Jet Propulsion Laboratory
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Fundamentals and Applications of Data Assimilation

What, how, and what for?

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Self Introduction

1) B.S. Geophysics, University of Tokyo（1983）

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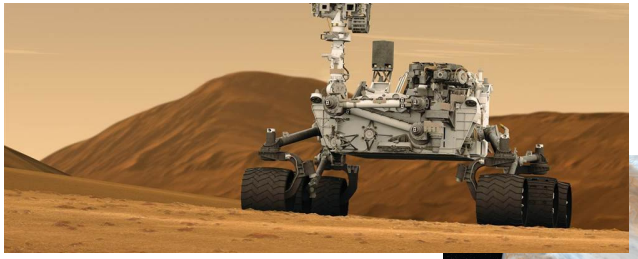
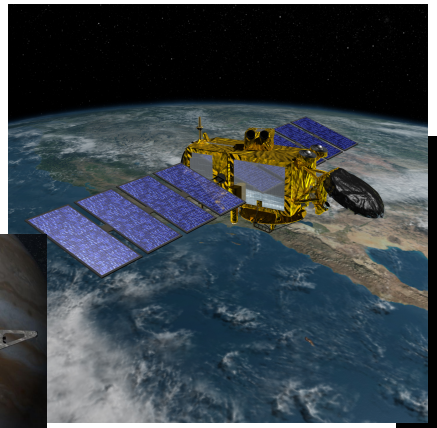
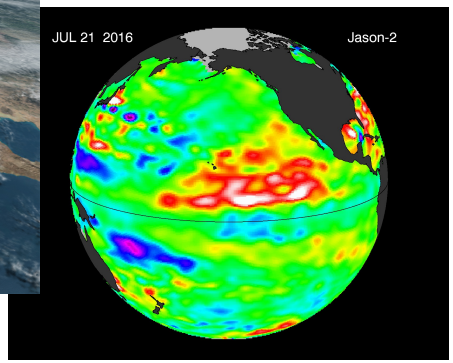
2) Ph.D. MIT-Woods Hole Oceanographic Institution, Joint Program in Oceanography (1989)

Carl Wunsch, Joseph Pedlosky, Steve Rintoul, S. Sakai, Y. Hayashi, M. Kawase, J. Yano

3) California Institute of Technology, Jet Propulsion Laboratory (1990-present)







Outline

1) Data Assimilation and Inverse Problem

1) ECCO Ocean Analysis

2) Applications of Adjoint Models

Gist to Solving Assimilation

Assimilation may seem complicated at first, due to its terminologies and equations, the gist to solving it is to recognize the simple underlying problem

Assimilation is nothing more than an inverse problem

Assimilation is to study what the state of the "ocean" is and to understand why that is so.

Observations $\mathbf{H}_t \mathbf{x}_t = \mathbf{y}_t$

Theory (model) $\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{G}\mathbf{u}_t$

data

$$\mathbf{y} = \begin{pmatrix} \vdots \\ \eta(x_1, y_1) \\ T(x_n, y_n, z_n) \\ \vdots \end{pmatrix}$$

state

$$\mathbf{x} = \begin{pmatrix} \vdots \\ u_{i,j,k} \\ v_{i,j,k} \\ T_{i,j,k} \\ S_{i,j,k} \\ \eta_{i,j} \\ \vdots \end{pmatrix}$$

external forcing and other inhomogeneous terms

$$\mathbf{u} = \begin{pmatrix} \vdots \\ \tau_{i,j} \\ q_{i,j} \\ r_{i,j} \\ \vdots \end{pmatrix}$$

Assimilation is nothing more than an inverse problem

$$\begin{pmatrix} \vdots \\ \mathbf{H}\mathbf{x}_t \\ \mathbf{H}\mathbf{x}_{t+1} \\ \vdots \\ \mathbf{x}_{t+1} - \mathbf{A}\mathbf{x}_t - \mathbf{G}\mathbf{u}_t \\ \mathbf{x}_{t+2} - \mathbf{A}\mathbf{x}_{t+1} - \mathbf{G}\mathbf{u}_{t+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \mathbf{y}_t \\ \mathbf{y}_{t+1} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \end{pmatrix}$$

Inverse Problem

Inverse Problem

$$\mathbf{E}\mathbf{a} = \mathbf{b}$$

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & \mathbf{H} & & & & & \cdots \\ \cdots & & \mathbf{H} & & & & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & -\mathbf{A} & \mathbf{I} & & -\mathbf{G} & & \cdots \\ \cdots & & -\mathbf{A} & \mathbf{I} & & -\mathbf{G} & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \vdots \\ \mathbf{x}_t \\ \mathbf{x}_{t+1} \\ \mathbf{x}_{t+2} \\ \vdots \\ \mathbf{u}_t \\ \mathbf{u}_{t+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \mathbf{y}_t \\ \mathbf{y}_{t+1} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \end{pmatrix}$$

Solving an Inverse Problem

Inverse Problem $\mathbf{E}\mathbf{a} = \mathbf{b}$

Most inverse problems in Earth Science are so-called "ill-posed problems" that have fewer equations than unknowns, and have an infinite number of possible solutions. Therefore, some form of assumptions are invariably made to derive a particular solution. Such solutions are valid so long as there are no inconsistencies, but those assumptions need to be recognized in interpreting the results.

For instance, consider the singular value decomposition $\mathbf{E} = \mathbf{U}\Lambda\mathbf{V}^T$ n x m matrix

$$\mathbf{U} = \begin{pmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_n \end{pmatrix}$$

n vector

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{pmatrix}$$

m vector

$$\mathbf{u}_i^T \mathbf{u}_j = \mathbf{v}_i^T \mathbf{v}_j = \delta_{ij}$$

$$\lambda_i = 0 \text{ for } i \geq r$$

$$\mathbf{U}^T\mathbf{U} = \mathbf{V}^T\mathbf{V} = \mathbf{I}$$

Rank of matrix **E**

inverse problem $\mathbf{Ea} = \mathbf{b}$ solution by singular value decomposition $\mathbf{E} = \mathbf{U}\Lambda\mathbf{V}^T$ n x m matrix

solution can generally be written as $\mathbf{a} = \sum_{i=1}^{r} c_i \mathbf{v}_i + \sum_{i=r+1}^{m} c_i \mathbf{v}_i$

$$\mathbf{Ea} = \mathbf{U}\Lambda\mathbf{V}^T \left(\sum_{i=1}^{r} c_i \mathbf{v}_i + \sum_{i=r+1}^{m} c_i \mathbf{v}_i \right)$$

$$= \mathbf{U}\Lambda\mathbf{V}^T \left(\mathbf{V}_r \mathbf{c}_r + \mathbf{V}_0 \mathbf{c}_0 \right)$$

$$= \mathbf{U}\Lambda\mathbf{c}_r = \mathbf{b}$$

$$\mathbf{U}^T\mathbf{U}\Lambda\mathbf{c}_r = \mathbf{U}^T\mathbf{b}$$

$$\mathbf{c}_r = \Lambda_r^{-1}\mathbf{U}_r^T\mathbf{b}$$

Ockham's razor $c_i = 0$ for all $\mathrm{i} \geq \mathrm{r}+1$

"nothing should be more complicated than necessary"

$$\mathbf{a} \approx \sum_{i=1}^{r} c_i \mathbf{v}_i \qquad c_i = \left(\Lambda_r^{-1}\mathbf{U}^T\mathbf{b} \right)_i$$

Minimum Length Solution $\min\left(|\mathbf{a}|\right)$

inverse problem $\mathbf{Ea} = \mathbf{b}$

Solve for a solution with minimum expected error using prior information

prior solution $\mathbf{a}_0$ $\left\langle \left(\mathbf{a}_0 - \bar{\mathbf{a}}\right)\left(\mathbf{a}_0 - \bar{\mathbf{a}}\right)^T \right\rangle \equiv \mathbf{P}_0$ $\bar{\mathbf{a}}, \bar{\mathbf{b}}$: true value

observation $\mathbf{b}$ $\left\langle \left(\mathbf{b} - \bar{\mathbf{b}}\right)\left(\mathbf{b} - \bar{\mathbf{b}}\right)^T \right\rangle \equiv \mathbf{R}$ in particular $\bar{\mathbf{b}} \equiv \mathbf{E}\bar{\mathbf{a}}$

In particular $\mathbf{P} \equiv \left\langle \left(\mathbf{a} - \bar{\mathbf{a}}\right)\left(\mathbf{a} - \bar{\mathbf{a}}\right)^T \right\rangle$ minimize each diagonal element

here $\mathbf{a} = \mathbf{a}_0 + \mathbf{B}\left(\mathbf{b} - \mathbf{Ea}_0\right)$ assume a linear solution

first
$$\begin{aligned}
\mathbf{a} - \bar{\mathbf{a}} &= \mathbf{a}_0 + \mathbf{B}\left(\mathbf{b} - \mathbf{Ea}_0\right) - \bar{\mathbf{a}} \\
&= \mathbf{B}\left(\mathbf{b} - \mathbf{Ea}_0 - \mathbf{E}\bar{\mathbf{a}} + \mathbf{E}\bar{\mathbf{a}}\right) + \left(\mathbf{a}_0 - \bar{\mathbf{a}}\right) \\
&= \mathbf{B}\left(\mathbf{b} - \mathbf{E}\bar{\mathbf{a}}\right) - \mathbf{B}\left(\mathbf{Ea}_0 - \mathbf{E}\bar{\mathbf{a}}\right) + \left(\mathbf{a}_0 - \bar{\mathbf{a}}\right) \\
&= \mathbf{B}\left(\mathbf{b} - \mathbf{E}\bar{\mathbf{a}}\right) + \left(\mathbf{I} - \mathbf{BE}\right)\left(\mathbf{a}_0 - \bar{\mathbf{a}}\right) = \boxed{\mathbf{B}\delta\mathbf{b} + \left(\mathbf{I} - \mathbf{BE}\right)\delta\mathbf{a}_0}
\end{aligned}$$

previously $\mathbf{a}-\bar{\mathbf{a}}=\mathbf{B}\delta\mathbf{b}+\left(\mathbf{I}-\mathbf{BE}\right)\delta\mathbf{a}_0$ use to expand P

$$\left\langle\left(\mathbf{a}-\bar{\mathbf{a}}\right)\left(\mathbf{a}-\bar{\mathbf{a}}\right)^T\right\rangle=\mathbf{B}\left\langle\delta\mathbf{b}\,\delta\mathbf{b}^T\right\rangle\mathbf{B}^T+\mathbf{B}\left\langle\cancel{\delta\mathbf{b}\,\delta\mathbf{a}_0{}^T}\right\rangle\left(\mathbf{I}-\mathbf{BE}\right)^T$$

$$+\left(\mathbf{I}-\mathbf{BE}\right)\left\langle\cancel{\delta\mathbf{a}_0\,\delta\mathbf{b}^T}\right\rangle\mathbf{B}^T+\left(\mathbf{I}-\mathbf{BE}\right)\left\langle\delta\mathbf{a}_0\,\delta\mathbf{a}_0{}^T\right\rangle\left(\mathbf{I}-\mathbf{BE}\right)^T$$

$$=\mathbf{BRB}^T+\left(\mathbf{I}-\mathbf{BE}\right)\mathbf{P}_0\left(\mathbf{I}-\mathbf{BE}\right)^T$$

$$=\mathbf{BRB}^T+\mathbf{P}_0-\mathbf{BEP}_0-\mathbf{P}_0\mathbf{E}^T\mathbf{B}^T+\mathbf{BEP}_0\mathbf{E}^T\mathbf{B}^T$$

$$=\mathbf{B}\left(\mathbf{EP}_0\mathbf{E}^T+\mathbf{R}\right)\mathbf{B}^T-\mathbf{BEP}_0-\mathbf{P}_0\mathbf{E}^T\mathbf{B}^T+\mathbf{P}_0$$

$$\boxed{\begin{aligned}&=\left(\mathbf{B}-\mathbf{P}_0\mathbf{E}^T\left(\mathbf{EP}_0\mathbf{E}^T+\mathbf{R}\right)^{-1}\right)\left(\mathbf{EP}_0\mathbf{E}^T+\mathbf{R}\right)\left(\mathbf{B}-\mathbf{P}_0\mathbf{E}^T\left(\mathbf{EP}_0\mathbf{E}^T+\mathbf{R}\right)^{-1}\right)^T\\&\quad-\mathbf{P}_0\mathbf{E}^T\left(\mathbf{EP}_0\mathbf{E}^T+\mathbf{R}\right)^{-1}\mathbf{EP}_0+\mathbf{P}_0\end{aligned}}$$

$\therefore\quad \mathbf{B}=\mathbf{P}_0\mathbf{E}^T\left(\mathbf{EP}_0\mathbf{E}^T+\mathbf{R}\right)^{-1}$ and $\left\langle\left(\mathbf{a}-\bar{\mathbf{a}}\right)\left(\mathbf{a}-\bar{\mathbf{a}}\right)^T\right\rangle=\mathbf{P}_0-\mathbf{P}_0\mathbf{E}^T\left(\mathbf{EP}_0\mathbf{E}^T+\mathbf{R}\right)^{-1}\mathbf{EP}_0$

Wunsch, (2006), *p128*.

minimum error variance solution

inverse problem $\mathbf{E}\mathbf{a} = \mathbf{b}$

$$\mathbf{a} = \mathbf{a}_0 + \mathbf{P}_0\mathbf{E}^T\left(\mathbf{E}\mathbf{P}_0\mathbf{E}^T + \mathbf{R}\right)^{-1}\left(\mathbf{b} - \mathbf{E}\mathbf{a}_0\right)$$

Kalman Filter (correct with data)

observation $\mathbf{H}_t\mathbf{x}_t = \mathbf{y}_t$

$$\hat{\mathbf{x}}_t^a = \hat{\mathbf{x}}_t^f + \mathbf{P}_t^f\mathbf{H}^T\left(\mathbf{H}\mathbf{P}_t^f\mathbf{H}^T + \mathbf{R}_t\right)^{-1}\left(\mathbf{y}_t - \mathbf{H}\hat{\mathbf{x}}_t^f\right)$$

RTS smoother

model $\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{G}\mathbf{u}_t$

$$\hat{\mathbf{x}}_t^s = \hat{\mathbf{x}}_t^a + \mathbf{P}_t^a\mathbf{A}^T\left(\mathbf{A}\mathbf{P}_t^a\mathbf{A}^T + \mathbf{G}\mathbf{Q}_t\mathbf{G}^T\right)^{-1}\left(\hat{\mathbf{x}}_{t+1}^s - \hat{\mathbf{x}}_{t+1}^f\right)$$

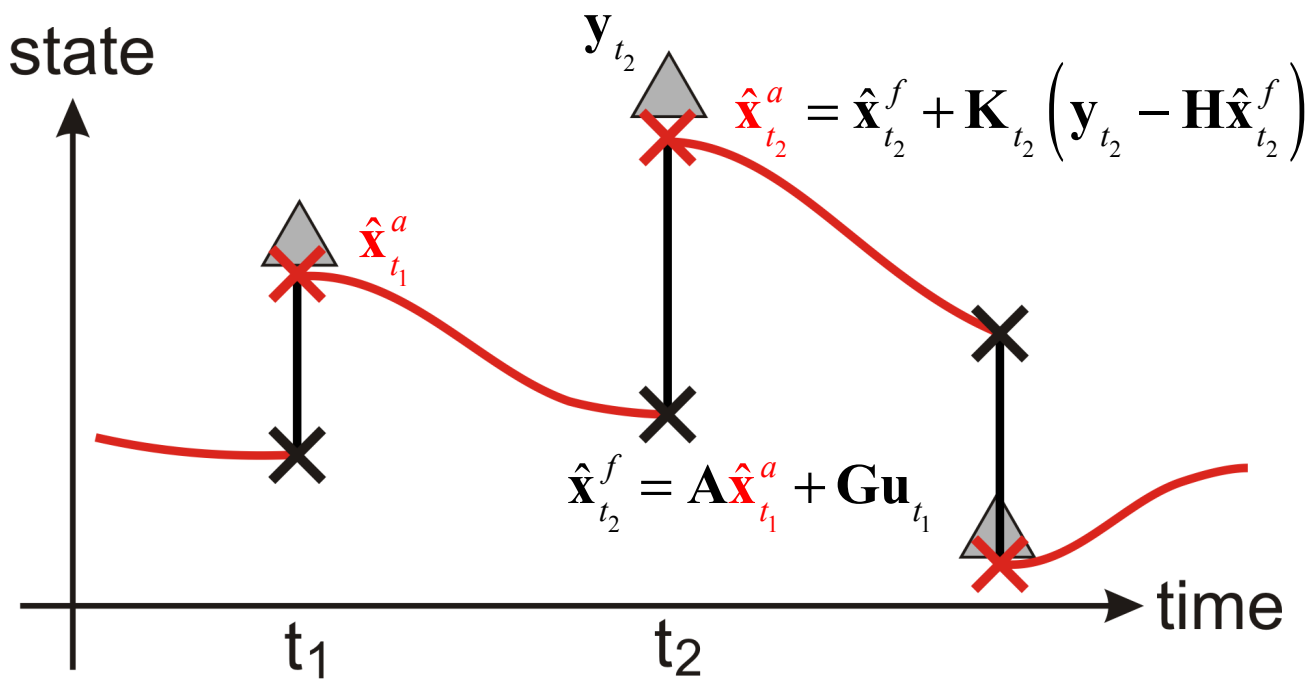
$$\hat{\mathbf{u}}_t^s = \mathbf{u}_t + \mathbf{Q}_t\mathbf{G}^T\left(\mathbf{G}\mathbf{Q}_t\mathbf{G}^T + \mathbf{A}\mathbf{P}_t^a\mathbf{A}^T\right)^{-1}\left(\hat{\mathbf{x}}_{t+1}^s - \hat{\mathbf{x}}_{t+1}^f\right)$$

$$\hat{\mathbf{x}}_{t+1}^s = \mathbf{A}\hat{\mathbf{x}}_t^s + \mathbf{G}\hat{\mathbf{u}}_t^s$$

notable points of data assimilation（1）

- data assimilation is an inverse problem and does not generally have a unique solution,
- Kalman filters, RTS smoothers are both minimum error variance solvers
- Kalman filters invert observations（**H**）, RTS smoothers invert the model（**A**, **G**）,
- Kalman filters do not assume normal distributions,
- Data error is not error of the observation but includes so-called representation error $\delta\mathbf{y} \equiv \mathbf{y} - \bar{\mathbf{y}} \equiv \mathbf{y} - \mathbf{H}\bar{\mathbf{x}}$
- Process noise（**G**, **u**）define the true solution $\bar{\mathbf{x}}$ and, therefore, is fundamental to data assimilation $\bar{\mathbf{x}}_{t+1} = \mathbf{A}\bar{\mathbf{x}}_t + \mathbf{G}\bar{\mathbf{u}}_t$ $\mathbf{u}_t = \bar{\mathbf{u}}_t + \delta\mathbf{u}_t$
- Assimilation is about extracting elements from data that are consistent with the model

Physical consistency (filter)



Due to data corrections, the filtered solutions' temporal evolution cannot be physically explained, but smoothers invert these corrections and, therefore, their solutions' evolution are physically consistent.

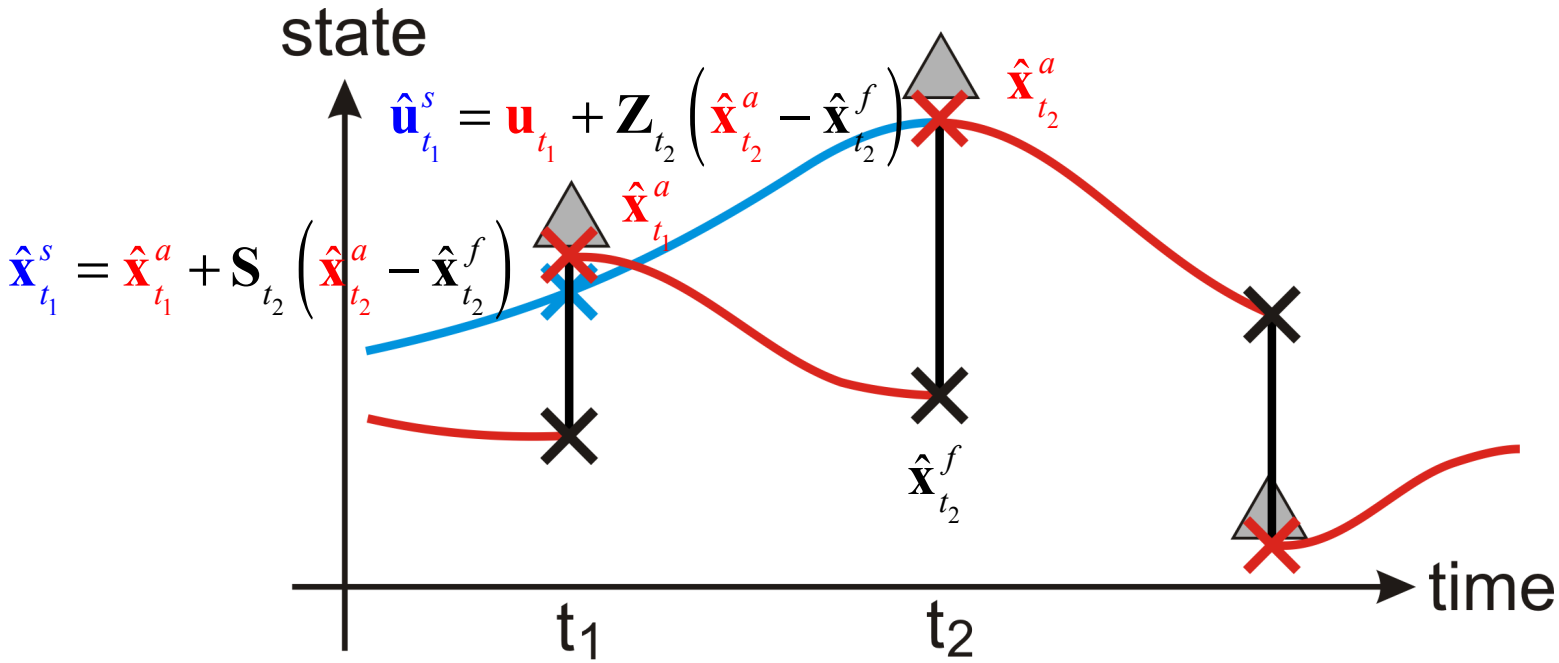
forecast vs analysis

data assimilation vs state estimation

reanalysis

Kalman smoother

Physical consistency (smoother #1)



Due to data corrections, the filtered solutions' temporal evolution cannot be physically explained, but smoothers invert these corrections and, therefore, their solutions' evolution are physically consistent.

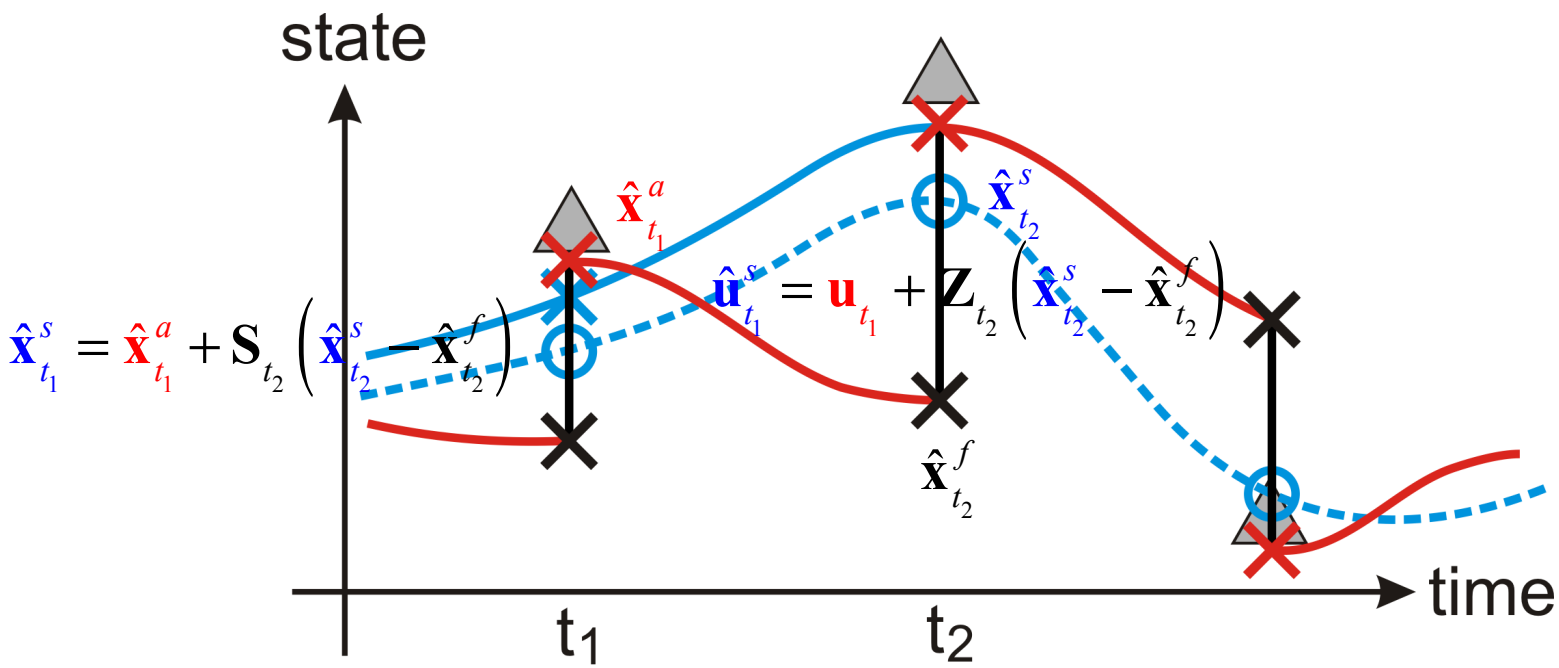
forecast vs analysis

data assimilation vs state estimation

reanalysis

Kalman smoother

Physical consistency (smoother #2)



Due to data corrections, the filtered solutions' temporal evolution cannot be physically explained, but smoothers invert these corrections and, therefore, their solutions' evolution are physically consistent.

forecast vs analysis

data assimilation vs state estimation

reanalysis

Kalman smoother

Solving an inverse problem (adjoint method)

data assimilation problem

$$\begin{pmatrix} \vdots \\ \mathbf{H}\mathbf{x}_t \\ \mathbf{H}\mathbf{x}_{t+1} \\ \vdots \\ \mathbf{x}_{t+1} - \mathbf{A}\mathbf{x}_t - \mathbf{G}\Delta\mathbf{u}_t \\ \mathbf{x}_{t+2} - \mathbf{A}\mathbf{x}_{t+1} - \mathbf{G}\Delta\mathbf{u}_{t+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \mathbf{y}_t \\ \mathbf{y}_{t+1} \\ \vdots \\ \mathbf{G}\mathbf{u}_t \\ \mathbf{G}\mathbf{u}_{t+1} \\ \vdots \end{pmatrix}$$

cost function

Lagrange multiplier

$$J = \sum_t \left(\mathbf{y}_t - \mathbf{H}_t\mathbf{x}_t\right)^T \mathbf{R}_t^{-1}\left(\mathbf{y}_t - \mathbf{H}_t\mathbf{x}_t\right) + \sum_t \Delta\mathbf{u}_t^T \mathbf{Q}_t^{-1}\Delta\mathbf{u}_t + \Delta\mathbf{x}_0^T \mathbf{P}_0^{-1}\Delta\mathbf{x}_0 + \sum_t \lambda_t\left(\mathbf{x}_{t+1} - \left(\mathbf{A}\mathbf{x}_t + \mathbf{G}\mathbf{u}_t + \mathbf{G}\Delta\mathbf{u}_t\right)\right)$$

Solve for a solution ($\Delta\mathbf{x}_0, \Delta\mathbf{u}_t$) that minimizes *J* based on its gradient using an optimization algorithm

$$\frac{1}{2}\frac{\partial J}{\partial \Delta\mathbf{x}_0} = \mathbf{P}_0^{-1}\Delta\mathbf{x}_0 + \mathbf{A}^T\lambda_0 = 0$$

$$\frac{1}{2}\frac{\partial J}{\partial \mathbf{x}_t} = -\mathbf{H}_t^T\mathbf{R}_t^{-1}\left(\mathbf{y}_t - \mathbf{H}_t\mathbf{x}_t\right) - \lambda_{t-1} + \mathbf{A}^T\lambda_t = 0$$

$$\frac{1}{2}\frac{\partial J}{\partial \Delta\mathbf{u}_t} = \mathbf{Q}_t^{-1}\Delta\mathbf{u}_t + \mathbf{G}^T\lambda_t = 0$$

$$\frac{1}{2}\frac{\partial J}{\partial \lambda_t} = \mathbf{x}_{t+1} - \left(\mathbf{A}\mathbf{x}_t + \mathbf{G}\mathbf{u}_t + \mathbf{G}\Delta\mathbf{u}_t\right) = 0$$

Wunsch (2006), *p217*.

notable points of data assimilation（2）

- Methods to solve inverse and data assimilation problems are almost all based on least squares,
- The filter/smoother combination solves the entire inverse problem by separately inverting observations and model incrementally in time to obtain a minimum error variance solution,
- The adjoint method utilizes the property of the model adjoint to solve the entire inverse problem for the minimum residual variance solution,
- The adjoint solution to the entire inverse problem is physically consistent as is the case for the smoother.

Outline

1) Data Assimilation and Inverse Problem

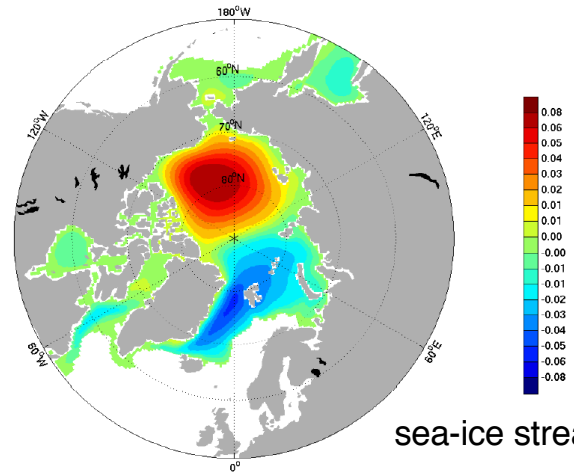
1) ECCO Ocean Analysis

2) Applications of Adjoint Models

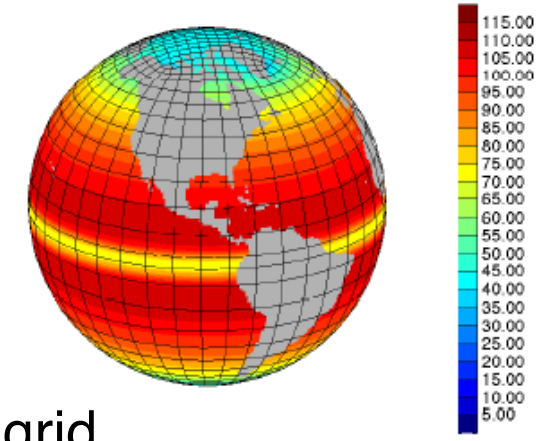
ECCO Version 4

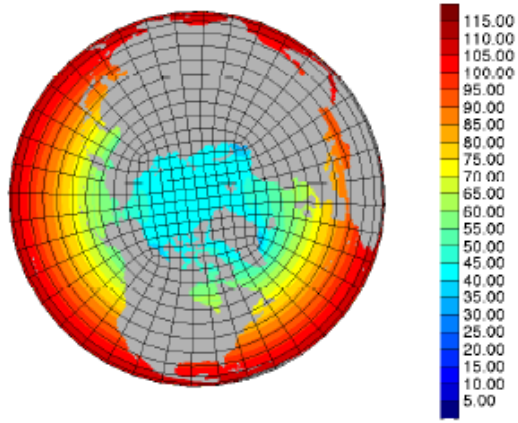
"Estimating the Circulation and Climate of the Ocean" (ECCO)
Latest ocean analysis

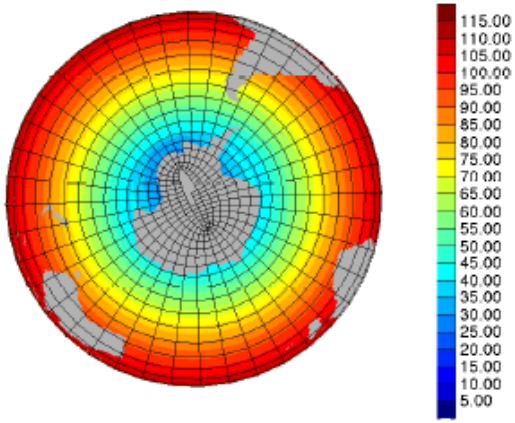
Characteristics	ECCO Version 4
grid system	latitude-longitude-cap
resolution	0.3˚~1˚, 50 levels
sea-ice model	prognostic
external forcing	bulk parameterization, geothermal forcing



sea-ice stream line







V4 grid

Forget, *et al.* (2015), *Geosci. Model Dev.*, *8*(10), 3071-3104, doi:10.5194/gmd-8-3071-2015.

ECCO Version 4 details

- physically consistent
- state estimation with adjoint method
- comprehensive ocean data usage
- analysis covering 1992 to present
- controls include initial condition (uvTSη), external forcing/atmospheric state, and mixing parameters
- accounting variables' temporal correlation
- accounting inhomogeneous grid
- near real-time extension with Kalman filter and RTS smoother
- adaptation to OpenAD

data used in ECCO Version 4

variable	observations
sea level	TOPEX/Poseidon (1992-2005), Jason-1 (2002-2008), Jason-2 (2008-2015), Geosat-Follow-On (2001-2008), CryoSat-2 (2011-2015), ERS-1/2 (1992-2001), ENVISAT (2002-2012), SARAL/AltiKa (2013-2015)
climatology	World Ocean Atlas 2009
temperature	Argo floats (1997-2015), XBTs (1992-2015), CTDs (1992-2015), TAO moorings (1992-2015), Southern Elephant seals as Oceanographic Samplers (SEaOS; 2004-2010), Ice-Tethered Profilers (ITP, 2004-2011)
salinity	Argo floats (1997-2015), CTDs (1992-2015), SEaOS (2004-2010)
sea surface temperature	AVHRR (1992-2015), AMSR-E (2002-2010)
sea surface salinity	Aquarius (2011-2015)
sea-ice concentration	SSM/I (1992-2009), SSMIS (2003-2015)
ocean bottom pressure	GRACE (2002-2015)

ECCO Version 4： accounting correlation

For simplicity, correlation is often ignored among the errors that define the cost function

$$\begin{pmatrix} \vdots \\ \mathbf{H}\mathbf{x}_t \\ \mathbf{H}\mathbf{x}_{t+1} \\ \vdots \\ \mathbf{x}_{t+1} - \mathbf{A}\mathbf{x}_t - \mathbf{G}\Delta\mathbf{u}_t \\ \mathbf{x}_{t+2} - \mathbf{A}\mathbf{x}_{t+1} - \mathbf{G}\Delta\mathbf{u}_{t+1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \mathbf{y}_t \\ \mathbf{y}_{t+1} \\ \vdots \\ \mathbf{G}\mathbf{u}_t \\ \mathbf{G}\mathbf{u}_{t+1} \\ \vdots \end{pmatrix}$$

Cost function

$$J = \Delta\mathbf{x}_0^T \mathbf{P}_0^{-1} \Delta\mathbf{x}_0 + \sum_t \left(\mathbf{y}_t - \mathbf{H}_t\mathbf{x}_t\right)^T \mathbf{R}_t^{-1} \left(\mathbf{y}_t - \mathbf{H}_t\mathbf{x}_t\right) + \sum_t \Delta\mathbf{u}_t^T \mathbf{Q}_t^{-1} \Delta\mathbf{u}_t + \sum_t \lambda_t \left(\mathbf{x}_{t+1} - \left(\mathbf{A}\mathbf{x}_t + \mathbf{G}\mathbf{u}_t + \mathbf{G}\Delta\mathbf{u}_t\right)\right)$$

$$\approx \sum_i \Delta x_0^{\ 2} / \sigma_x^{\ 2} + \sum_{t,i} \left(y_t - Hx_t\right)^2 / \sigma_y^{\ 2} + \sum_{t,i} u_t^{\ 2} / \sigma_u^{\ 2} + \sum_t \lambda_t \left(\mathbf{x}_{t+1} - \left(\mathbf{A}\mathbf{x}_t + \mathbf{G}\mathbf{u}_t + \mathbf{G}\Delta\mathbf{u}_t\right)\right)$$

ECCO Version 4： accounting correlation

Cost is biased when correlation is ignored

$$\mathbf{c} = \begin{pmatrix} 1.1 \\ 0.9 \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix}$$

$$\mathbf{c}^T \mathbf{Q}^{-1} \mathbf{c} = 1.25$$ as opposed to $$\sum c_i^{\,2} / Q_{ii} = 2.0$$

In comparison, if the problem is re-written to one with minimal correlation

$$\mathbf{c}' \equiv \begin{pmatrix} c_1 - \overline{c} \\ \overline{c} \end{pmatrix}, \quad \mathbf{Q}' \approx \begin{pmatrix} 0.2/4 & 0 \\ 0 & 3.8/4 \end{pmatrix}$$

$$\sum c_i'^{\,2} / Q'_{ii} = 1.25$$

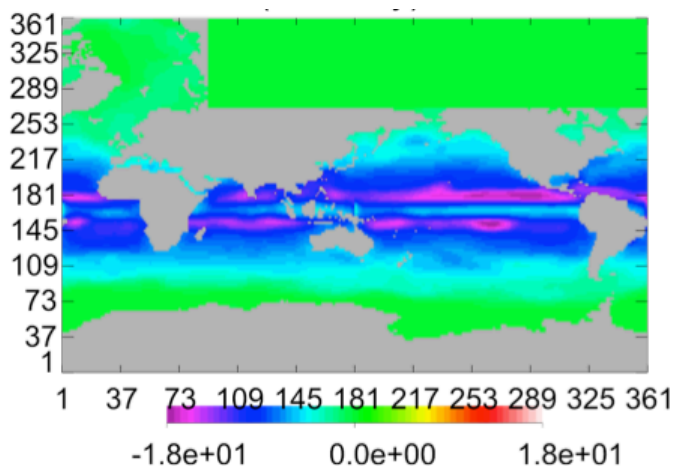
Biased evaluations when grid inhomogeneities are ignored

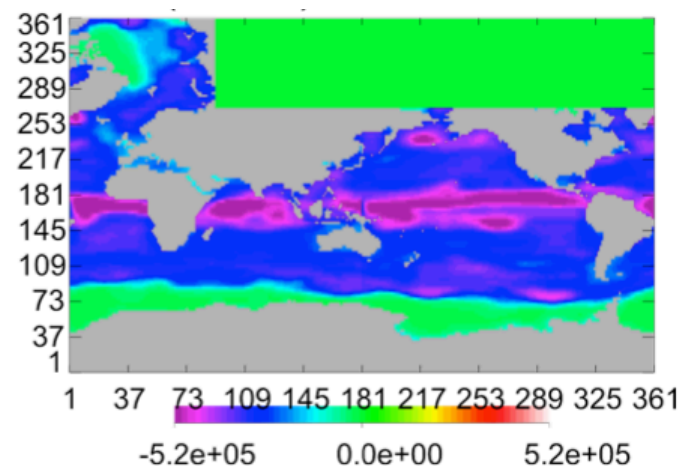
$$c = \frac{a+b}{2}$$

$$\delta c^2 = \frac{\delta a^2 + \delta b^2}{4} = \frac{\delta a^2}{2}$$

Sensitivity of global mean sea level (J) to rain (r)

$\partial J / \partial r$

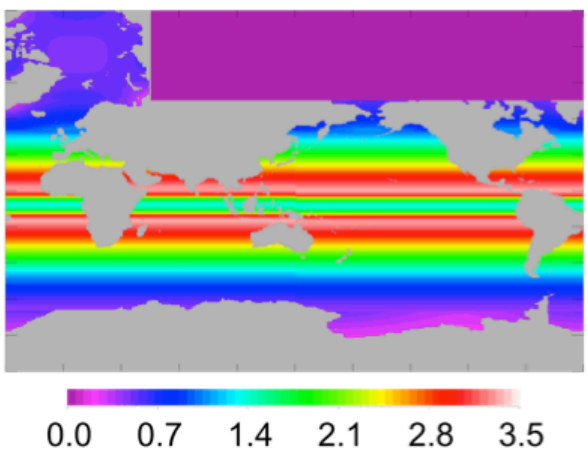




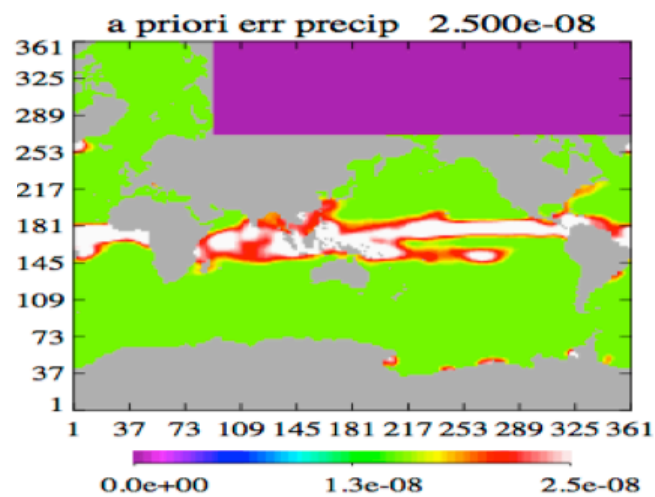
$$\frac{\partial J / \partial r}{\sqrt{\gamma}}$$

grid area

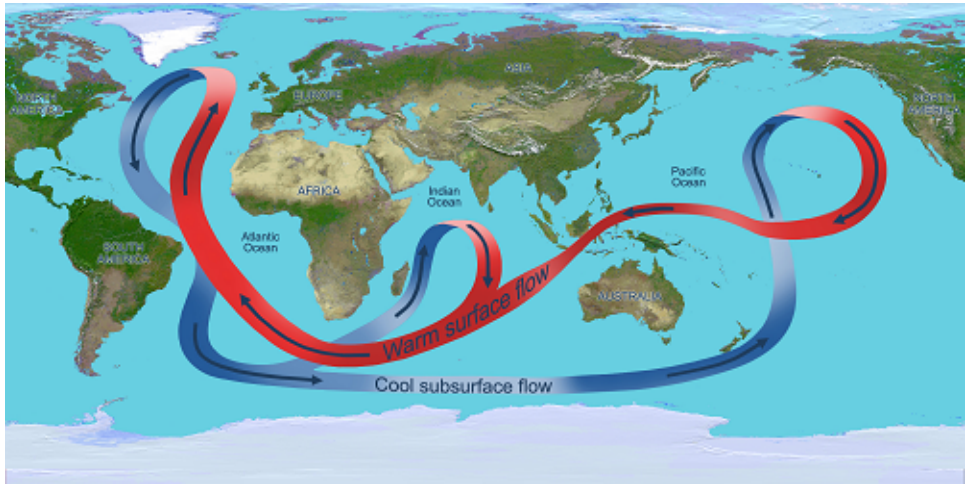
γ

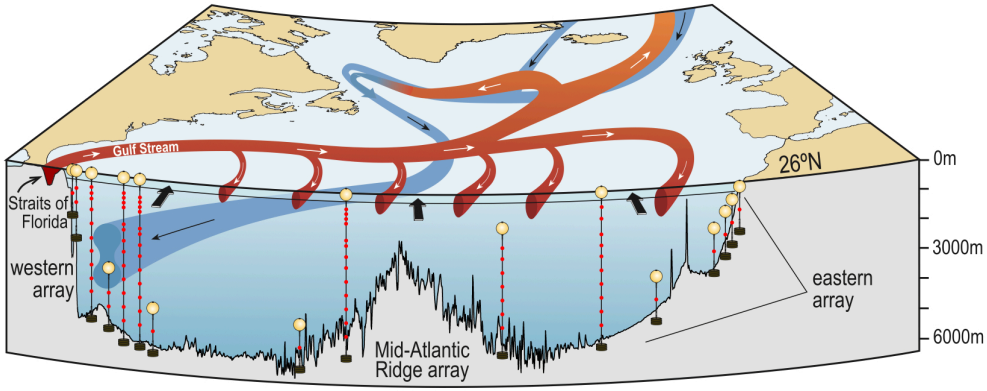


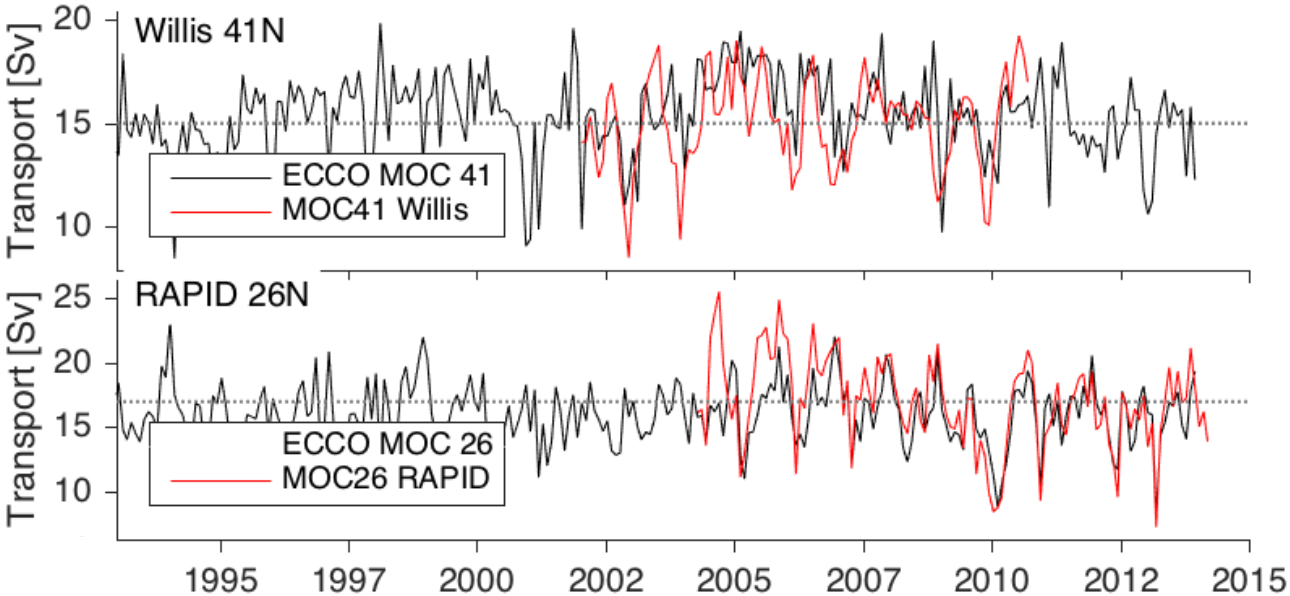
rain error



ECCO Version 4: North Atlantic Meridional Ocean Circulation

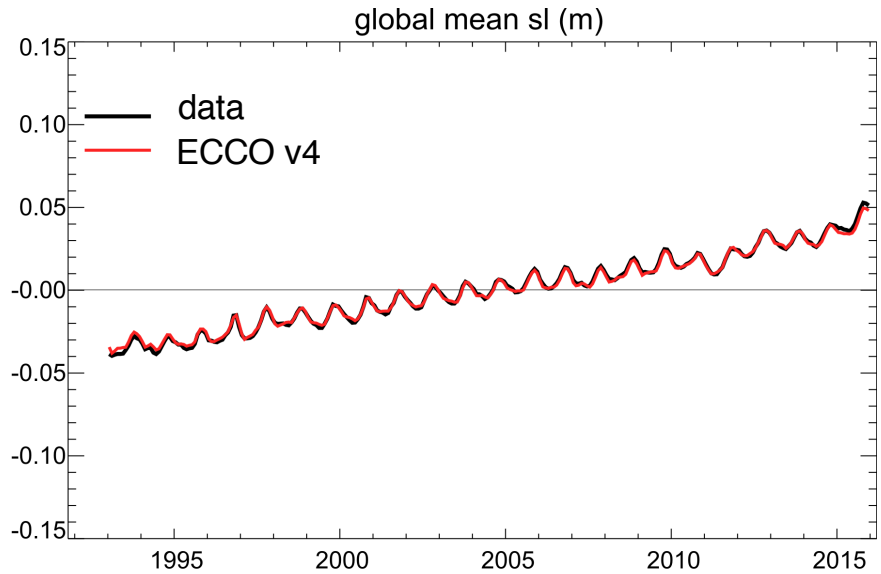




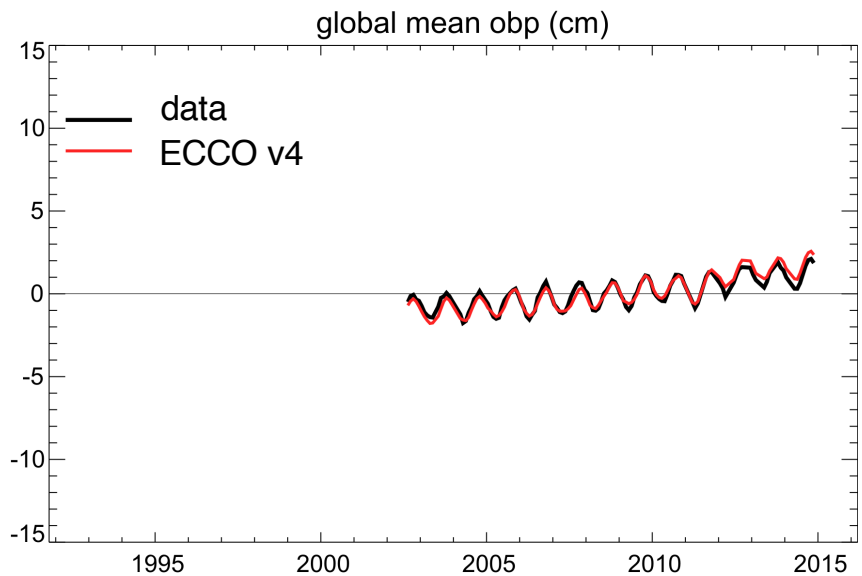


ECCO Version 4： sea level rise

mean sea level



mean mass

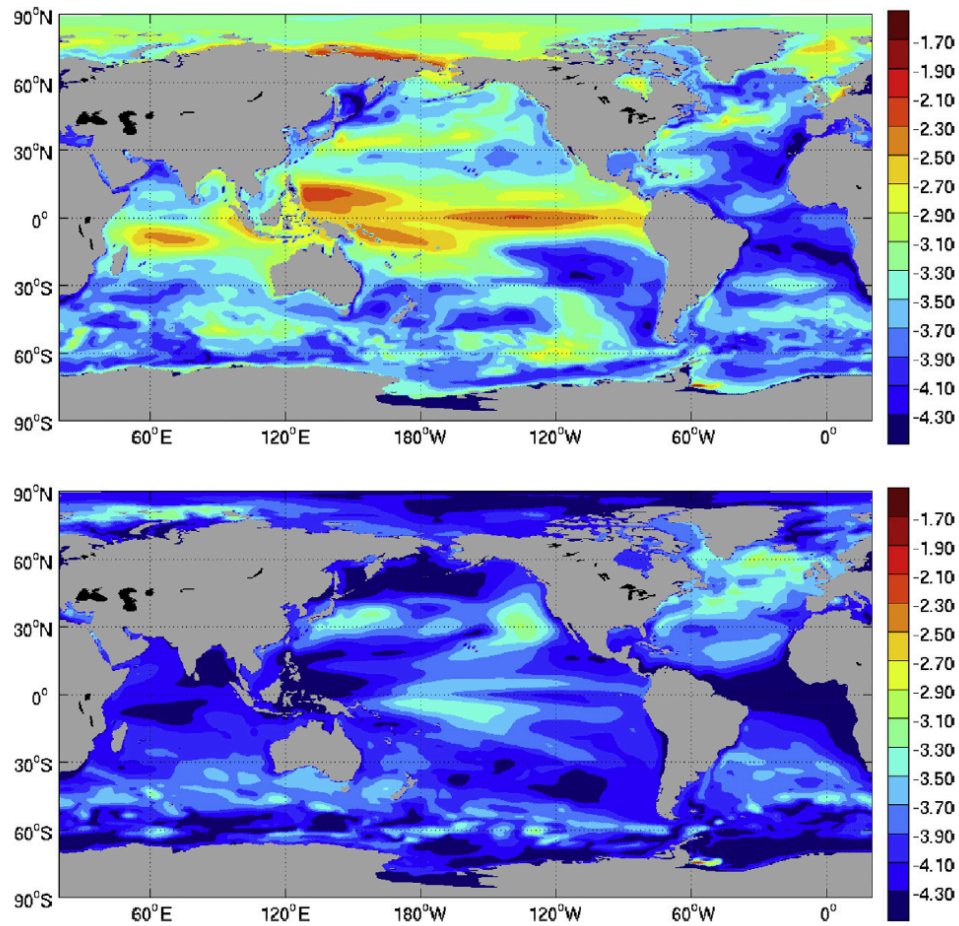


ECCO Version 4： sea level budget

sea level variance （logarithmic value）

wind contribution

thermohaline forcing

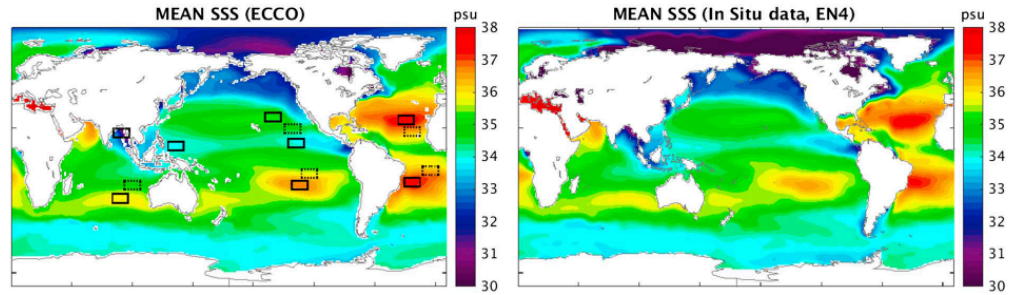


Forget, G., and R. M. Ponte (2015), *Prog Oceanogr*, *137, Part A*, 173-195, doi:10.1016/j.pocean.2015.06.002.

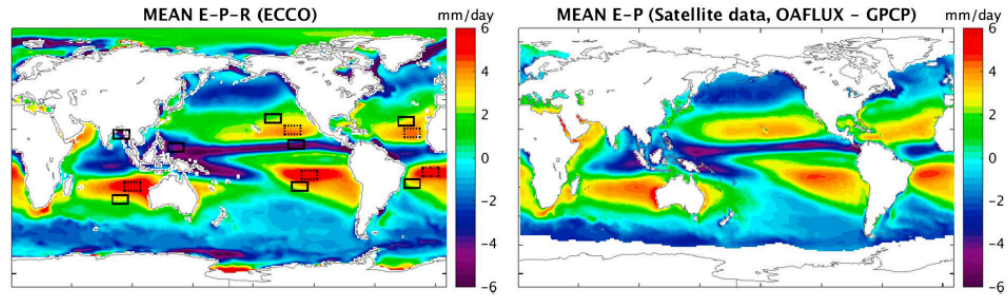
ECCO Version 4 : sea surface salinity budget

ECCO v4 observation

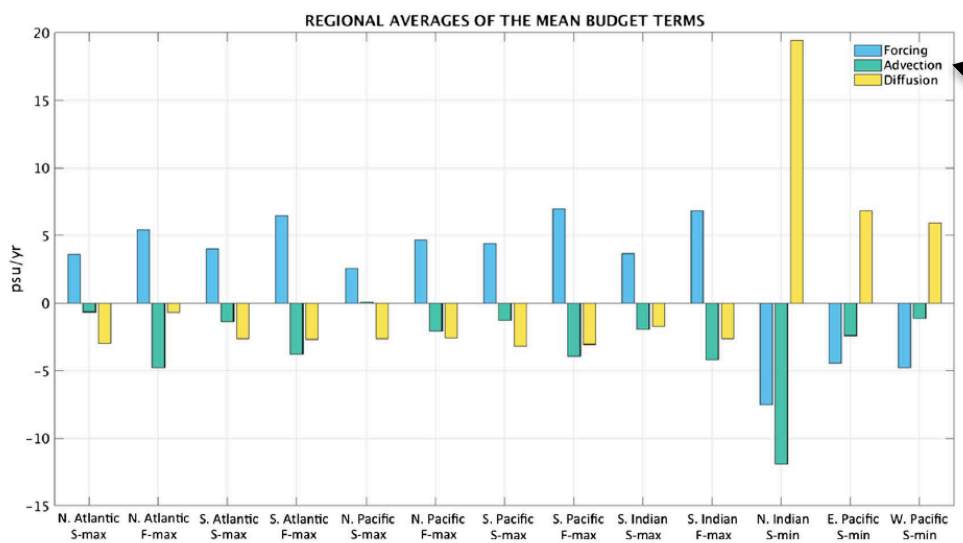
sea surface salinity



evaporation-precipitation-runoff



salinity budget



external forcing

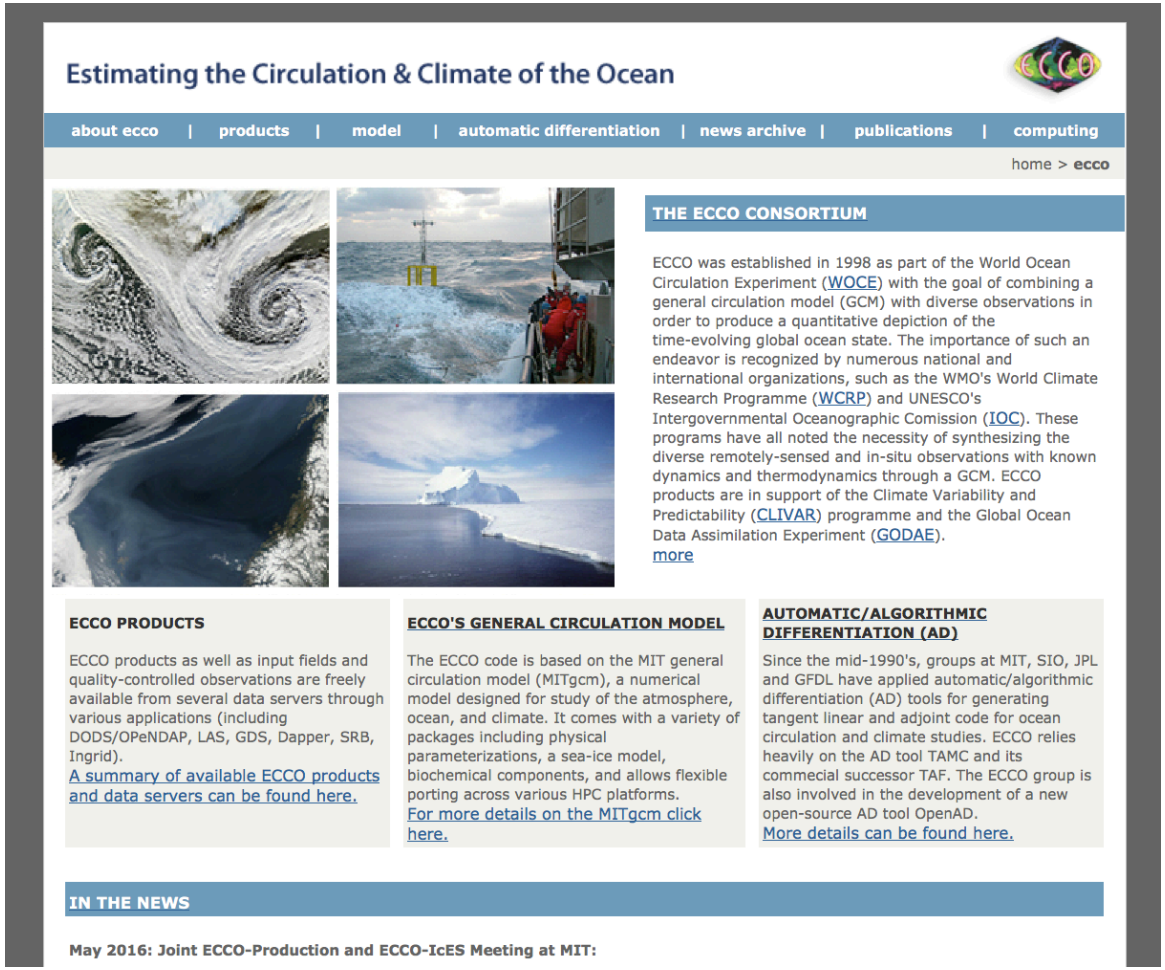
advection

mixing

Ponte, R. M., and N. T. Vinogradova (2016), *Geophys Res Lett*, doi:10.1002/2016GL069857.

ECCO Version 4

http://www.ecco-group.org/



Estimating the Circulation & Climate of the Ocean

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THE ECCO CONSORTIUM

ECCO was established in 1998 as part of the World Ocean Circulation Experiment (WOCE) with the goal of combining a general circulation model (GCM) with diverse observations in order to produce a quantitative depiction of the time-evolving global ocean state. The importance of such an endeavor is recognized by numerous national and international organizations, such as the WMO's World Climate Research Programme (WCRP) and UNESCO's Intergovernmental Oceanographic Comission (IOC). These programs have all noted the necessity of synthesizing the diverse remotely-sensed and in-situ observations with known dynamics and thermodynamics through a GCM. ECCO products are in support of the Climate Variability and Predictability (CLIVAR) programme and the Global Ocean Data Assimilation Experiment (GODAE).
more

ECCO PRODUCTS

ECCO products as well as input fields and quality-controlled observations are freely available from several data servers through various applications (including DODS/OPeNDAP, LAS, GDS, Dapper, SRB, Ingrid).
A summary of available ECCO products and data servers can be found here.

ECCO'S GENERAL CIRCULATION MODEL

The ECCO code is based on the MIT general circulation model (MITgcm), a numerical model designed for study of the atmosphere, ocean, and climate. It comes with a variety of packages including physical parameterizations, a sea-ice model, biochemical components, and allows flexible porting across various HPC platforms.
For more details on the MITgcm click here.

AUTOMATIC/ALGORITHMIC DIFFERENTIATION (AD)

Since the mid-1990's, groups at MIT, SIO, JPL and GFDL have applied automatic/algorithmic differentiation (AD) tools for generating tangent linear and adjoint code for ocean circulation and climate studies. ECCO relies heavily on the AD tool TAMC and its commecial successor TAF. The ECCO group is also involved in the development of a new open-source AD tool OpenAD.
More details can be found here.

IN THE NEWS

May 2016: Joint ECCO-Production and ECCO-IcES Meeting at MIT:

Outline

1) Data Assimilation and Inverse Problem

1) ECCO Ocean Analysis

2) Applications of Adjoint Models

“Correlation does not imply causation”

but adjoints do.

Utility of Adjoints

Adjoint models can quantify sensitivity of arbitrary model quantities and are effective in analyzing various processes

model $\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{G}\mathbf{u}_t$

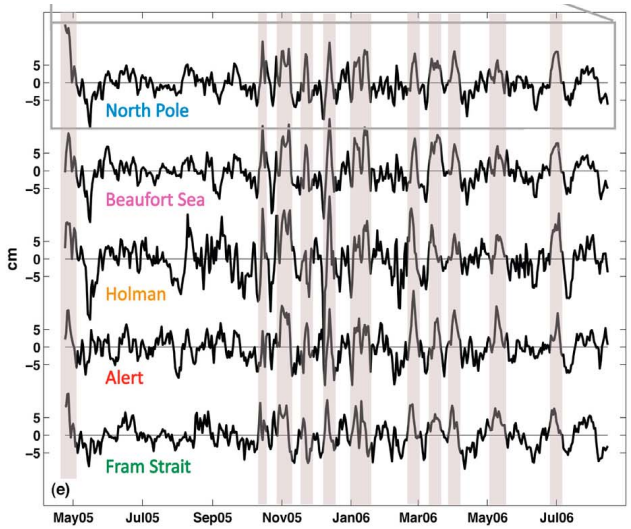
$$\frac{\partial J(t)}{\partial \mathbf{u}(t-N)} = \frac{\partial \mathbf{x}^T(t-N+1)}{\partial \mathbf{u}(t-N)} \frac{\partial \mathbf{x}^T(t-N+2)}{\partial \mathbf{x}(t-N+1)} \cdots \frac{\partial \mathbf{x}^T(t)}{\partial \mathbf{x}(t-1)} \frac{\partial J(t)}{\partial \mathbf{x}(t)}$$

$$= \mathbf{G}^T(t-N)\mathbf{A}^T(t-N+1)\cdots\mathbf{A}^T(t-1)\frac{\partial J(t)}{\partial \mathbf{x}(t)}$$

Example: Near-uniform fluctuation of Arctic ocean bottom pressure

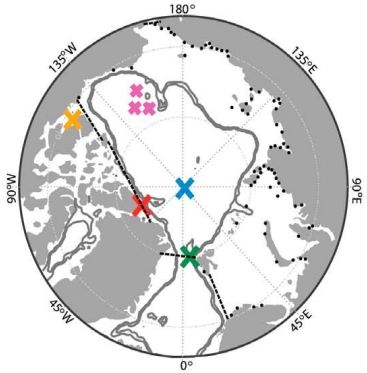
A spatially near-uniform fluctuation dominates ocean bottom pressure variations in the Arctic Ocean.

±5 cm

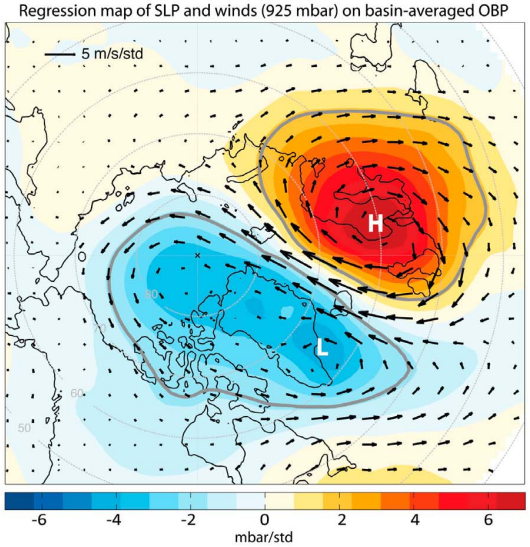


May 05 Jul 06

Time-series of Arctic OBP (ocean bottom pressure)



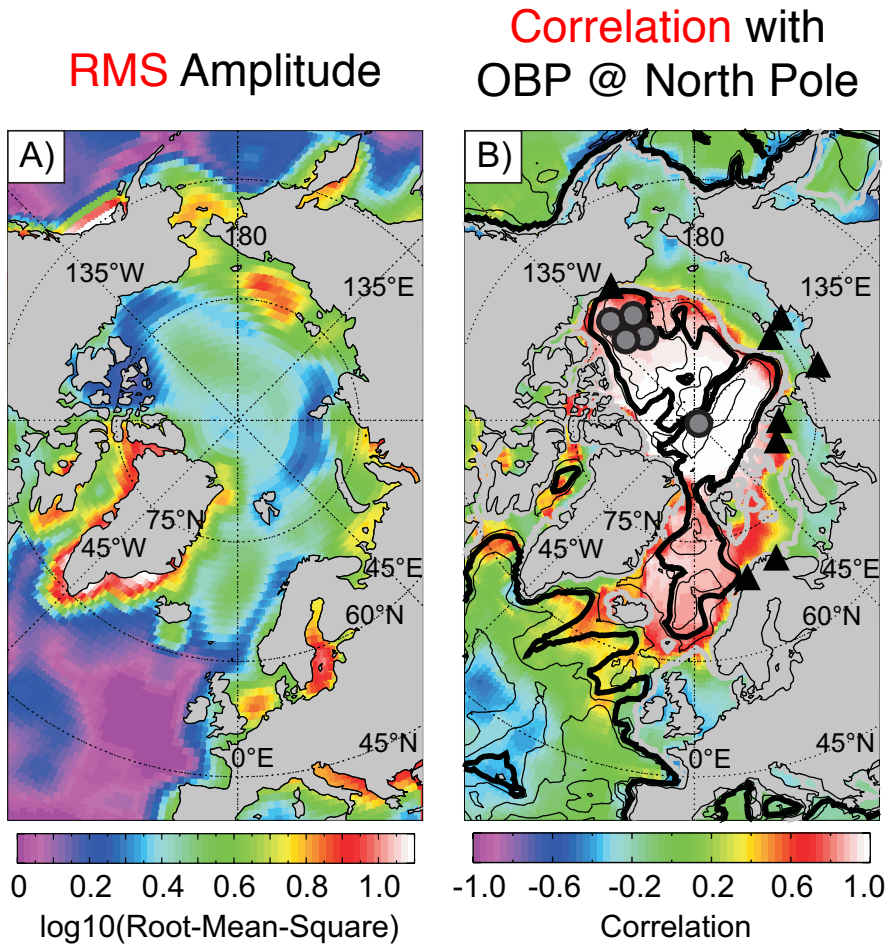
Regression of Arctic OBP with winds and atmospheric pressure



Peralta-Ferriz et al. (2011)

GRACE Observations

Ocean bottom pressure (OBP) variations are nearly uniform in amplitude & phase across the deep Arctic basins.

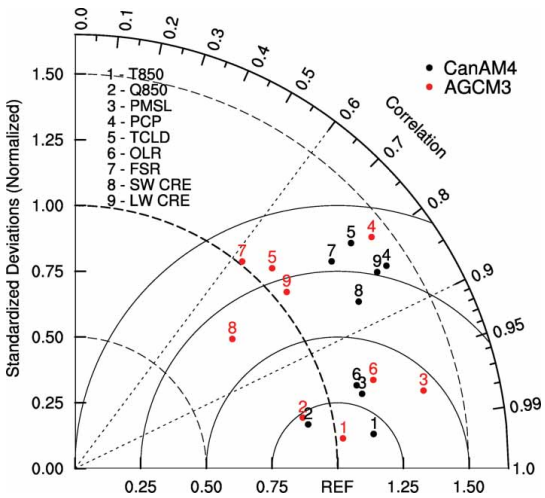


Explained Variance

Similarity of amplitude & phase can be simultaneously quantified in terms of "explained variance."

$$E = 1 - \frac{\mathrm{var}(P-Q)}{\mathrm{var}(P)}$$

$\mathrm{var}(P-Q)$ can be recognized as the amount of P's variance not explained by Q. Then, E describes the amount of P's variance that can be explained by Q.



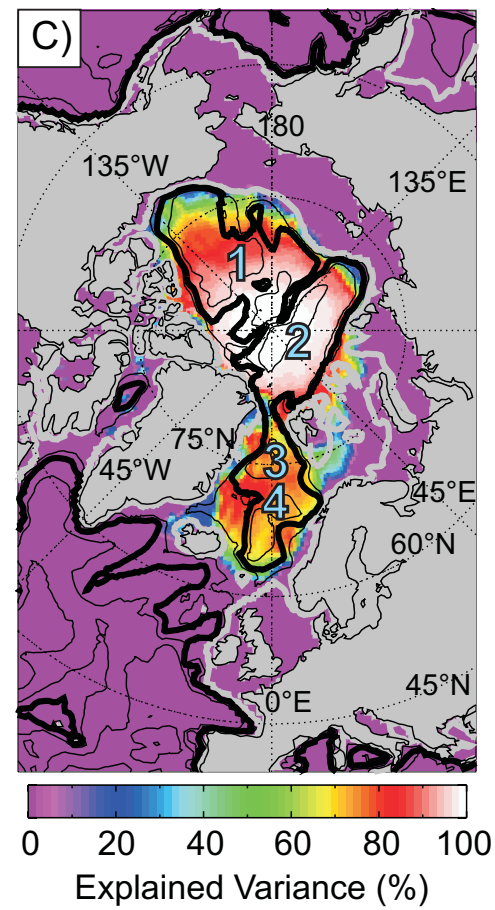
Taylor Diagram

(von Salzen et al, 2013)

$$E = -a^2 + 2a\,\mathrm{corr}(P,Q)$$

$$a = \sqrt{\mathrm{var}(Q)} / \sqrt{\mathrm{var}(P)}$$

OBP variance explained by OBP @ North Pole



Explained Variance

Explained variance is a useful quantity in comparing relative contribution of constituents.

$$P = \sum_i Q_i \qquad E_k = 1 - \frac{\mathrm{var}\left(P - Q_k\right)}{\mathrm{var}\left(P\right)}$$

For example, empirical orthogonal functions

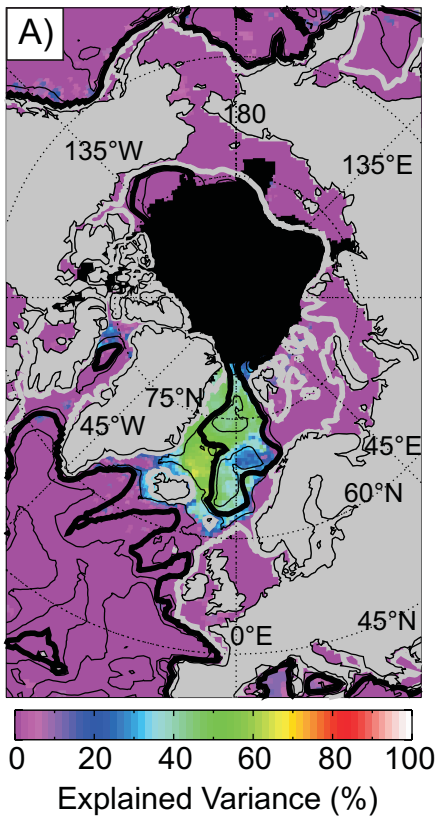
$$P = \sum_i \lambda_i \mathbf{e}_i \qquad \mathrm{var}\left(P\right) = \sum_i \lambda_i^{\,2} \qquad \mathrm{var}\left(P\right) - \mathrm{var}\left(P - Q_k\right) = \lambda_k^{\,2}$$

$$E_k = \frac{\lambda_k^{\,2}}{\sum_i \lambda_i^{\,2}}$$

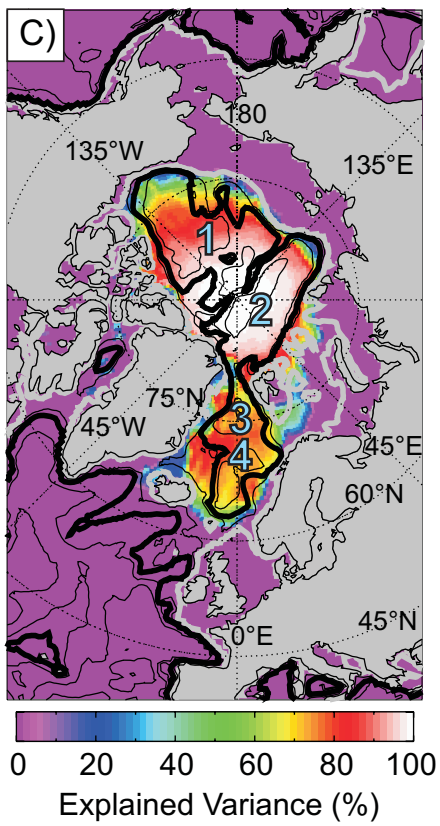
Altimeter vs GRACE Observations

Much of sea level variability is similarly near-uniform and can also be explained by bottom pressure variations.

SSH variance explained by OBP @ North Pole



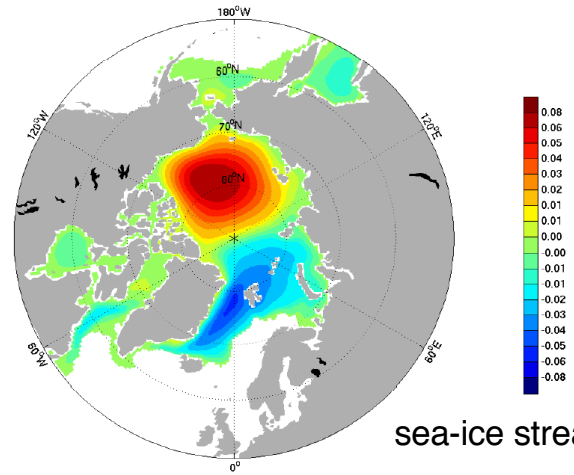
OBP variance explained by OBP @ North Pole



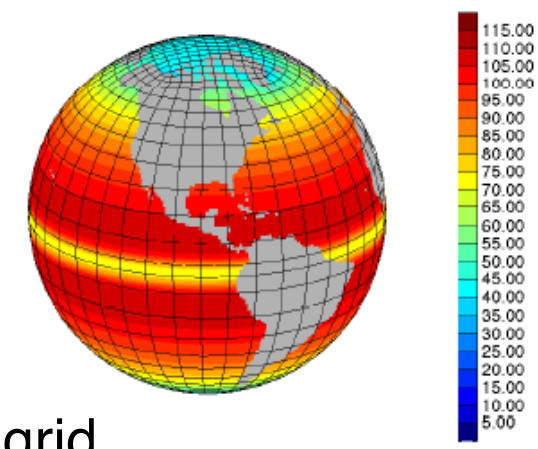
ECCO Version 4

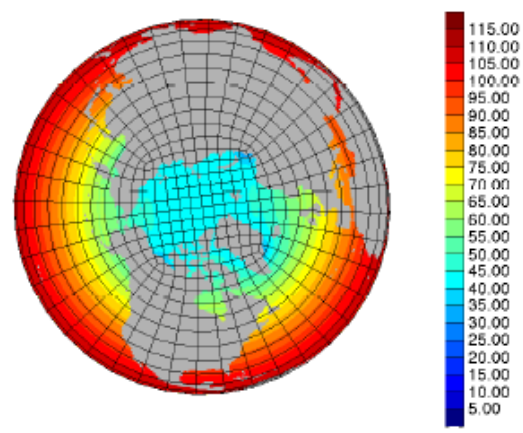
"Estimating the Circulation and Climate of the Ocean" (ECCO)
Latest ocean analysis

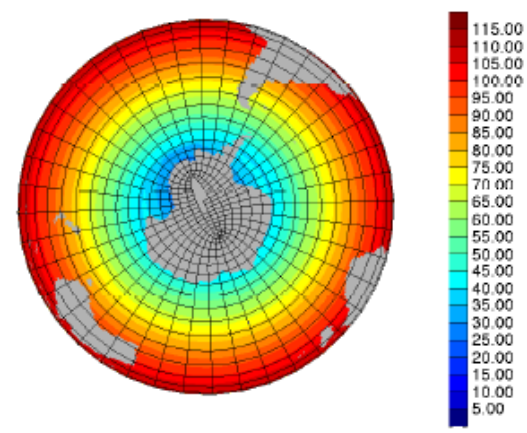
Characteristics	ECCO Version 4
grid system	latitude-longitude-cap
resolution	0.3˚~1˚, 50 levels
sea-ice model	prognostic
external forcing	bulk parameterization, geothermal forcing



sea-ice stream line







V4 grid

Forget, *et al.* (2015), *Geosci. Model Dev.*, *8*(10), 3071-3104, doi:10.5194/gmd-8-3071-2015.

Variation in ECCO v4（space）

Model simulated variations are spatially near-uniform across the Arctic basins similar to observations.

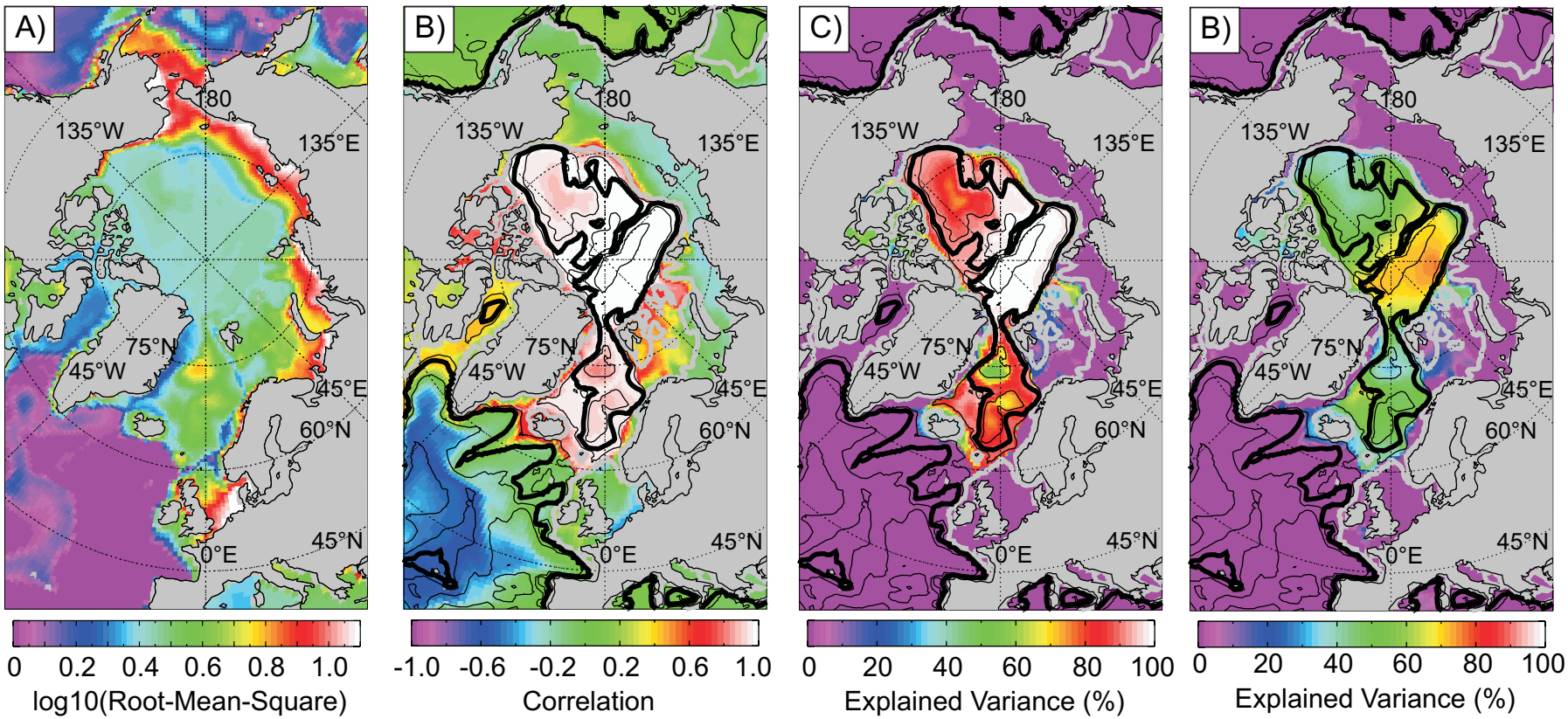
ocean bottom pressure

RMS Amplitude

Correlation with OBP @ North Pole

Variance explained by OBP @ North Pole

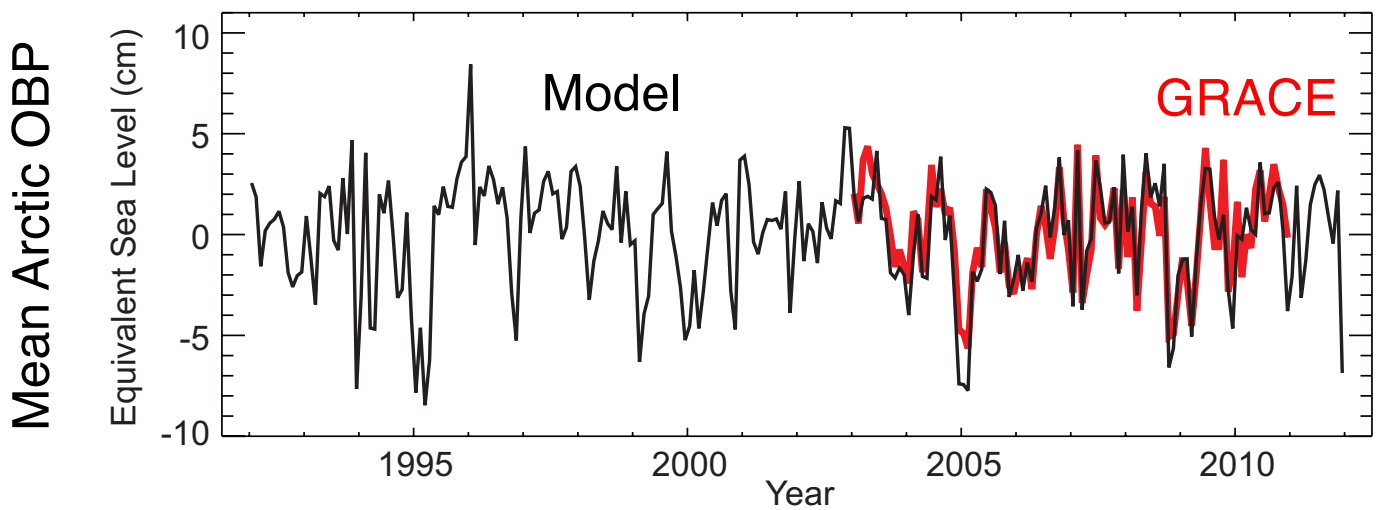
SSH variance explained by OBP @ North Pole



Variation in ECCO v4（time）

Model is also consistent with observations in time.

time-series of mean ocean bottom pressure across the Arctic Ocean



Adjoint Gradient Decomposition

Model adjoints provide an effective means to identify the forcings responsible for quantities of interest.

To first approximation, any quantity can be expanded in terms of its forcings as,

adjoint gradient

cause (forcing, initial condition, parameters etc)

$$J(t) \approx \sum_i \sum_{\mathbf{x}} \sum_s \frac{\partial J(t)}{\partial \phi_i(\mathbf{x},s)} \delta\phi_i(\mathbf{x},s)$$

time

time

space

Assuming stationarity $\dfrac{\partial J(t)}{\partial \phi_i(\mathbf{x},s)} \approx \dfrac{\partial J}{\partial \phi_i(\mathbf{x},\Delta t)}$ where $\Delta t = t - s$

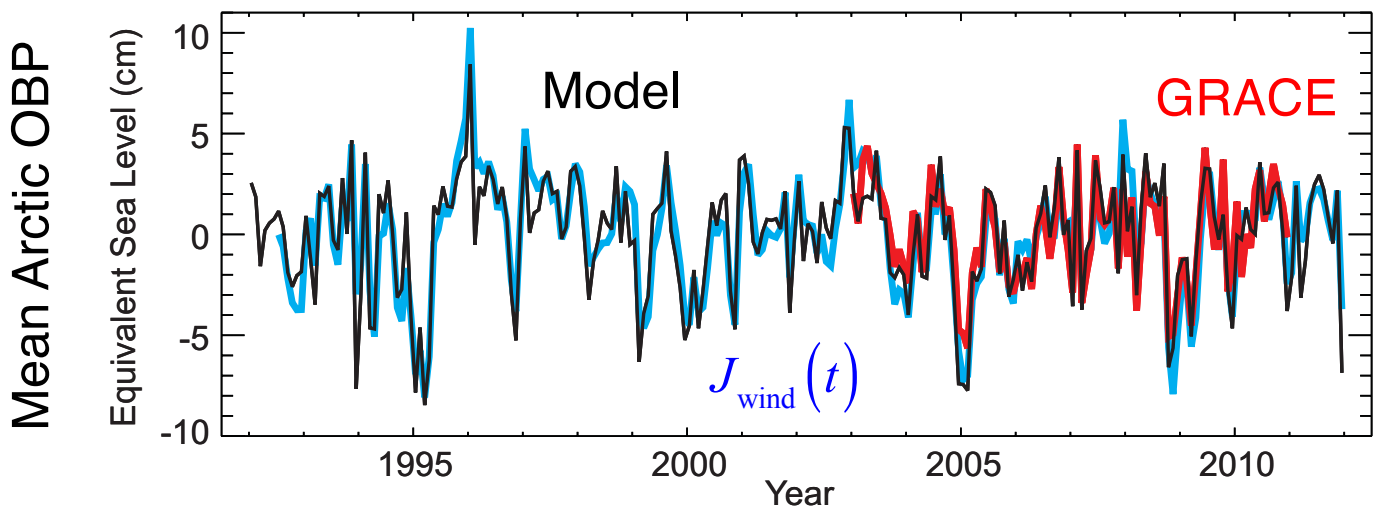
$$J(t) \approx \sum_i \sum_{\mathbf{x}} \sum_{\Delta t} \frac{\partial J}{\partial \phi_i(\mathbf{x},\Delta t)} \delta\phi_i(\mathbf{x},t-\Delta t)$$

- $\dfrac{\partial J}{\partial \phi_i(\mathbf{x},\Delta t)}$ can be computed by the model's adjoint.
- The largest term on the RHS is the dominant forcing of J.

Causal Mechanism (forcing)

Wind-driven contribution estimated by the adjoint gradient reconstruction;

$$J_{\text{wind}}(t) \equiv \sum_{i=\text{wind}} \sum_{\mathbf{x}} \sum_{\Delta t} \frac{\partial J}{\partial \phi_i(\mathbf{x}, \Delta t)} \delta\phi_i(\mathbf{x}, t - \Delta t)$$

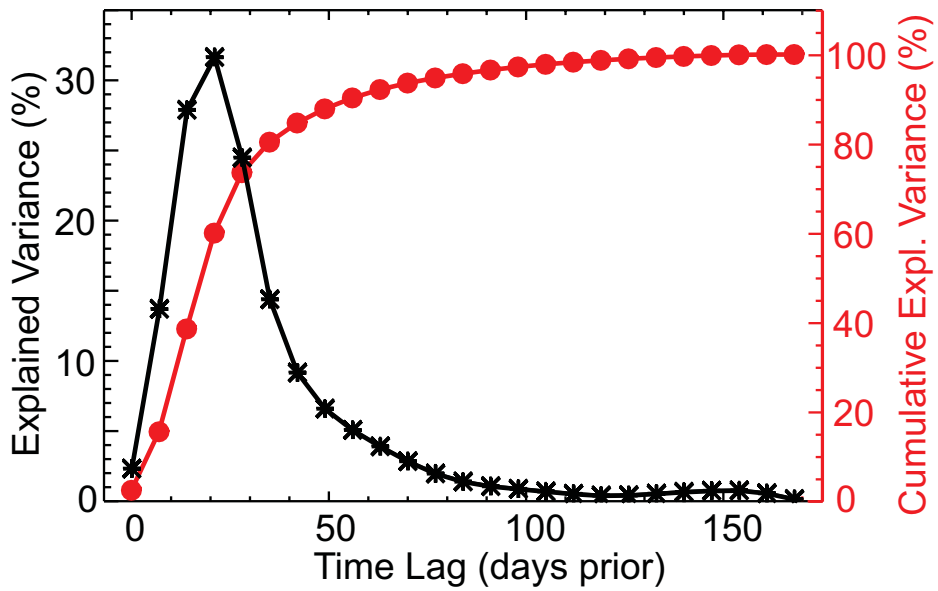


The fluctuation is wind-driven.

Causal Mechanism (timing)

Explained variance as a function of the wind's temporal lag:

$$E_{wind}(\Delta t) \equiv 1 - \frac{\mathrm{var}\left\{ J_{wind}(t) - \sum_{i=wind} \sum_{\mathbf{x}} \frac{\partial J}{\partial \phi_i(\mathbf{x}, \Delta t)} \delta\phi_i(\mathbf{x}, t - \Delta t) \right\}}{\mathrm{var}\left\{ J_{wind}(t) \right\}}$$



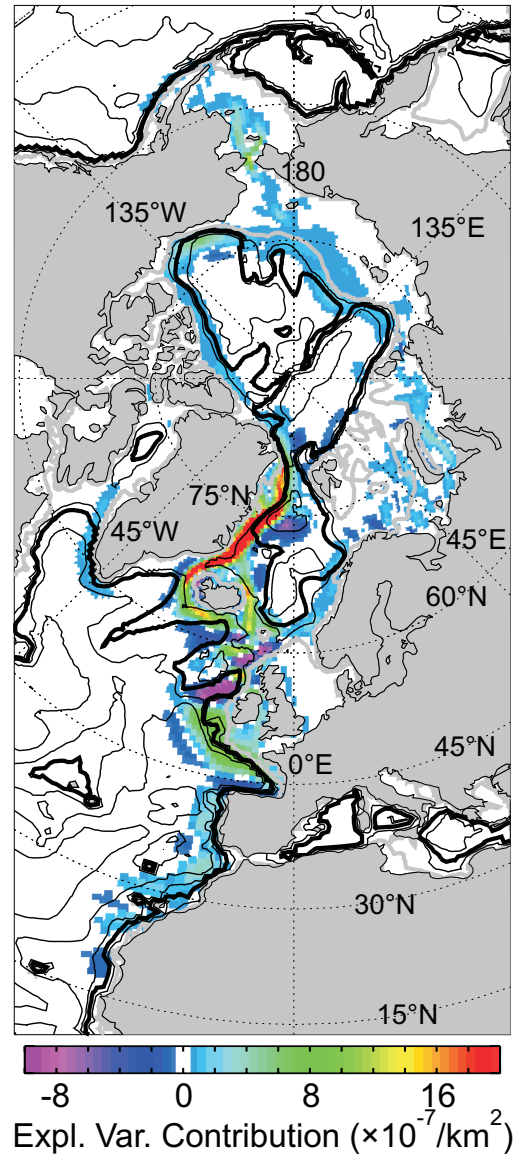
Coincident winds drive the fluctuation.

Causal Mechanism (location)

Explained variance as a function of the wind's location:

$$E_{\text{wind}}(\mathbf{x}) \equiv \frac{1}{dS(\mathbf{x})} \times \left[1 - \frac{\operatorname{var}\left\{ J_{\text{wind}}(t) - \sum_{i=\text{wind}} \sum_{\Delta t} \frac{\partial J}{\partial \phi_i(\mathbf{x}, \Delta t)} \delta\phi_i(\mathbf{x}, t - \Delta t) \right\}}{\operatorname{var}\left\{ J_{\text{wind}}(t) \right\}} \right]$$

Winds along the continental slope are responsible for the fluctuation.

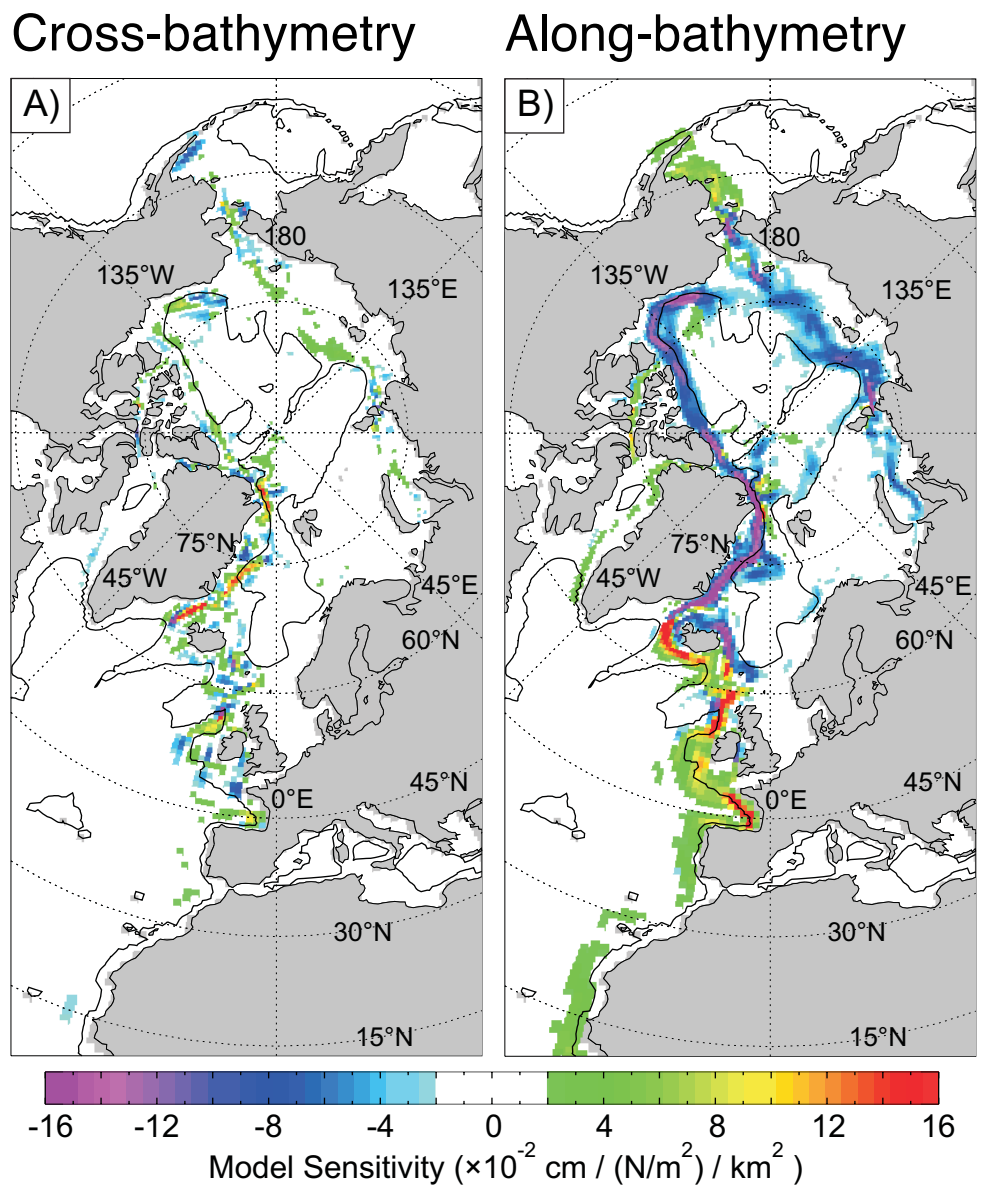


Causal Mechanism (which wind)

Sensitivity of mean Arctic Mediterranean OBP to winds:

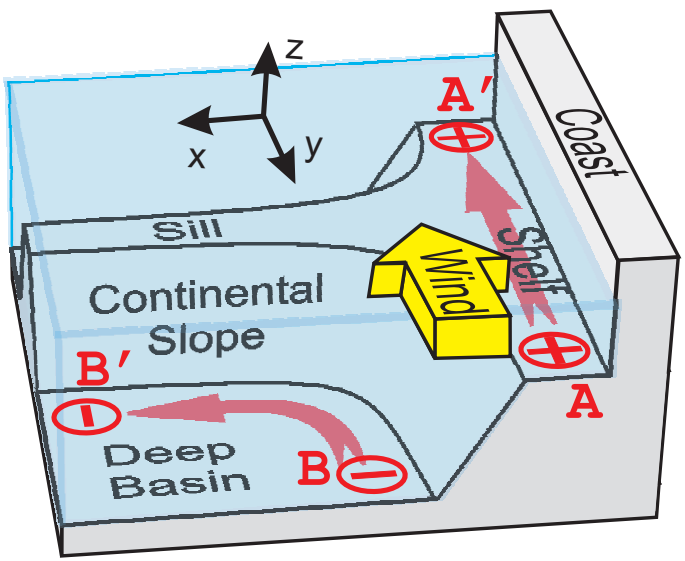
$$G_{\text{wind}}(\mathbf{x},\Delta t) \equiv \frac{1}{dS(\mathbf{x})} \frac{\partial J}{\partial \phi_{\text{wind}}(\mathbf{x},\Delta t)}$$

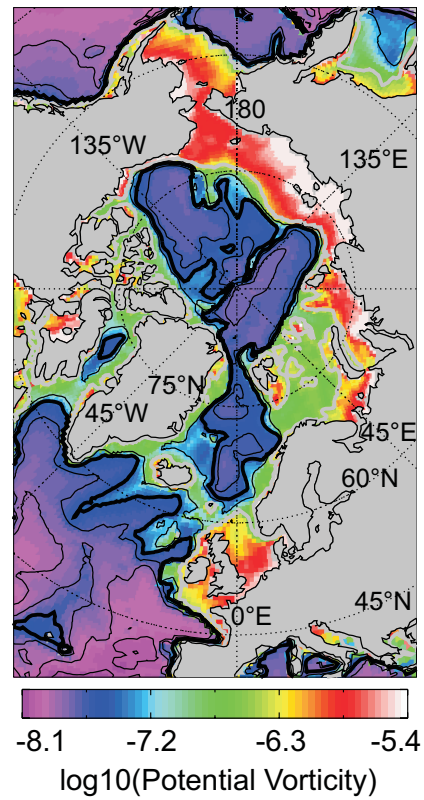
- Along-bathymetry component is dominant,
- Opposite signs inside & outside the Arctic,
- Eastern boundary dominates outside the Arctic.



Causal Mechanism (Ocean's Response)

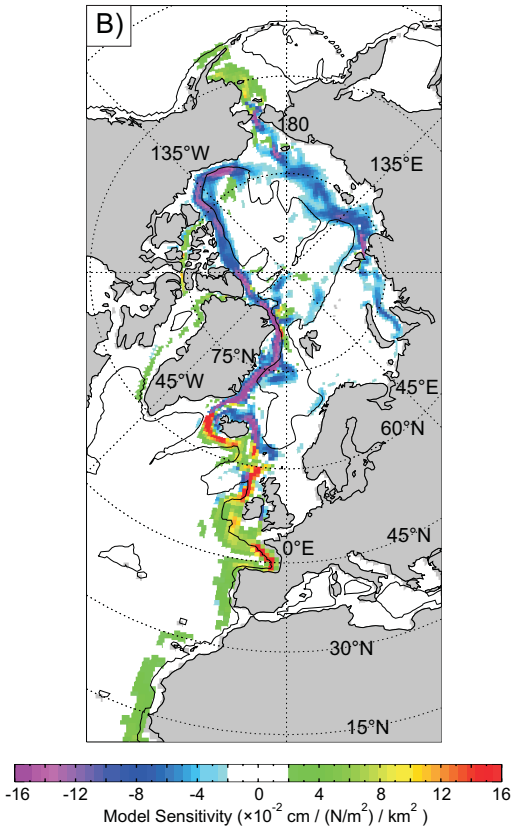
Wind-driven coastally trapped waves bifurcate at the shallow straits, allowing anomalies to persist in the deep Arctic basins shielded by steep gradients of planetary potential vorticity, f/H.



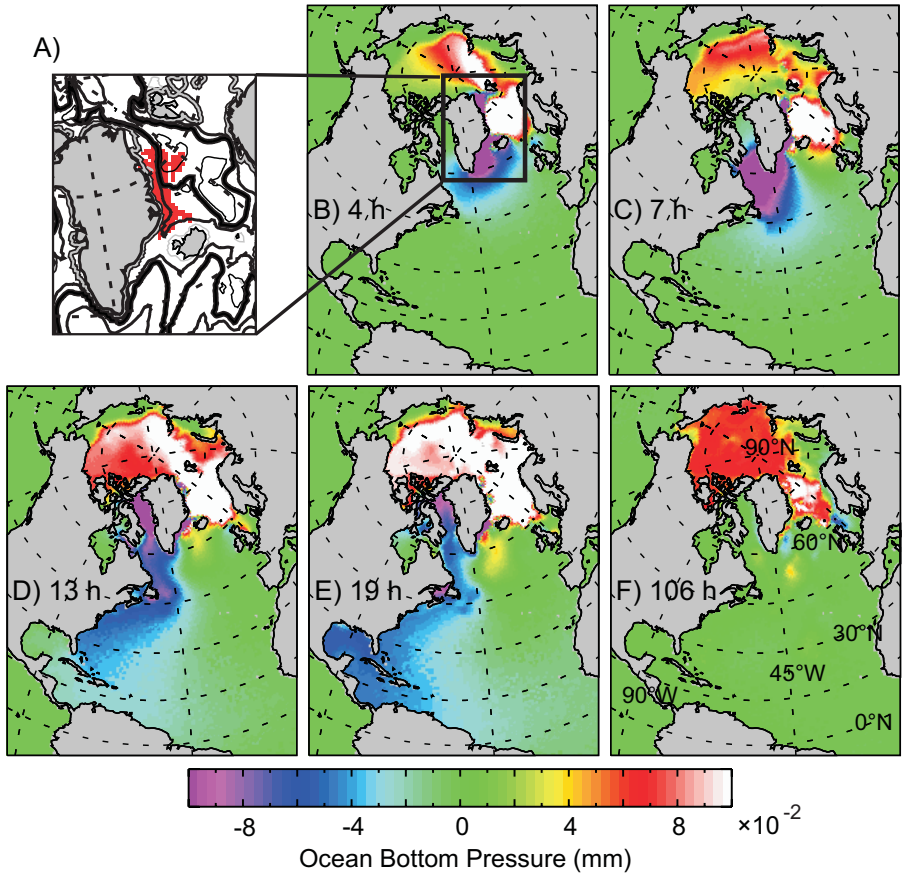


Perturbation Experiment

Along-bathymetry sensitivity



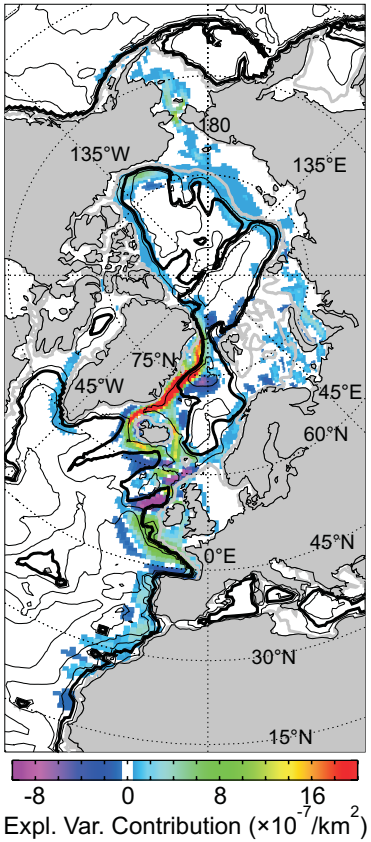
OBP response to meridional wind perturbation off Greenland



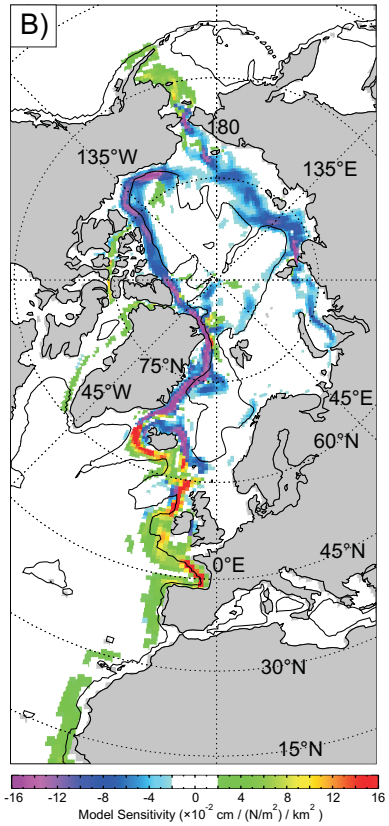
Causal Mechanism (which wind)

Differences in attribution can be related to correlation within atmospheric fields and the relative magnitude of different contributions.
"Correlation does not imply causation"

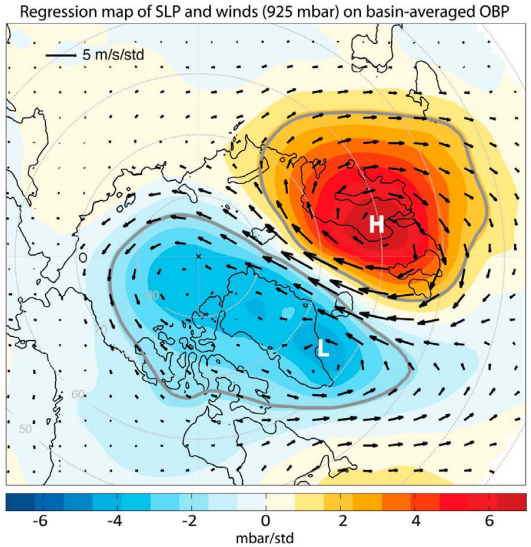
Wind Contribution



Along-Bath. Sensitivity



Regression
(Peralta-Ferriz et al., 2011)

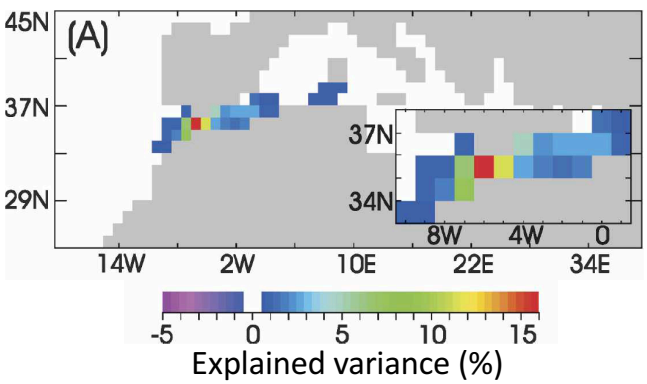


Examples of adjoint gradient decomposition

ocean bottom pressure and sea level fluctuations across the deep Arctic Ocean

Fukumori et al. (2015), *Prog Oceanogr*, ***134***, 152-172, doi:10.1016/j.pocean.2015.01.013.

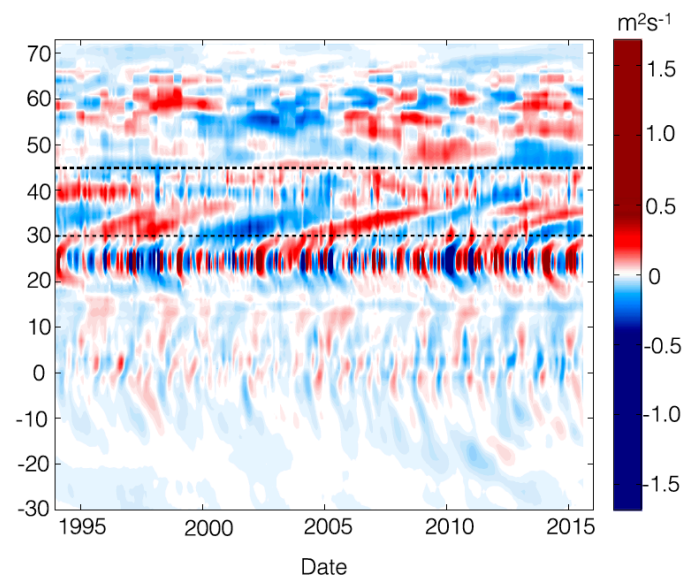
uniform sea level variations across the Mediterranean Sea



Fukumori, I., D. Menemenlis, and T. Lee (2007), *J Phys Oceanogr*, ***37***(2), 338-358, doi:10.1175/Jpo3016.1.

variations of Atlantic meridional ocean circulation across 25N

Pillar et al. (2016), *Journal of Climate*, ***29***(9), 3339-3352, doi:10.1175/JCLI-D-15-0727.1.



summary

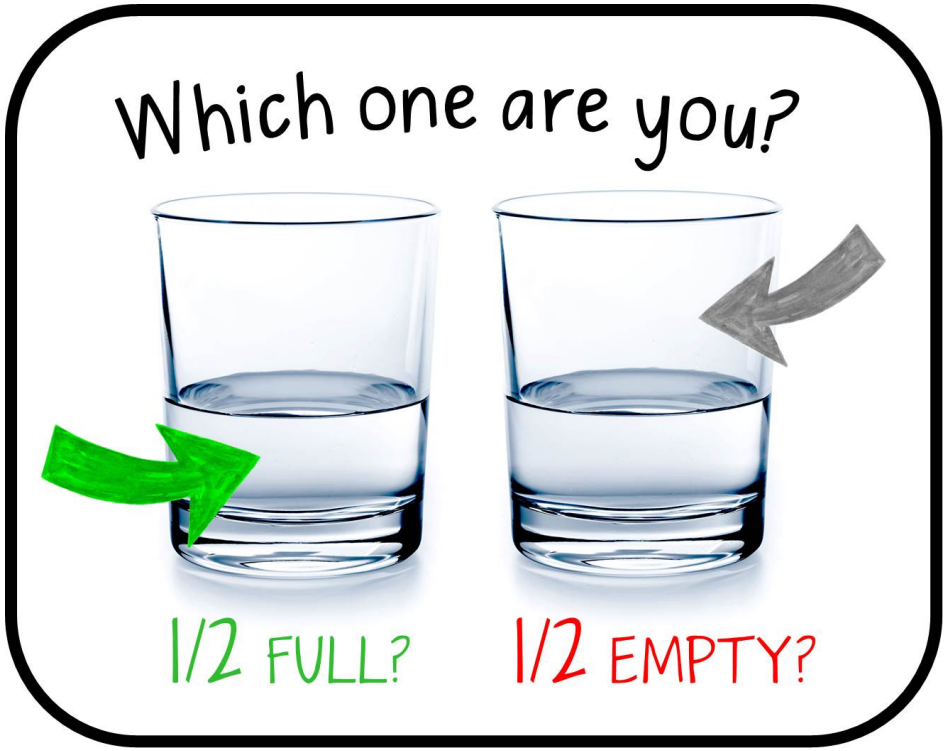
1) Mathematically, data assimilation is an inverse problem that theoretically belongs to estimation and control theory advanced in engineering,

2) Practically all methods of assimilation are least-squares,

3) In practice, assumptions are made (often implicitly) to make the problem tractable. Results are inaccurate if such assumptions are incorrect. In such situations, assumptions need to be revisited on the one hand, and one must recognize such assumptions in applying the results on the other,

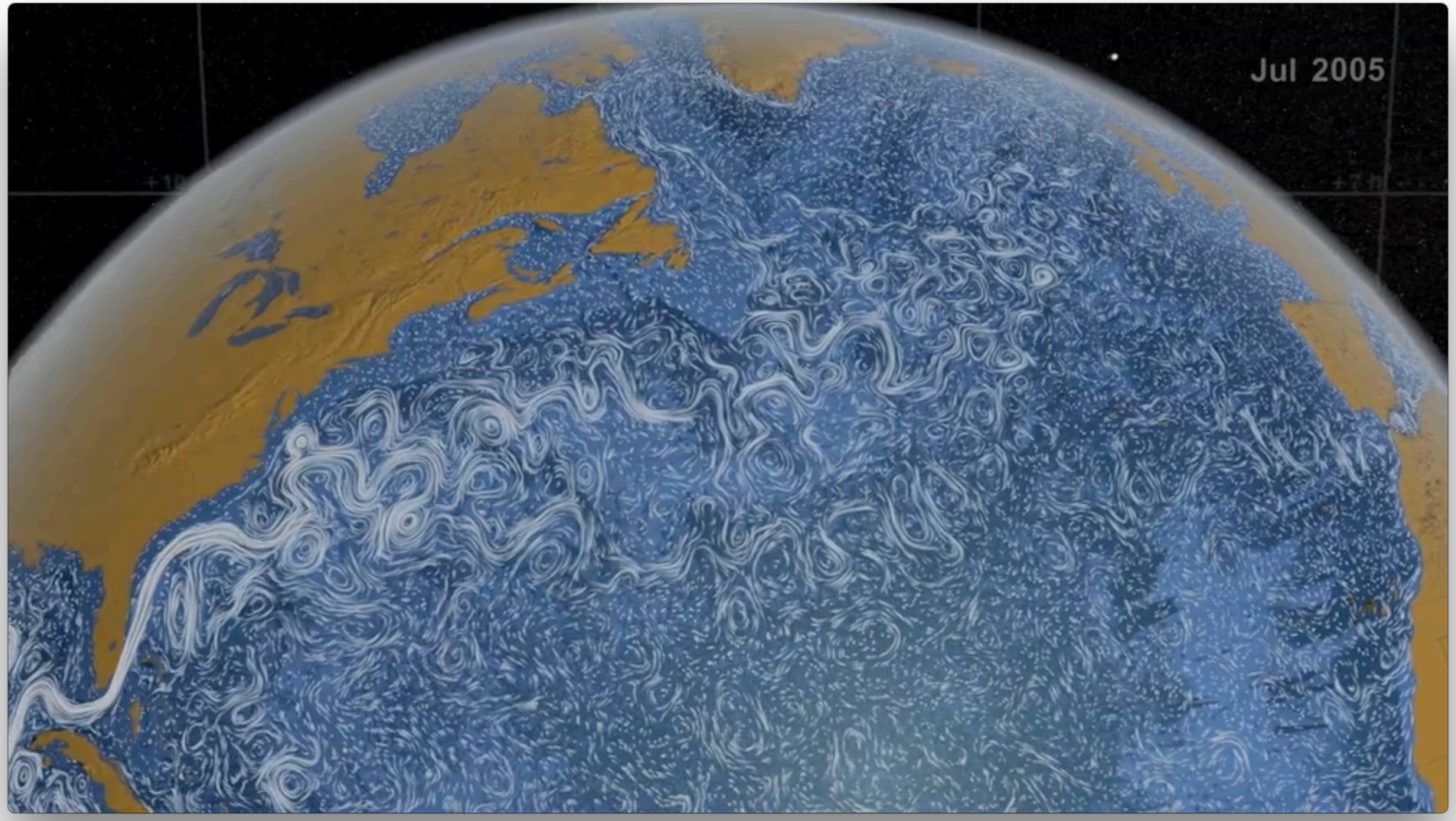
4) Adjoints are not only useful for data assimilation, but are effective in analyzing various causal mechanisms.

Advice



ECCO2

The highest resolution ocean state estimate to date.



"Perpetual Ocean"
https://www.youtube.com/watch?v=CCmTY0PKGDs

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