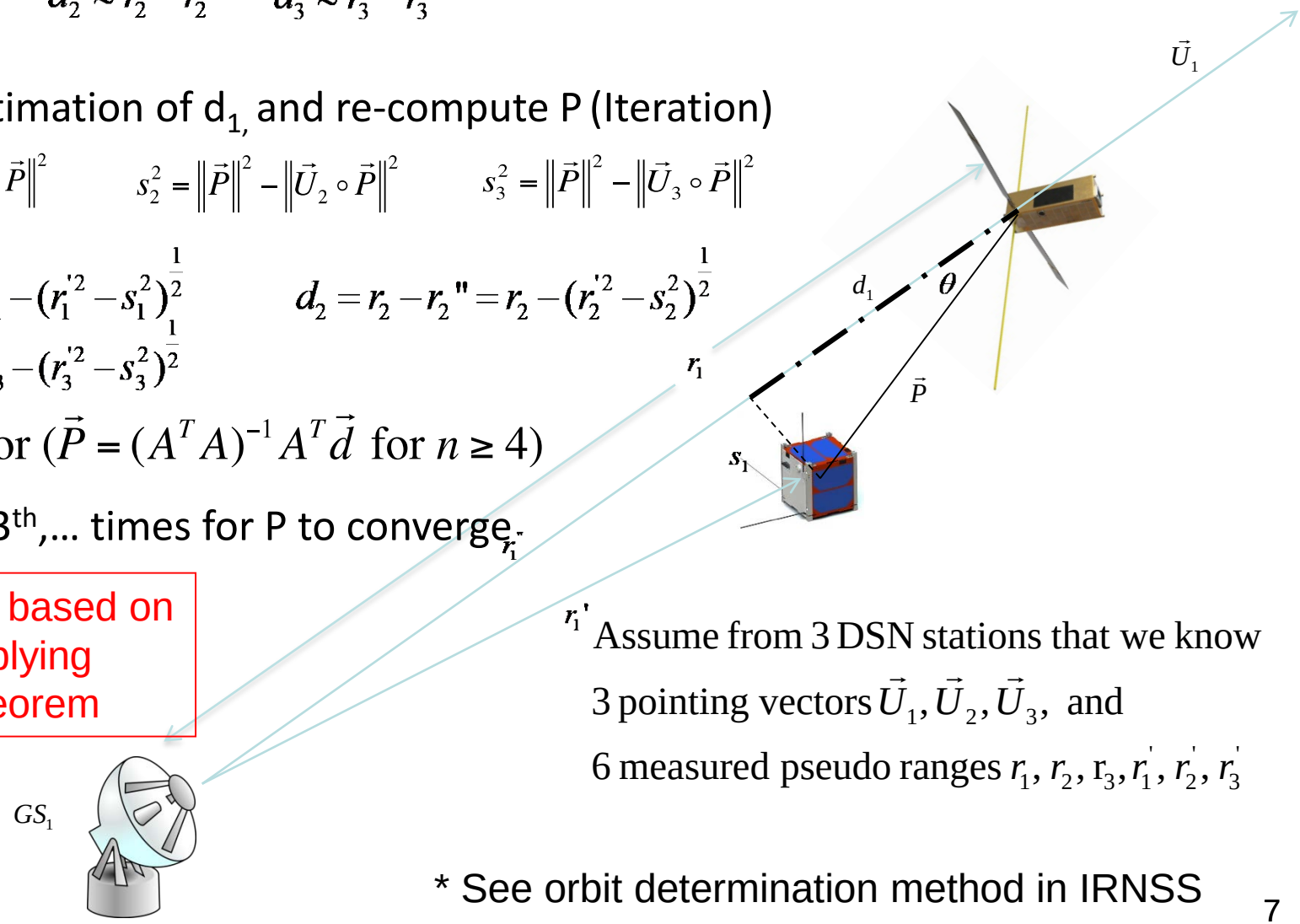
$$d_1 = r_1 - r_1'' = r_1 - (r_1'^2 - s_1^2)^{\frac{1}{2}} \qquad d_2 = r_2 - r_2'' = r_2 - (r_2'^2 - s_2^2)^{\frac{1}{2}}$$

$$d_3 = r_3 - r_3'' = r_3 - (r_3'^2 - s_3^2)^{\frac{1}{2}}$$

$$\vec{P} = A^{-1}\vec{d} \quad \text{or } (\vec{P} = (A^T A)^{-1} A^T \vec{d} \text{ for } n \geq 4)$$

- Iterate 2nd, 3th,… times for P to converge.

Convergence is based on alternatively applying Pythagoras' Theorem



Assume from 3 DSN stations that we know 3 pointing vectors $\vec{U}_1, \vec{U}_2, \vec{U}_3$, and 6 measured pseudo ranges $r_1, r_2, r_3, r_1', r_2', r_3'$

* See orbit determination method in IRNSS

Theory/Analysis: Different Trilateration Schemes (6)
New Scheme: with Clock Bias btw Reference and Target

Assume from 4 DSN stations (Gold, Madr, Mala, Kour) that we know

4 pointing vectors $\vec{U}_1, \vec{U}_2, \vec{U}_3, \vec{U}_4$, and 8 measured pseudo ranges $r_1, r_2, r_3, r_4, r_1', r_2', r_3', r_4'$,

where r_i' includes the delay Δ for $i = 1,\ldots,4$

then $-\vec{U}_i \circ \vec{P} = d_i = r_i - r_i'' = r_i - \sqrt{(r_i' - \Delta)^2 - s_i^2}$

or $-\vec{U}_i \circ \vec{P} - \Delta = d_i = r_i - r_i'' = r_i - \sqrt{(r_i' - \Delta)^2 - s_i^2} - \Delta$

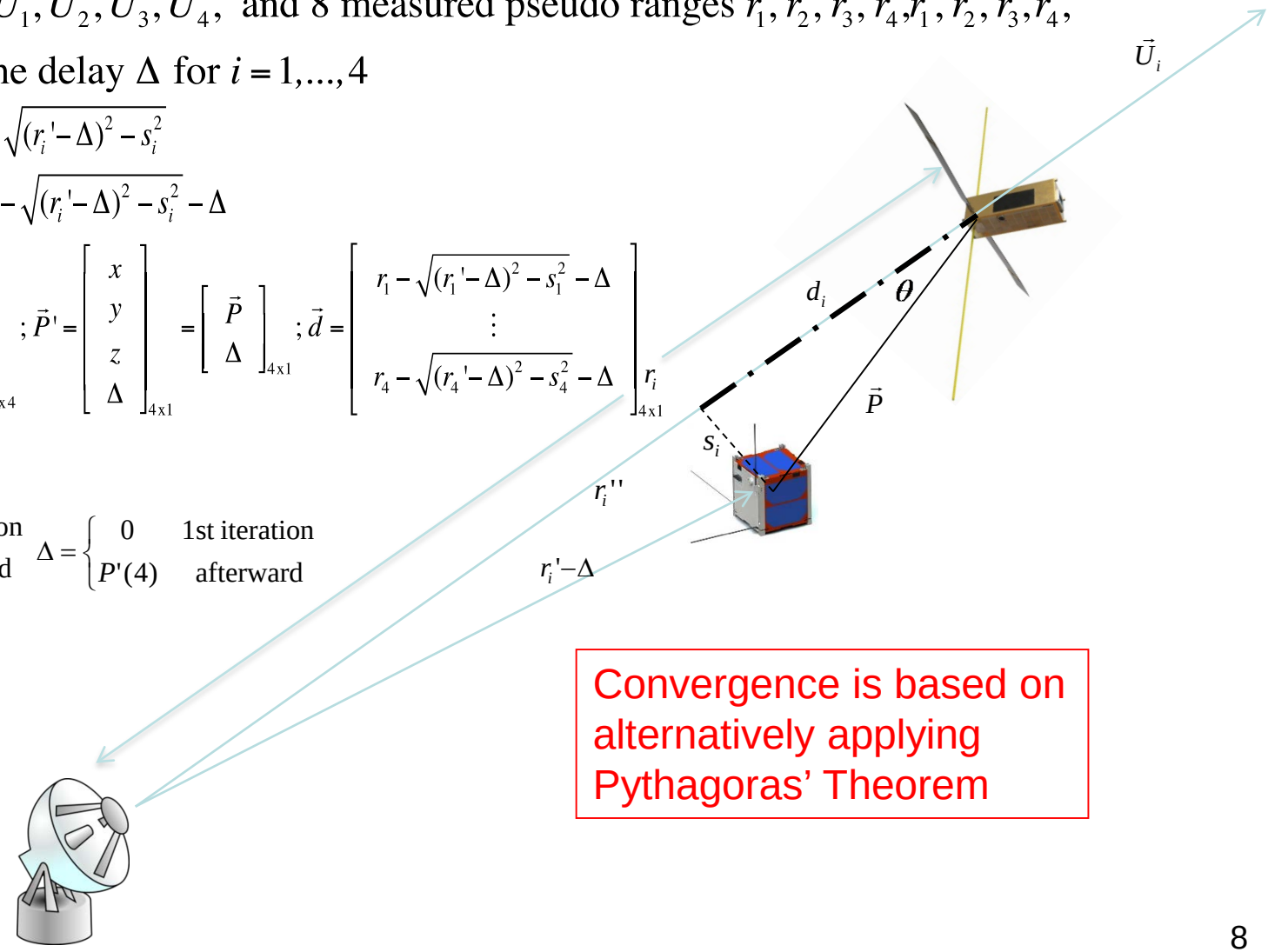
$$A\vec{P}' = \vec{d}; \quad \text{where} \quad A = \begin{bmatrix} -\vec{U}_1^T & -1 \\ \vdots & \\ -\vec{U}_4^T & -1 \end{bmatrix}_{4\text{x}4}; \vec{P}' = \begin{bmatrix} x \\ y \\ z \\ \Delta \end{bmatrix}_{4\text{x}1} = \begin{bmatrix} \vec{P} \\ \Delta \end{bmatrix}_{4\text{x}1}; \vec{d} = \begin{bmatrix} r_1 - \sqrt{(r_1' - \Delta)^2 - s_1^2} - \Delta \\ \vdots \\ r_4 - \sqrt{(r_4' - \Delta)^2 - s_4^2} - \Delta \end{bmatrix}_{4\text{x}1} r_i$$

Scheme:

$$s_i^2 = \begin{cases} 0 & \text{1st iteration} \\ \|\vec{P}\|^2 - (\vec{U}_i \circ \vec{P})^2 & \text{afterward} \end{cases} \quad \Delta = \begin{cases} 0 & \text{1st iteration} \\ P'(4) & \text{afterward} \end{cases}$$

$$d_i = r_i - \sqrt{(r_i' - \Delta)^2 - s_i^2} - \Delta$$

$$\vec{P}' = (A^T A)^{-1} A^T \vec{d}$$



Convergence is based on alternatively applying Pythagoras' Theorem

Theory/Analysis: Major Error Sources in Trilateration (1)

- Raw-range measurement between an anchor (e.g. GPS satellite, ground station) and a user can include the following error sources:
  - Random measurement errors: assume to be zero mean and Gaussian
  - Media delays: applicable if signals go through atmosphere (tropospheric delay and ionospheric delay). For relative positioning analysis, we need to consider both the common portion and the relative portion
  - Anchors' ephemeris errors: location errors of the infrastructure
  - Clock biases relative to the anchors: both reference and target spacecraft can have different clock biases relative to the anchors
- The trilateration scheme "amplify" the raw-range measurement errors in a manner somewhat similar to the dilution of precision (DOP) analysis in standard GPS literatures

Theory/Analysis: Major Error Sources in Trilateration (2)

- This analysis considers the location error only, and does not include clock bias

$$A\vec{P} = \vec{d}; \quad \text{where} \quad A = \begin{bmatrix} \vec{U}_1^T \\ \vdots \\ \vec{U}_n^T \end{bmatrix}_{n\text{x}3}; \vec{P} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\text{x}1}; \vec{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}_{n\text{x}1}$$

$$A(\vec{P} + \vec{e}_p) = \vec{d} + \vec{e}_d; \quad A\vec{e}_p = \vec{e}_d; \quad \vec{e}_p = (A^T A)^{-1} A^T \vec{e}_d;$$

$$\begin{aligned}
\| \vec{e}_p \|^2 &= ((A^T A)^{-1} A^T \vec{e}_d)^T (A^T A)^{-1} A^T \vec{e}_d = \vec{e}_d^T ((A^T A)^{-1} A^T)^T (A^T A)^{-1} A^T \vec{e}_d \\
&= \vec{e}_d^T A((A^T A)^{-1})^T (A^T A)^{-1} A^T \vec{e}_d = \vec{e}_d^T A((A^{-1}(A^T)^{-1})^T (A^T A)^{-1} A^T \vec{e}_d \\
&= \vec{e}_d^T A A^{-1} (A^T)^{-1} A^{-1} (A^T)^{-1} A^T \vec{e}_d = \vec{e}_d^T (A^T)^{-1} A^{-1} \vec{e}_d \\
&= \vec{e}_d^T (A A^T)^{-1} \vec{e}_d
\end{aligned}$$

Theory/Analysis: Major Error Sources in Trilateration (3)

- Properties of AA^T
  - AA^T is a symmetric matrix (and positive-definite)
  - All eigenvalues are positive
  - $[AA^T]_{ij}$ is the Cosine of the angle sustained by U_i and U_j
  - Largest and smallest eigenvalue of $(AA^T)^{-1}$ are $1/\lambda_{min}$ and $1/\lambda_{max}$, where λ_{min} *and* λ_{max} are the smallest and latest eigenvalues of AA^T

- Note that error amplification ratio $$\frac{\|\vec{e}_p\|^2}{\|\vec{e}_d\|^2} = \frac{\vec{e}_d^T\left(AA^T\right)^{-1}\vec{e}_d}{\vec{e}_d^T\vec{e}_d}$$

- Applying min-max theorem (Rayleigh's Quotient), we have

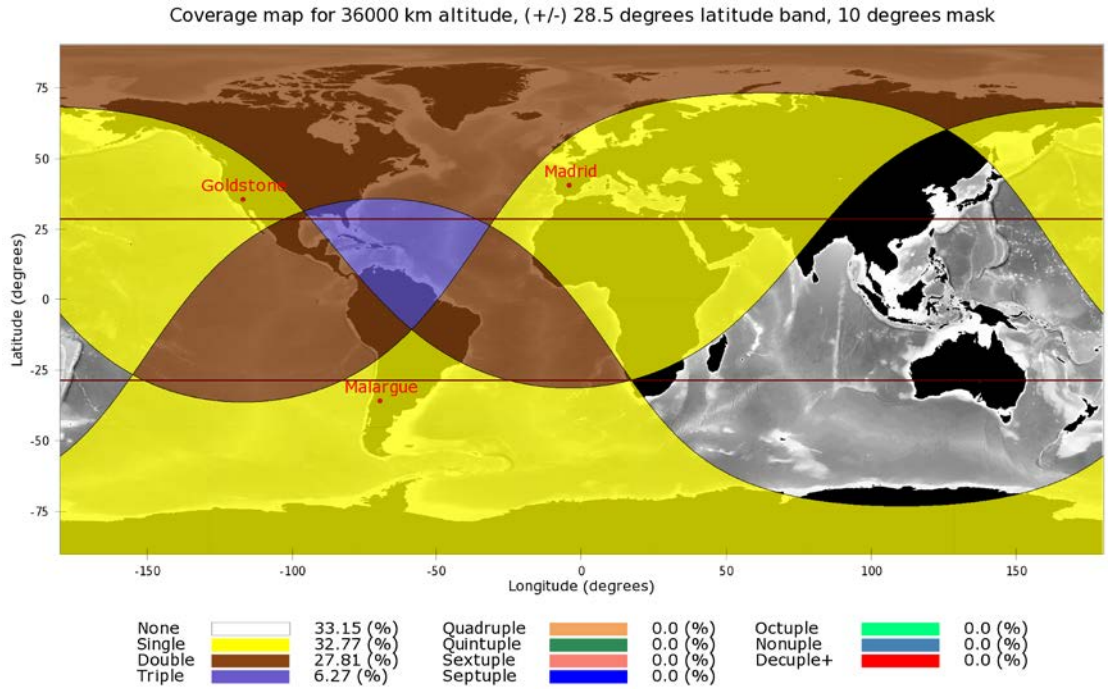
$$\frac{1}{\lambda_{\max}} \le \frac{\vec{x}^T\left(AA^T\right)^{-1}\vec{x}}{\vec{x}^T\vec{x}} \le \frac{1}{\lambda_{\min}} \quad \forall \vec{x}$$

- Thus a loose upper bound of relative localization mean square error is given by

$$\|\vec{e}_p\|^2 \le \frac{1}{\lambda_{\min}}\|\vec{e}_d\|^2$$

Simulation – Ground Stations Tracking Spacecraft at GEO: No Clock Bias Case (1)

- Consider a baseline network of ground stations at Goldstone, Madrid, and Marlargue, with the following visibility



- Simulation Setup:
  - Introduce an error of 0.1 millidegree in directional cosines U_1, U_2, and U_3
  - Assume random raw-range measurement errors with Gaussian distribution with zero mean and standard deviations of 2.5 mm and 5.0 mm respectively
  - Assume the raw-range measurements include <u>the same</u> media error σ of 10 or 100 meters that is unknown to the users
  - Consider the distance between the reference and the target spacecraft Δ to be 10, 100, or 500 kilometers

Simulation – Ground Stations Tracking Spacecraft at GEO: No Clock Bias Case (2)

- RMSE at GEO distance (36000 km), random RMS error 2.5 mm

	$\|\vec{P}\|$= 10 km	$\|\vec{P}\|$=100 km	$\|\vec{P}\|$=500 km
σ = 10 m	4.69 cm	8.56 cm	4.90 cm
σ = 100 m	5.63 cm	6.70 cm	5.09 cm

- RMSE at GEO distance (36000 km), random RMS error 5.0 mm

	$\|\vec{P}\|$= 10 km	$\|\vec{P}\|$=100 km	$\|\vec{P}\|$=500 km
σ = 10 m	17.86 cm	9.90 cm	9.07 cm
σ = 100 m	9.81 cm	13.34 cm	13.83 cm

Simulation – Ground Stations Tracking Spacecraft at GEO: With Clock Bias Case (1)

- Consider a baseline network of ground stations at Goldstone, Madrid, and Marlargue, and Kourou, and using the same error assumptions as before
- Compare with Montebruck's Scheme, for RMSE = 2.5 mm

Table 3. RMSE at GEO distance (4 stations), random RMS error = 2.5 mm

	$\|\vec{P}\|$ = 10 km	$\|\vec{P}\|$ =100 km	$\|\vec{P}\|$ =500 km
σ = 10 m, Δ = 100 m	57.77 cm	70.14 cm	53 m
σ = 100 m, Δ = 100 m	95.80 cm	49.30 cm	53 m
σ = 10 m, Δ = 1,000 m	52.64 cm	61.04 cm	54 m
σ = 100 m, Δ = 1,000 m	62.19 cm	123.68 cm	55 m
σ = 10 m, Δ = 10,000 m	53.39 cm	79.62 cm	54 m
σ = 100 m, Δ = 10,000 m	86.38 cm	94.82 cm	56 m

Table 5. RMSE Using Montenbruck's Scheme, random errors = 2.5 mm

	$\|\vec{P}\|$ = 10 km	$\|\vec{P}\|$ =100 km	$\|\vec{P}\|$ =500 km
σ = 10 m, Δ = 100 m	16 m	1633 m	41578 m
σ = 100 m, Δ = 100 m	17 m	1633 m	41661 m
σ = 10 m, Δ = 1,000 m	16 m	1633 m	41599 m
σ = 100 m, Δ = 1,000 m	17 m	1631 m	41559 m
σ = 10 m, Δ = 10,000 m	16 m	1633 m	41602 m
σ = 100 m, Δ = 10,000 m	16 m	1633 m	41628 m

Simulation – Ground Stations Tracking Spacecraft at GEO: With Clock Bias Case (2)

- Compare with Montebruck's Scheme, for RMSE = 5.0 mm

Unusual error behavior

Table 4. RMSE at GEO distance (4 stations), random RMS error = 5.0 mm

	$\|\vec{P}\|$ = 10 km	$\|\vec{P}\|$ =100 km	$\|\vec{P}\|$ =500 km
σ = 10 m, Δ = 100 m	118.03 cm	97.83 cm	54 m
σ = 100 m, Δ = 100 m	149.18 cm	96.22 cm	54 m
σ = 10 m, Δ = 1,000 m	121.40 cm	106.63 cm	54 m
σ = 100 m, Δ = 1,000 m	97.36 cm	111.03 cm	52 m
σ = 10 m, Δ = 10,000 m	189.30 cm	114.80 cm	53 m
σ = 100 m, Δ = 10,000 m	200.00 cm	149.33 cm	54 m

Table 6. RMSE Using Montenbruck's Scheme, random errors = 5.0 mm

	$\|\vec{P}\|$ = 10 km	$\|\vec{P}\|$ =100 km	$\|\vec{P}\|$ =500 km
σ = 10 m, Δ = 100 m	16 m	1632 m	41619 m
σ = 100 m, Δ = 100 m	17 m	1633 m	41629 m
σ = 10 m, Δ = 1,000 m	17 m	1633 m	41654 m
σ = 100 m, Δ = 1,000 m	18 m	1635 m	41688 m
σ = 10 m, Δ = 10,000 m	15 m	1631 m	41702 m
σ = 100 m, Δ = 10,000 m	17 m	1632 m	41628 m

Concluding Remarks

- In this paper, we introduce a trilateration scheme that evaluates the 3-D relative position between a reference spacecraft and a target spacecraft using Pythagoras' Theorem instead of the GPS's Newton's Method

- We consider at GEO distance the cases when a) the clocks on the reference spacecraft and the target spacecraft are perfectly synchronized, and b) there is a clock bias between the reference spacecraft and target spacecraft

- We demonstrate that under reasonable conservative error assumptions the algorithms converge to centimeter-level accuracy in the case of no clock bias, and meter-level accuracy in the presence of clock bias

- Note that we do not assume the use of any advanced GPS measurement techniques, signal-processing algorithms, and location-specific meteorological measurements. We believe that if we were to implement this scheme using the current state-of-the-art GPS capabilities, we could achieve a considerably better accuracy performance than what we show in this paper

Latest Results and Ongoing Work

- This paper assumes fixed atmospheric delays. But we found that accuracy can be sensitive to tropospheric and ionospheric delay differentials between the reference and target spacecraft, with additional accuracy degradation of a few centimeters

- Instead of using ground stations as “anchors”, one can use the side-lobes of GPS signals on the opposite side of Earth to perform better relative positioning for a constellation of spacecraft at GEO; analogous to MMS tracking of weak GPS signals

- Using similar problem formulation, we found that the same scheme can be used for GPS positioning. The accuracy performance is indistinguishable compared to the traditional GPS Newton-Raphson method, but much less computationally intensive

- We investigated the scenario of using GPS satellites “looking downward” to perform real-time relative positioning between two moving vehicles on or in the vicinity of the Earth’s surface, without the need of a reference station, and achieve decimeter-level accuracy for distance of 100 m
  - Enable applications like formation flying and collision avoidance for high-density and dynamic applications (e.g. Uber’s Elevate)