First-principles Modeling of the IAT-driven Anomalous Resistivity in Hollow Cathode Discharges I: Theory

Benjamin A. Jorns* Alejandro Lopez Ortega† and Ioannis Mikellides‡

Jet Propulsion Laboratory, California Institute of Technology, Pasadena, CA 91109

Self-consistent expressions are derived for the impact of ion acoustic turbulence (IAT) on the fluid properties of electrons and ions in the plume of a hollow cathode. This instability is believed to have a dominant role in the plasmadynamics in this region. The random phase approximation is applied to the perturbed Boltzmann equation to determine the effective drag and heating of the plasma species induced by the IAT. These driving terms are then incorporated into momentum and energy fluid equations for the ions and electrons. A fluid-like relation governing the evolution of the total IAT wave energy density is also presented. The physical significance of the IAT-based terms is discussed in the context of energy and momentum conservation, and a path forward is outlined for implementing the effects of IAT in the fluid-based model in Part II.

Nomenclature

$\mathcal{C}$	Collision operator
c_s	Ion sound speed
$\vec{E}$	Electric field
E_k	Energy density of the electrostatic field of the kth mode
$\vec{F}_{AN(s)}$	Rate change in momentum density for species (s) due to electrostatic waves
ϵ_0	Permittivity of free space
$\varepsilon(\omega, \vec{k})$	Dielectric tensor of electrostatic mode at frequency ω and wave vector $\vec{k}$
$f_{(s)}$	Distribution function for the species (s)
$\vec{k}$	Wave vector of electrostatic mode
κ_s	Thermal conductivity of species (s)
$\lambda_{d(s)}$	Debye length for species (s)
m_s	Mass of species (s)
n_s	Density of species (s)
N_k	Wave action of kth component of electrostatic mode
ν_s	Species collision frequency
ν_{ei}	Total electron-ion collision frequency
ν_{ie}	Total ion-electron collision frequency
ν_{ei}^c	Electron-ion Coulomb collision frequency
ν_{ie}^c	Ion-electron Coulomb collision frequency
ν_a	Anomalous collision frequency of electrons due to IAT
$\nu_{a(i)}$	Anomalous collision frequency of ions due to IAT
ν_{in}	Ion-neutral collision frequency
ν_{en}	Electron-neutral collision frequency
ϕ_k	Plasma potential amplitude of kth component of electrostatic mode
q_s	Fundamental charge of the (s) species
$Q_{AN(s)}$	Rate change in energy density for species (s) due to electrostatic waves

*Engineer, Electric Propulsion Group, AIAA Senior Member, benjamin.a.jorns@jpl.nasa.gov
†Engineer, Electric Propulsion Group, AIAA Member
‡Principal Engineer, Electric Propulsion Group, AIAA Associate Fellow

T_s	Temperature of (s) species in units of energy
$\vec{V}_s$	Average fluid velocity for species (s)
$\vec{V}_D$	Relative drift velocity between electrons and ions
W_k	Wave energy density of kth component of electrostatic mode
ω	Angular frequency of electrostatic mode
ω_0	Lower cut-off frequency for IAT
ω_{ps}	Plasma frequency for species (s)
Z_0	Plasma dispersion function

I. Introduction

Predictive modeling is an important tool for qualifying electric propulsion systems for the next generation of deep space missions. Validated models can help overcome concerns associated with qualification through ground testing such as the problem of facility effects and high costs incurred by long duration tests. The path to qualification through modeling most recently has been identified as a tool for assessing the ion propulsion system for the proposed Asteroid Robotic Redirect Mission (ARRM) with its high power level—12.5 kW per thruster—and multiyear design life—50,000 hours.[1] For these unprecedented powers and run times, models will be necessary to help evaluate the life of each element in the electric propulsion system—including the hollow cathode. Indeed, while hollow cathodes are essential components to nearly all of the forms of electric propulsion currently flown, they also can exhibit a number of critical failure modes over time such as emitter evaporation[2–4] and keeper erosion.[5–8]

Concerns about cathode life have led to an on-going effort to develop first-principles models for these devices. At the Jet Propulsion Laboratory (JPL), the investigation largely has centered on the 2D Orificed Cathode (OrCa2D) numerical code. This in-house, fluid-based model was originally developed in the mid-2000s to provide insight into internal processes in hollow cathodes that could not be accessed easily with probes.[2,8–10] The classically-dominated, dense plasma inside these devices lends itself to a fluid description, and as a result, OrCa2D has proven successful to date in yielding high fidelity results that agree with internal measurements on a number of cathodes. Non-classical processes that occur in the cathode plume, however, have proven more difficult to capture accurately. Previous efforts to model the plume with OrCa2D have revealed that electron resistivity due to classical processes such as electron-neutral and Coulomb collisions are orders of magnitude too low to explain the trends in experimentally-measured plasma parameters.[9] It is only after an anomalous resistivity is added to the model that the correct plasma parameters are captured.

Mikellides et al first postulated that the anomalous resistivity in the cathode plume might be attributed to the onset of ion acoustic turbulence (IAT).[2,8,9] The existence of this electrostatic instability and its large contribution to resistivity was later confirmed experimentally by Jorns et al.[11] In an effort to incorporate the IAT-driven resistivity into OrCa2D, Mikellides et al adopted an effective electron collision frequency based on the Sagdeev scaling.[12] By assuming that the IAT saturates through non-linear wave-particle interactions, this scaling reduces the collision frequency to a simple algebraic function of the background, macroscopic plasma parameters. The use of the Sagdeev formulation in OrCa2D led to the correct modeling of experimentally-measured plasma parameters along the centerlines of the NSTAR[9] and NEXIS cathodes,[8] operating at maximum discharge currents of 13.3 A and 27.5 A respectively, as well as a 100-A LaB_6 laboratory cathode.[10] Despite this success, however, our most recent work indicates that OrCa2D with the Sagdeev scaling fails to capture the correct plasma trends when modeling higher-current cathodes.[13] The implication of this result is that the simplified saturation model for the IAT that leads to the Sagdeev scaling is not correct in these devices. Instead, it is necessary to develop a more sophisticated model that self-consistently captures the evolution of the IAT in the cathode plume and the associated anomalous collision frequency driven by it.

Recent experimental and analytical work[14–16] has provided new insight about IAT in hollow cathode plumes and suggests a path forward for how its impact may be modeled self-consistently in a fluid-based code. The purpose of this two part investigation is to leverage the results from these previous studies to implement a first-principles description of the IAT in OrCa2D. The first part of the study, presented here, provides a self-consistent series of governing equations for how the IAT evolves in the unmagnetized plume of a hollow cathode and how it acts as both a drag and heating source for the plasma species. The second part, presented in our companion paper, Ref. 13, implements the governing relations into OrCa2D and validates

the output of the improved model against experimental measurements of a LaB_6 hollow cathode operating at 140 A.

This paper is organized in the following way. In the first section, we present a hierarchy to describe how the fluctuations associated with electrostatic modes exchange momentum and energy with the plasma species. In the second section, we summarize the properties of the ion acoustic modes in the cathode plume. In the third second, we use the properties of the IAT to evaluate the driving terms as functions of the total wave energy in the spectrum. In the fourth section, we derive a governing equation for how the total wave energy evolves in the cathode plume. In the fifth section, we use the results from the previous parts to find modified fluid equations for the electrons and ions that incorporate the effects of the IAT. In the sixth and final sections, we summarize the key assumptions for our model and the physical significance of the governing equations we have derived.

II. Impact of electrostatic turbulence on macroscopic parameters

We estimate the impact of electrostatic oscillations on the macroscopic properties of the ions and electrons by taking moments of a kinetic description of the plasma. This technique follow the treatment of Dum[17] and Davidson and Krall[18] in finding fluid models for anomalous transport in high-temperature plasmas. For our approach, we assume an unmagnetized plasma—as is consistent with our previous experimental and numerical work on high-current cathodes[10,11,15,16]— and use the Boltzmann hierarchy to describe each species:

$$\frac{\partial f_{(s)}}{\partial t} + \vec{v} \cdot \frac{\partial f_{(s)}}{\partial \vec{r}} + \frac{q_s}{m_s} \vec{E} \cdot \frac{\partial f_{(s)}}{\partial \vec{v}} = \mathcal{C}\left[f_s\right]. \tag{1}$$

Here $f_{(s)}$ denotes the distribution function in velocity space for species (s), $\vec{E}$ is the electric field, q_s is the species charge, m_s is the species mass, and $\mathcal{C}[]$ denotes a collision operator. We treat the electrostatic fluctuations as perturbations to the background electric field and equilibrium distribution function:

$$\begin{aligned} f_{(s)} &= f_{0(s)} + \sum_k f_{1k(s)} e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)} \\ \vec{E} &= \vec{E}_0 + \sum_k \vec{E}_{1k} e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}, \end{aligned} \tag{2}$$

where we have invoked an eikonal approximation to represent the perturbations as a sum over modes with frequency ω and wavevector $\vec{k}$. The zeroth order component $f_{0(s)}$ describes the macroscopic properties in the plasma which evolve on a fluid-time scale. The perturbations, represented by $f_{1k(s)}$, vary more rapidly.

As we are interested in the evolution of the more slowly-evolving macroscopic parameters, we substitute the relations from Eq. 2 into Eq. 1 and average over the shorter length scales of the oscillations. This is the equivalent of invoking the random phase approximation such that only the amplitudes of the oscillating terms survive the averaging:

$$\frac{\partial f_{0(s)}}{\partial t} + \vec{v} \cdot \frac{\partial f_{0(s)}}{\partial \vec{r}} + \frac{q_s}{m_s} \vec{E}_0 \cdot \frac{\partial f_{0(s)}}{\partial \vec{v}} = \mathcal{C}\left[f_{s(0)}\right] - \frac{q_s}{m_s} \frac{\partial}{\partial \vec{v}} \cdot \left[\sum_k \vec{E}_{1k} f_{1k(s)}\right]. \tag{3}$$

In the absence of fluctuations, the zeroth, first, and second moments of Eq. 3 lead to the standard fluid equations for continuity, momentum, and energy where the collision operator, $\mathcal{C}$, is modeled with the Brangiskii formulation.[19] When fluctuations are present, on the other hand, we can see that the first and second moments of the second term on the right hand side act as IAT-driven source or sink terms for the macroscopic fluid properties:

$$\begin{aligned} \vec{F}_{AN(s)} &= -q_s \int \vec{v} \frac{\partial}{\partial \vec{v}} \cdot \left[\sum_k \vec{E}_{1k} f_{1k(s)}\right] d\vec{v} \\ Q_{AN(s)} &= -\frac{q_s}{2} \int v^2 \frac{\partial}{\partial \vec{v}} \cdot \left[\sum_k \vec{E}_{1k} f_{1k(s)}\right] d\vec{v}. \end{aligned} \tag{4}$$

The first term in Eq. 4, $\vec{F}_{AN(s)}$, represents the rate at which momentum is added or subtracted to species (s) by the interaction with the electrostatic modes, and the second term, $Q_{AN(s)}$ is the rate at which energy is added or subtracted. We simplify these terms by using the electrostatic approximation to express the kth component of the perturbing electric field in terms of a scalar potential: $\vec{E}_{1k} = -i\vec{k}\phi_k \exp[i(\vec{k}\cdot\vec{r} - \omega t)]$. Integrating Eq. 4 by parts thus allows us to write the driving terms as

$$\begin{aligned} \vec{F}_{AN(s)} &= -q_s \sum_k \vec{k}\phi_k \mathrm{Im}\left[n_{1k(s)}\right] \\ Q_{AN(s)} &= -q_s \sum_k \phi_k \mathrm{Im}\left[\omega n_{1k(s)}\right]. \end{aligned} \tag{5}$$

Here we have defined the zeroth moment of the first-order distribution as the perturbed plasma density, $n_{1k(s)} = \int f_{1k(s)} d\vec{v}$, and we have simplified the second line by using the zeroth moment (continuity) of the perturbed form of Eq. 1, $\int \vec{k}\cdot\vec{v} f_{1k(s)} d\vec{v} = \omega \int f_{1k(s)} d\vec{v}$.

We can gain insight into how these driving terms for each species in Eq. 5 relate to one another by taking the real and imaginary components of Poisson's equation for the kth mode:

$$\begin{aligned} \mathrm{Im}\left[n_{1k(i)}\right] &= \mathrm{Im}\left[n_{1k(e)}\right] \\ \mathrm{Re}\left[n_{1k(i)}\right] &= \mathrm{Re}\left[n_{1k(e)}\right] + \frac{\epsilon_0 \phi_k k^2}{q}. \end{aligned} \tag{6}$$

From Eqs. 5 and 6, we thus find

$$\vec{F}_{AN(e)} = -\vec{F}_{AN(i)} = q \sum_k \vec{k}\phi_k \mathrm{Im}\left[n_{1k(e)}\right]. \tag{7}$$

This is a general statement of the conservation of momentum for the wave-driven process. Ion acoustic modes are characterized by oscillations in both particle species as well as electrostatic fields (c.f. Ref. 20). Since electrostatic fields have no momentum and the electrons are light, however, the wave momentum is primarily carried by the oscillations in the species with the most inertia, the ions. An increase in wave momentum thus results in an average increase in momentum of the ions, $\vec{F}_{AN(i)}$. By conservation law, this wave growth must be balanced by a loss of of the bulk electron momentum, i.e. $\vec{F}_{AN(i)} + \vec{F}_{AN(e)} = 0$. As we will see in Sec. III, the wave growth and subsequent loss in electron momentum stems from the kinetic process of inverse electron Landau damping.

For the heating terms in Eq. 5, we prescribe that the wavector $\vec{k}$ is real but the frequency of each kth mode has an imaginary component $\omega = \omega_r + i\omega_i$ where $\omega_i \ll \omega_r$. The relations in Eq. 6 combined with the second line of Eq. 5 thus yield

$$\begin{aligned} Q_{AN(e)} &= q \sum_k \phi_k \left(\omega_r \mathrm{Im}\left[n_{1k(e)}\right] + \omega_i \mathrm{Re}\left[n_{1k(e)}\right]\right) \\ Q_{AN(i)} &= -\sum_k \left[q\phi_k \left(\omega_r \mathrm{Im}\left[n_{1k(e)}\right] + \omega_i \mathrm{Re}\left[n_{1k(e)}\right]\right) + 2\omega_i E_k\right], \end{aligned} \tag{8}$$

where we have defined the total energy density of the electrostatic field,

$$E_k = \frac{1}{2} n_0 \epsilon_0 \left|\vec{E}_k\right|^2 = \frac{1}{2} n_0 \epsilon_0 k^2 \phi_k^2 \tag{9}$$

and used quasineutrality to write $n_{e(0)} = n_{i(0)} = n_0$. The terms in Eq. 8 represent the change in kinetic energy of the electrons and ions associated with the growth, damping, and propagation of the electrostatic waves. Unlike in the case of momentum, however, not only do the ion oscillations have nonnegligible kinetic energy but so too do the electron oscillations and the electric field associated with the waves, E_k. Therefore, energy is only conserved by taking into account both the particle kinetic energy and field energy contributions: $Q_{AN(e)} + Q_{AN(i)} + dE_k/dt = 0$ where we have defined the rate in increase of the electric field energy density of the kth mode as $dE_k/dt = 2\omega_i E_k$.

In summary, Eqs. 7 and 8 express the driving terms for the change in momentum and energy induced by the electrostatic oscillations. In order to evaluate these explicitly for the case of ion acoustic waves, it is necessary to find expressions for the perturbed electron density $n_{1k(e)}$, the real and imaginary components of the mode frequency ω_r, ω_i, the wave vector $\vec{k}$, and the potential amplitude ϕ_k. We consider the explicit forms of these parameters for ion acoustic waves in the next section.

III. Ion acoustic modes

A. Perturbed density

Expanding Eq. 1 to first order, we find the following equation governing the component of the distribution function oscillating at wavevector $\vec{k}$:

$$f_{1k(s)}\left(\omega + i\nu_s - \vec{k}\cdot v\right) = -\frac{q_s\phi_k}{m_s}\vec{k}\cdot\frac{\partial f_{0(s)}}{\partial\vec{v}} + i\nu_s\frac{n_{1k(s)}}{n_0}f_{0(s)}, \tag{10}$$

where ν_s denotes the species collision frequency, and we have invoked the BGK operator (c.f. Refs. 11 and 21) to approximate the impact of the collision operator on the fluctuations, $\mathcal{C}\left[f_{1k(s)}\right]$. In keeping with a fluid description of the plasma, we assume each species is thermalized and therefore can be described by isotropic drifting Maxwellian distributions:

$$f_{s(0)}\left(\vec{v}\right) = n_{0(s)}\left(\frac{m_s}{2\pi T_s}\right)e^{-\frac{m_s}{2T_s}\left(\vec{v}-\vec{V}_s\right)^2}, \tag{11}$$

where T_s denotes the temperature in energy units. Here we only consider a distribution in two-dimensions as hollow cathodes are axisymmetric. Subject to the Maxwellian assumption, we solve Eq. 10 for $f_{1k(s)}$ and take the zeroth moment to find the perturbed density:

$$n_{1k(s)} = \frac{q_s n_0\phi_k}{T_s}\frac{1+\zeta_s Z_0\left(\zeta_s\right)}{1+i\left(\nu_s/kv_s\right)Z_0\left(\zeta^s\right)}, \tag{12}$$

where $v_s = \sqrt{2T_s/m_s}$ is the species thermal speed and we have introduced the plasma dispersion function

$$Z_0\left(\zeta_s\right) = i\sqrt{\pi}e^{-\left(\zeta^s\right)^2}\left[1+\mathrm{erf}\left(i\zeta^s\right)\right] \qquad \zeta_s = \frac{\omega - \vec{k}\cdot\vec{V}_s}{v_s k}. \tag{13}$$

We can simplify Eq. 12 by considering the typical ordering of parameters appropriate for hollow cathode plasmas: $v_i \ll c_s \ll V_e \ll v_e$, where $c_s = \sqrt{T_e/m_i}$ is the ion sound speed. In this limit, the plasma dispersion functions in Eq. 12 satisfy $\left|\zeta^e\right| \ll 1$ and $\left|\zeta^i\right| \gg 1$ such that we can expand asymptotically[20] to find

$$n_{1k(e)} = \frac{q\phi_k n_0}{T_e}\left(1 + i\left(\frac{\pi}{2}\right)^{1/2}\left[\frac{\omega - \vec{k}\cdot\vec{V}_e}{kv_e}\right]\right) \tag{14}$$

$$n_{1k(i)} = \frac{q\phi_k n_0}{m_i}\left[\left(\frac{k}{\left(\omega - \vec{k}\cdot\vec{V}_i\right)}\right)^2 + i\left(\left(\frac{\nu_i}{\omega - \vec{k}\cdot\vec{V}_i}\right)\left(\frac{k}{\omega - \vec{k}\cdot\vec{V}_i}\right)^2 + \left(\frac{\pi}{2}\right)^{1/2}\left(\frac{m_i}{T_i}\right)^{3/2}\frac{\omega - \vec{k}\cdot\vec{V}_i}{k}e^{-\frac{m_i}{2T_i}\left(\frac{\omega - \vec{k}\cdot\vec{V}_i}{k}\right)^2}\right)\right],$$

where per Refs. 11 and 21 we have invoked the assumption that $\nu_e/\left(kv_e\right) \ll 1$ to neglect contributions from the electron collisions.

B. Dispersion Relation

Writing Poisson's equation for the kth plasma mode yields the dielectric response, $\epsilon(\omega,\vec{k})$, of the plasma

$$0 = k^2\epsilon\left(\omega,\vec{k}\right) = k^2 - \frac{1}{\epsilon_0\phi_k}\sum_s q_s n_{1k(s)}. \tag{15}$$

Assuming the growth is small $\omega_i \ll \omega_r$, we prescribe the wavevector $\vec{k}$ to be real and solve for the real and imaginary components of the frequency by performing a Taylor expansion of the dielectric around $\omega = \omega_r$:

$$0 = \epsilon\left(\omega,\vec{k}\right) = \epsilon_r\left(\omega_r,\vec{k}\right) + i\left[\epsilon_i\left(\omega_r,\vec{k}\right) + \omega_i\frac{\partial\epsilon_r\left(\omega_r,\vec{k}\right)}{\partial\omega}\right], \tag{16}$$

where ϵ_r and ϵ_i denote the real and imaginary components of the dielectric tensor respectively. Setting the real component of Eq. 16 to zero and using the expressions for perturbed density from Eq. 14, we find the dispersion relation for the acoustic modes:

$$\left(\omega_r - \vec{k}\cdot\vec{V}_i\right)^2 = k^2 c_s^2 \left(\frac{1}{\left(\lambda_{d(e)}k\right)^2 + 1}\right), \tag{17}$$

where $\lambda_{d(e)}$ denotes the electron Debye length. For this investigation, and as has been experimentally observed in high-current hollow cathode plumes,[11] the wave frequencies where the waves propagate are lower than the ion plasma frequency, $\omega_r \ll \omega_{pi}$. We therefore consider the limit $\lambda_{d(e)}k \ll 1$ such that the simplified dispersion relation is

$$\omega_r = kc_s + \vec{k}\cdot\vec{V}_i, \tag{18}$$

where $c_s = \sqrt{T_e/m_i}$ denotes the ion sound speed. This expression shows that in the ion frame of reference the propagating modes are acoustic in nature scaling linearly with the ion sound speed. Setting the imaginary component of Eq. 16 to zero and using the simplified dispersion relation from Eq. 18, we similarly can find the growth rate:

$$\omega_i = \left(\frac{\pi}{8}\right)^{1/2} c_s k \left[\left(\frac{\vec{k}\cdot\left(\vec{V}_e - \vec{V}_i\right) - c_s}{kv_e}\right) - \left(\frac{T_e}{T_i}\right)^{3/2} e^{-T_e/2T_i}\right] - \frac{1}{2}\nu_i. \tag{19}$$

This result for the imaginary component of the frequency provides insight into the terms that damp or grow the acoustic modes. The first term on the right hand side of Eq. 19 represents the growth of the wave from inverse Landau damping on the electron species. This is a collisionless process whereby the acoustic waves can gain energy at the expense of electron drift $\vec{V}_e$ provided that a resonant condition is satisfied: the electron drift in the direction of the phase velocity must exceed the wave speed in the ion frame of reference, c_s. This is inherently a kinetic effect and depends on an interaction of the waves with a small subset of the electron distribution of particles. We note that the criterion for electron growth is typically satisfied for hollow cathode plasmas as they exhibit high electron drift in the plume. The second term in Eq. 19 represents collisionless damping of the waves from ion Landau damping. In a process that directly mirrors that of the electrons, the bulk ion population can remove energy from the acoustic turbulence due to the resonant interaction of a small subset of the ions in the distribution function moving at the wave phase velocity. The last term in Eq. 19 represents damping of the wave due to ion collisions.

C. Potential Amplitude

Since the perturbed density scales with potential amplitude (Eq. 14), the summations over $\vec{k}$ in the expressions for momentum and energy rates, Eqs. 7 and 8, are weighted by the square of the potential amplitude, ϕ_k^2. Without a fully self-consistent description of the dominant processes governing the interaction of the ion acoustic modes—including nonlinear effects such as wave-wave coupling (c.f. Ref. 22) — it is not possible to know *a priori* how ϕ_k^2 depends on the wavevector. We circumvent this limitation by assuming a functional form of the potential amplitude that is informed by experimental measurements and is consistent with theories for saturated acoustic turbulent spectra:

$$\phi_k^2 = \begin{cases} 0 & \omega_r > \beta\omega_{pi} \\ \delta\left(\hat{k} - \hat{V}_D\right) C_0 \left(\omega_r\right)^{-p} & \beta\omega_{pi} > \omega_r > \omega_0, \\ 0 & \omega_r < \omega_0 \end{cases} \tag{20}$$

where C_0 and β are constants, $p = 1$, δ denotes the Dirac delta function, and $\vec{V}_D = \left(\vec{V}_e - \vec{V}_i\right) / \left|\vec{V}_e - \vec{V}_i\right|$ is the direction of relative drift between the species.

The inverse power law for the magnitude of the energy spectrum is consistent with a number of theoretical and experimental results of saturated IAT. It is the result of a redistribution of energy in the spectra to longer length scales by nonlinear processes.[22,23] The value $p = 1$ is in keeping with the nonlinear ion Landau damping saturation mechanism originally proposed by Sagdeev and Galeev.[12] The need for a lower cutoff

frequency ω_0 for the spectrum is based on previous measurements of the IAT spectra in a 100-A LaB_6 cathode (Fig. 3 in Ref. 11 and Fig. 5 in Ref. 16) and is physically attributed to damping of acoustic modes at long lengthscales by processes such as geometric cutoffs or ion collisions.[16] The choice to place an upper bound on the spectrum at $\omega_r = \beta\omega_{pi}$ represents the theoretical cutoff of ion acoustic waves above this frequency, and the scaling factor $\beta < 1$ captures the experimental observations from Ref. 16 that the acoustic modes actually dissipate at some fraction of this theoretical limit. The multiplication by the Dirac delta only selects wave vectors in the direction of relative drift. This choice is motivated by Eq. 19 which shows that the growth rate is maximized when $\vec{k}$ is collinear with $\vec{V}_e - \vec{V}_i$. We therefore assume that the wave direction is confined to a very narrow cone in this dominant direction.

D. Energy Density

Applying the definition for wave energy density of electrostatic modes[20] to the dielectric response (Eq. 15), we find for the kth mode of the ion acoustic waves

$$W_k = \epsilon_0 \frac{k^2\phi_k^2}{2}\omega_r \frac{\partial\epsilon_r}{\omega_r} = \frac{\omega_r}{kc_s}\left(\frac{n_0 q^2 \phi_k^2}{T_e}\right). \tag{21}$$

As we will find in the next section, it will be necessary to determine moments of the summation, $\sum_k (\omega_r)^\gamma W_k$, where γ is an integer. To approximate this quantity, we convert the summation over k to an integral by multiplying by a differential Δk such that

$$\sum_k (\omega_r)^\gamma W_k = \frac{1}{\Delta k}\int_{k_{min}}^{k_{max}} W_k (\omega_r)^\gamma \, dk. \tag{22}$$

Because we have assumed a distribution where $\vec{k}||\vec{V}_D$ (Eq. 20), the frequency and wavenumber are linearly related by $\omega_r = kc_s[1 + \hat{V}_D \cdot \vec{V}_i/c_s]$. Therefore, without loss of generality, we can make the transformation of Eq. 22 to an integral over frequency space

$$\sum_k (\omega_r)^\gamma W_k = \frac{1}{\Delta k\left(1 + \hat{V}_D \cdot \vec{V}_i/c_s\right)}\int_{\omega_0}^{\beta\omega_{pi}} W_k (\omega_r)^\gamma \, d\omega. \tag{23}$$

Evaluating this integral with the specified form of ϕ_k^2 from Eq. 20 under the assumption that $\beta\omega_{pi} \gg \omega_0$ (the upper bound for the frequency cutoff is much higher than the lower bound), we find the result

$$\sum_k (\omega_r)^\gamma W_k = \begin{cases} W_T \frac{\beta\omega_{pi}}{\ln[\beta\omega_{pi}/\omega_0]} & \gamma = 1 \\ W_T & \gamma = 0 \\ W_T \frac{1}{\omega_0 \ln[\beta\omega_{pi}/\omega_0]} & \gamma = -1 \end{cases} \tag{24}$$

where W_T denotes the total wave energy density summed over all k modes of the spectrum.

IV. Driving terms for the case of ion acoustic waves

Armed with the simplified expressions for the density perturbations, dispersion, wave growth, and potential amplitude distributions from the previous section, we now can evaluate the driving terms in Eqs. 7 and 8. It should be noted here that the density perturbations are explicit functions of both the real and imaginary components of the wave frequency $n_{1k(s)}(\omega)$. When estimating the driving terms then, we make the substitution $\omega = \omega_r + i\omega_i$ and only retain the contributions from ω_i to first order.

A. Momentum

For the change in momentum of the electrons, Eq. 7 yields

$$\vec{F}_{AN(e)} = -m_e n_0 \nu_a \left(\vec{V}_e - \vec{V}_i\right), \tag{25}$$

where we have applied the assumed distribution of ϕ_k (Eq. 20) in setting $\vec{k}||(\vec{V}_e - \vec{V}_i)$ and we have defined an effective collision frequency

$$\nu_a = \sum_k \frac{kq^2\phi_k^2}{m_e T_e}\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{m_e}{T_e}\right)^{1/2}. \tag{26}$$

The physical implication of this result is that the relative drift between species leads to the growth of the acoustic modes through inverse Landau damping on the electrons. The transfer of momentum to the turbulence from the electrons is represented in a macroscopic, fluid way as an enhanced drag on this species with anomalous collision frequency, ν_a.

We use the definition of wave energy density from Eq. 21 and the moments for W_k from Eq. 24 to express the anomalous collision frequency as

$$\nu_a = \left[\frac{\beta}{\ln\left[\beta\omega_{pi}/\omega_0\right]}\left(\frac{\pi}{2}\right)^{1/2}\frac{1}{\left(1+\hat{V}_D\cdot\vec{V}_i/c_s\right)^2}\right]\omega_{pe}\frac{W_T}{T_e n_0}. \tag{27}$$

The parameter $\beta < 1$ is assumed to be constant, and we know empirically for the IAT in the cathode plasma (c.f. Fig. 3 in Ref. 11), the logarithm of the ratio of the upperbound to lowerbound cutoff frequencies is $\sim \mathcal{O}[1]$. Similarly, typical ion velocities in cathode plumes have been measured to be $2-5$ km/s[24,25] with electron temperatures of $T_e = 2-5$ eV.[11,26,27] We therefore have $\left(1+\frac{\hat{V}_D\cdot\vec{V}_i}{c_s}\right) \sim \mathcal{O}[1]$. With these order of magnitude estimates, we can reduce the term in brackets to a single parameter $\alpha < 1$ such that

$$\nu_a = \alpha\omega_{pe}\frac{W_T}{T_e n_0}. \tag{28}$$

This is the same Sagdeev and Galeev[12] scaling that has been shown to be valid for a number of experiments with empirically-determined constant $\alpha \approx 10^{-2}$. Turning now to the ions, we have from Eq. 7 that

$$\vec{F}_{AN(i)} = m_i n_0 \nu_{a(i)}\left(\vec{V}_i - \vec{V}_e\right), \tag{29}$$

where $\nu_{a(i)} = (m_e/m_i)\,\nu_a$. Just as with classical electron-ion collisions, we thus see that the effective collision frequency driven by the ion acoustic turbulence is elastic.

B. Energy

Using the relations from Secs. II and III, the heating terms for the electrons and ions are given by Eq. 8:

$$\begin{aligned} Q_{AN(e)} &= \frac{1}{\left(1+\hat{V}_D\cdot\vec{V}_i/c_s\right)}\sum_k\left[\frac{1}{2}\frac{dW_k}{dt} - 2\omega_{i(e)}W_k\right] \\ Q_{AN(i)} &= \frac{1}{\left(1+\hat{V}_D\cdot\vec{V}_i/c_s\right)}\sum_k\left[\frac{1}{2}\frac{dW_k}{dt} - 2\omega_{i(i)}W_k\right] - \frac{dE_k}{dt}, \end{aligned} \tag{30}$$

where we have written the total change in wave energy density as $dW_k/dt = 2\omega_i W_k$ and isolated the contributions to the wave growth from Eq. 19:

$$\begin{aligned} \omega_{i(e)} &= \left(\frac{\pi}{8}\right)^{1/2} c_s k\left(\frac{\vec{k}\cdot\left(\vec{V}_e - \vec{V}_i\right) - c_s}{k v_e}\right) \\ \omega_{i(i)} &= -\left(\frac{\pi}{8}\right)^{1/2} c_s k\left(\left(\frac{T_e}{T_i}\right)^{3/2} e^{-T_e/2T_i} - \nu_i\right). \end{aligned} \tag{31}$$

Considering that the multiplying term is of order unity, the expressions in Eq. 30 lend themselves to a physical interpretation of the kinetic energy of each species associated with the wave dynamics. In particular, the first term in both $Q_{AN(e)}$ and $Q_{AN(i)}$ represents the sloshing motion of the particles necessary to carry the waves.

For the ions, this oscillatory motion gives rise to the wave momentum, whereas for the electrons the sloshing results from the need to preserve quasineutrality. In both cases, the magnitude of the particle kinetic energy for the wave motion is directly tied to the wave energy: with larger amplitude waves, the higher the energy from oscillations of both species. The rate change in the kinetic energy from this motion therefore scales with the rate change of the wave energy, dW_k/dt, leading to an effective heating—termed non-resonant or reversible (c.f. Ref. 28)— of the bulk populations. The second terms in both lines of Eq. 30 captures the resonant, irreversible heating processes associated with each species. For the electrons, this takes the form in Eq. 30 of inverse Landau damping on this species. This is a kinetic process where a population of electrons in the distribution that are resonant with the phase velocity lose energy to the wave growth, thereby causing an effective cooling of the electrons. For the ions, the contribution of irreversible heating is due to ion Landau damping and ion collisions. The former is a kinetic process where a small population of ions in the tail of distribution that are resonant with the phase velocity experience an irreversible gain in energy. This removes energy from the waves and adds it to the ions. The latter effect is due to frictional heating of the ions on the other species. The statement of energy conservation we showed in Sec. II now can be interpreted in the context of the irreversible and reversible heating processes outlined here. Namely, the change in wave energy as carried by particle motion is balanced by the growth driven by the electrons and damping resulting from the ions.

By the same arguments from the previous section we employed to simplify the anomalous collision frequency (Eq. 28), we can approximate Eq. 30 as

$$Q_{AN(e)} = -\frac{\alpha}{2}\omega_{pi}W_T\left(\frac{\left|\vec{V}_e-\vec{V}_i\right|-c_s}{v_e}+\left(\frac{T_e}{T_i}\right)^{3/2}e^{-T_e/2T_i}\right) \tag{32}$$

$$Q_{AN(i)} = \frac{\alpha}{2}\omega_{pi}W_T\left(\frac{\left|\vec{V}_e-\vec{V}_i\right|-c_s}{v_e}+\left(\frac{T_e}{T_i}\right)^{3/2}e^{-T_e/2T_i}\right), \tag{33}$$

where we have make the physically reasonable assumption that ion collision frequency is lower than the ion plasma frequency $\nu_i/\omega_{pi} \ll 1$ and we have dropped the contributions of the electrostatic terms in $Q_{AN(i)}$ under the assumption that the growth rate $\omega_i \ll \omega_{pi}$ is lower than the ion plasma frequency. In these simplified forms, we can see that the ion heating approximately balances the electron heating. Moreover, since the hollow cathode is typically characterized by large electron drift where $|\vec{V}_e - \vec{V}_i| \gg c_s$, Eq. 32 is negative definite for most locations in the cathode plume. The electrons therefore always lose energy to the growth of the waves—serving as their free source of energy. The ion population conversely always will be heated.

This section has yielded expressions for the effective drag and heating due to the acoustic modes as functions of typical fluid plasma parameters such as species temperature, average drift velocity, and density. We also have introduced a new variable, the total wave energy, that both the drag and heating terms scale with. In order to develop a self-consistent model for the role of the acoustic turbulence, it is necessary to introduce a governing equation for this total wave energy. We turn to this in the next section.

V. Wave energy equation

The plasma wave kinetic equation (c.f. Ref. 20) for the evolution of the kth electrostatic mode in the plasma is given by

$$\frac{\partial N_k}{\partial t}+\frac{\partial}{\partial \vec{r}}\cdot\left(N_k\frac{\partial\omega_r}{\partial\vec{k}}\right)-\frac{\partial}{\partial\vec{k}}\cdot\left(N_k\frac{\partial\omega_r}{\partial\vec{r}}\right)=2\omega_i N_k, \tag{34}$$

where $N_k = W_k/\omega_r$ denotes the wave action. This expression can be interpreted as a conservation equation for the density of quasi-particles, phonons, at each wavevector $\vec{k}$. In direct analog to the Vlasov equation (Eq. 1 in the collisionless limit), $\vec{k}$ replaces the velocity vector $\vec{v}$ and N_k is a distribution function of phonons rather than a distribution function of particles, $f_{(s)}$. The first two terms on the left hand side of Eq. 34 represent the flux of phonons in physical space where they are convected out of the volume at the mode group velocity, $\partial\omega/\partial\vec{k}$. The third term is an effective acceleration that results as varying background plasma parameters change the dispersion of the modes propagating at $\vec{k}$. This parallels the way the electric field term

in the Vlasov equation changes the velocity of particles in a given distribution function through acceleration. The terms on the right side of Eq. 34 represent sources and sinks due to interactions with the plasma species. These include linear effects such as Landau damping and weak collisions (c.f. Eq. 19), and higher-order, nonlinear terms due to wave-wave and wave-particle coupling effects[12,28] that we have neglected. As an aside, we note that Eq. 34 shows that the phonon, or wave packet, with maximum growth is the one in the direction where ω_i is maximized. This is further justification of the form for the potential amplitude we outlined in Sec. III.C.

Using the assumed distribution of wave amplitude and the explicit forms for frequency and growth rate from Sec. III.B, we integrate Eq. 34 over all values of $\vec{k}$ to find

$$\frac{\partial N_T}{\partial t} + \frac{\partial}{\partial \vec{r}} \cdot \left(N_T \left[c_s \hat{V}_D + \vec{V}_i \right] \right) = c_s \left(\int k N_k dk \right) \left(\frac{\pi}{2} \right)^{1/2} \left[\frac{\left| \vec{V}_e - \vec{V}_i \right| - c_s}{v_e} - \left(\frac{T_e}{T_i} \right)^{3/2} e^{-T_e/2T_i} \right] - \nu_i N_T, \tag{35}$$

where we have defined the total wave action $N_T = \int N_k dk$. Employing Eq. 24 for the moments of the energy distribution, we can re-write this expression as a function of total wave energy density:

$$\frac{\partial W_T}{\partial t} + \frac{\partial}{\partial \vec{r}} \cdot \left(W_T \left[c_s \hat{V}_D + \vec{V}_i \right] \right) = W_T \left(\frac{\omega_0 \ln \left[\beta \omega_{pi} / \omega_0 \right]}{1 + \hat{V}_D \cdot \vec{V}_i / c_s} \left(\frac{\pi}{2} \right)^{1/2} \left[\frac{\left| \vec{V}_e - \vec{V}_i \right| - c_s}{v_e} - \left(\frac{T_e}{T_i} \right)^{3/2} e^{-T_e/2T_i} \right] \right.$$
$$\left. -\nu_i - \omega_0 \ln \left[\beta \omega_{pi} / \omega_0 \right] \left(\frac{\partial}{\partial t} \left[\frac{1}{\omega_0 \ln \left[\beta \omega_{pi} / \omega_0 \right]} \right] + \frac{\partial}{\partial \vec{r}} \cdot \left[\frac{c_s \hat{V}_D + \vec{V}_i}{\omega_0 \ln \left[\beta \omega_{pi} / \omega_0 \right]} \right) \right] \right). \tag{36}$$

Since the acoustic modes propagate in the laboratory frame with group velocity $\partial \omega / \partial \vec{k} = c_s \hat{V}_D + \vec{V}_i$, the last term in Eq. 36 represents the characteristic timescale in the wave frame of reference over which the parameter $\omega_0 \ln \left[\beta \omega_{pi} / \omega_0 \right]$ varies. For this investigation, we assume that the cutoff frequency ω_0 is approximately constant and that the logarithmic contribution, $\ln \left[\beta \omega_{pi} / \omega_0 \right]$, is only a weakly varying parameter in the plume. We therefore can neglect this last term compared to the collisional damping and the growth. This allows us to write Eq. 36 in the simplified form

$$\frac{\partial W_T}{\partial t} + \nabla \cdot \left(W_T \left[c_s \hat{V}_D + \vec{V}_i \right] \right) = \omega_0 W_T \left(\frac{\pi}{2} \right)^{1/2} \left[\frac{\left| \vec{V}_e - \vec{V}_i \right| - c_s}{v_e} - \left(\frac{T_e}{T_i} \right)^{3/2} e^{-T_e/2T_i} - \frac{\nu_{in}}{\omega_0} \right], \tag{37}$$

where per our discussion in Sec. IV.A, we have used an order of magnitude analysis to fold the coefficients of $\mathcal{O}[1]$ into the parameter ω_0. The interpretation of this equation is that the wave energy propagates along characteristics parallel to the average group velocity, $\partial \omega / \partial \vec{k}$. This energy changes as it convects through the plasma depending on the interchange between growth from the electron drift and damping from resonant heating of ions and collisions.

VI. Fluid equations for incorporation into OrCa2D

We now use the results from the previous sections to find governing equations for the evolution of the macroscopic fluid properties of the ions and electrons. The formalism for these equations has been reported in previous OrCa2D publications (c.f. Ref. 2,8,10), and we adopt the same conventions here to illustrate how the equations are augmented by the anomalous terms. For the ion continuity, we take the zeroth moment of Eq. 3, the phase-averaged Boltzmann. This yields

$$\frac{\partial n}{\partial t} + \nabla \cdot \left[n \vec{V}_i \right] = \dot{n}, \tag{38}$$

where we have dropped the subscript 0 and denoted the ionization frequency due to electron impact collisions with neutrals as $\dot{n}$. For the ion momentum, the first moment of Eq. 3 multiplied by ion mass yields

$$\frac{\partial \left(m_i n \vec{V}_i \right)}{\partial t} + \nabla \cdot \left[m_i n \vec{V}_i \vec{V}_i \right] = nq\vec{E} - \nabla p_i + m_i \dot{n} \vec{V}_n - n m_i \left[\nu_{ie} \left(\vec{V}_i - \vec{V}_e \right) + \nu_{in} \left(\vec{V}_i - \vec{V}_n \right) \right], \tag{39}$$

where $p_i = nT_i$ is the ion pressure; we have incorporated the effect of newly created ions adding to the fluid momentum through $\dot{n}\vec{V}_n$; and we have represented the impact of charge-exchange collisions with the neutrals through $\nu_{in}(\vec{V}_i - \vec{V}_n)$. Similarly, we have defined $\nu_{ie} = \frac{m_e}{m_i}(\nu_{ei}^c + \nu_a)$ where ν_a is the anomalous collision frequency given by Eq. 28 and ν_{ei}^c is the Coulomb collision frequency found from the standard form of the collision operator (c.f. Ref. 19). We note here that while we do not provide an explicit expression for the neutral dynamics, $(n_n, \vec{V}_n)$, this equation is given in our companion paper, Ref. 13. For the electron momentum equation, we take the first moment of Eq. 3 and follow the standard treatment of neglecting the inertial contributions to arrive at an Ohm's Law:

$$0 = -nq\vec{E} - \nabla p_e - nm_e\left[\nu_{ei}\left(\vec{V}_e - \vec{V}_i\right) + \nu_{en}\vec{V}_e\right], \tag{40}$$

where $p_e = nT_e$ is the electron pressure, ν_{en} is the electron-neutral collision frequency, and we have defined $\nu_{ei} = (\nu_{ei}^c + \nu_a)$.

Finally, we consider equations for the evolution of the species temperatures. These are found by taking the variance moment of Eq. 3, i.e. multiplying through by $(1/2)m_s|\vec{v} - \vec{V}_{(s)}|^2$ and integrating over velocity space. Using the definition for pressure of the electrons this yields

$$\frac{3}{2}\frac{\partial p_e}{\partial t} + \nabla\cdot\left(\frac{5}{2}p_e\vec{V}_e - \kappa_e\nabla T_e\right) - \vec{V}_e\cdot\nabla p_e = Q_e^T + Q_{AN(e)} + m_e n\nu_{ei}\vec{V}_e\cdot\left(\vec{V}_e - \vec{V}_i\right) - \dot{n}q\varepsilon_{ip}, \tag{41}$$

where ε_{ip} is the ionization energy for xenon, κ_e denotes the thermal conductivity, and Q_e^T is heat exchange due to thermal non-equilibrium of electrons with ions and neutrals. The left hand side of Eq. 41 represents the flux and conduction of thermal energy of the electrons. The right hand side includes source terms for the electron energy where the first term was already described, the second term is the anomalous cooling due to the growth of acoustic modes, the third term is from frictional heating of the electrons due to collisions with the ions (both classical and anomalous), and the last term is the inelastic loss of energy due to ionization.

We can simplify Eq. 41 by doing a relative comparison of the anomalous contributions. In particular, assuming that $|\vec{V}_e| \gg |\vec{V}_i|$, we see from Eqs. 28 and 32 that

$$\left|\frac{Q_{AN(e)}}{m_e n\nu_a\vec{V}_e\cdot\left(\vec{V}_e - \vec{V}_i\right)}\right| = \frac{1}{2}\left(\frac{m_e}{m_i}\right)^{1/2}\left[\frac{\sqrt{T_e/m_e}}{\left|\vec{V}_e\right|} + \frac{T_e/m_e}{\left|\vec{V}_e\right|^2}\left(\frac{T_e}{T_i}\right)^{3/2} e^{-\frac{T_e}{2T_i}}\right]. \tag{42}$$

For cathode plasmas we have investigated in previous work,[15] we have measured the electron Mach number to be $|\vec{V}_e|/\sqrt{T_e/m_e} \sim 0.14$, and since the square root of the mass ratios is ~ 480 and the maximum value of the exponential contribution is $(T_e/T_i)^{3/2}\exp\left[-T_e/2T_i\right] \approx 1.16$, the anomalous heating, $Q_{AN(e)}$, is only 12% of the frictional contribution. We therefore neglect the anomalous heating compared to the frictional heating by the anomalous collision frequency, which allows us to write the simplified form for the electron temperature equation:

$$\frac{3}{2}\frac{\partial p_e}{\partial t} + \nabla\cdot\left(\frac{5}{2}p_e\vec{V}_e - \kappa_e\nabla T_e\right) - \vec{V}_e\cdot\nabla p_e = Q_e^T + m_e n\nu_{ei}\vec{V}_e\cdot\left(\vec{V}_e - \vec{V}_i\right) - \dot{n}q\varepsilon_{ip}. \tag{43}$$

We note here briefly that this energy equation indicates that the effect of the waves (through the anomalous friction term) is to cause an increase in temperature. This is an apparent contradiction to our discussion from previous sections where we showed that the electrons should lose energy to the wave growth. This discrepancy is resolved by noting that even though the internal energy, temperature, increases because of the "rubbing" against the ions, the total energy (including kinetic energy) of the electrons decreases, $dE_T/dt = d/dt[(1/2)m_e|\vec{V}_e|^2 + (3/2)T_e] = Q_{AN(e)} < 0$.

Finally, we apply the same treatment as above to derive the ion heating equation. However, this time we find that since $\left|\vec{V}_e\right| \gg \left|\vec{V}_i\right|$, we can neglect the resistive heating term in favor of the anomalous ion heating:

$$\frac{3}{2}\frac{\partial p_i}{\partial t} + \nabla\cdot\left(\frac{5}{2}p_i\vec{V}_i - \kappa_i\nabla T_i\right) - \vec{V}_i\cdot\nabla p_i = Q_i^T + Q_{AN(i)}. \tag{44}$$

We can simplify this expression further by making a number of assumptions informed by experimental measurements in the cathode plume. In particular, a recent campaign to measure ion velocities along the

centerline of a 100-A LaB_6 hollow cathode[25] has shown that the magnitude of the ion flow velocity varies slowly in the plume. We therefore make the assumption that $\nabla P_i/P_i \gg \nabla \cdot \vec{V}_i/|\vec{V}_i|$ Similarly, if we assume that the electron drift is sufficiently high that the growth from electrons dominates over the ion Landau damping (a valid approximation in most regions near the cathode), we can use Eq. 37 to write Eq. 44 as

$$\frac{3}{2}\frac{\partial p_i}{\partial t} + \nabla \cdot \left(\frac{3}{2} p_i \vec{V}_i\right) = \frac{\alpha}{2}\frac{\omega_{pi}}{\omega_0}\left(\frac{2}{\pi}\right)^{1/2}\left[\frac{\partial W_T}{\partial t} + \nabla \cdot \left(W_T\left[c_s \hat{V}_D + \vec{V}_i\right]\right)\right]. \tag{45}$$

At steady-state, which ultimately is the condition where the cathode equilibrates, and subject to the previously discussed assumption that $c_s \approx \left|\vec{V}_i\right|$ we further can reduce this expression to

$$\nabla\left[nT_i\right] \approx \frac{\alpha}{e}\frac{\omega_{pi}}{\omega_0}\left(\frac{2}{\pi}\right)^{1/2}\nabla W_T. \tag{46}$$

For the slowly varying plasma parameters in the cathode plume, we make the weak assumption that $\frac{\omega_{pi}}{\omega_0}$ is constant. In which case, we find the relation:

$$T_i \approx \Gamma \frac{W_T}{n}, \tag{47}$$

where $\Gamma \sim \mathcal{O}[1-100]$ is a constant depending on the ratio of the plasma frequency to the cutoff frequency. This expression is in keeping with a physical description of the wave dynamics. Oscillations in the plasma density and ion velocity that correspond to the propagation of the acoustic modes lead to a time-averaged broadening in the standard deviation of the velocity, i.e. the temperature. With higher wave energy density, there thus is a correspondingly larger ion temperature. This effective broadening is the direct consequence of the so-called reversible heating discussed in Sec. IV.B. The scaling indicated by Eq. 47 has been experimentally validated in the only three studies to date that have examined the relationship between IAT and ion energy in the cathode plume.[15,16,25]

VII. Summary of Key Assumptions

Before we conclude with a discussion, we summarize briefly here the major empirically-informed and literature-based assumptions we have employed to arrive at our above formulation.

- The following ordering for velocities is valid for the hollow cathode plume: $v_i \ll c_s \ll V_e \ll v_e$.
- With the exception of assuming a shape for the wave energy spectrum, nonlinear effects such as wave-wave and wave-wave-particle coupling are neglected.
- The acoustic modes at each location propagate in the direction of maximum growth, i.e. $\vec{k}|| \left(\vec{V}_e - \vec{V}_i\right)$.
- The propagating modes in the plasma are cutoff below $\omega_r < \omega_{pi}$ such that we can assume the dispersion relation is linear for the entire spectrum of excited modes: $\omega_r = kc_s + \vec{k} \cdot \vec{V}_i$.
- The random phase approximation (RPA) can be employed to eliminate the fast time-dependence of the oscillations when deriving governing equations for the bulk flow properties. The RPA is typically valid for turbulent like spectra as exhibited by the IAT.
- The spectrum of the oscillations has an inverse power law dependence on frequency with lower and upper bound cutoffs in frequency. The upper bound cutoff occurs at some fraction of the ion plasma frequency and the lower cutoff is approximately constant in the plume.
- The logarithm of the ratio of the plasma frequency to the cutoff frequency varies slowly in the wave frame of reference compared to the growth terms driving or damping the waves.
- The electron growth dominates over ion Landau damping in most areas of the plume, and the plasma parameters vary sufficiently slowly with respect to the wave energy such that the relation $T_i \approx W_T/n$ holds.

VIII. Discussion

The relations we have derived in Secs. V and VI provide a self-consistent and first-principles formalism through which the impact of ion acoustic turbulence can be evaluated with a fluid-based code. Physically, our results quantify how the inverse Landau-driven growth of the waves occurs at the expense of the energy and momentum in the drifting electrons. In turn, because the wave momentum is carried primarily by fluctuations in the ion density, the loss in electron momentum is balanced by an increase in average ion momentum and energy. This occurs irreversibly through ion Landau damping on the resonant species as well as through the increased sloshing of non-resonant ions that carry the waves.

Previous modeling efforts with OrCa2D have attempted to capture one of the anomalous terms we have examined in this investigation, the enhanced collision frequency on the electrons.[9,10] Our work here expands on these previous attempts in two major ways. First, we have derived self-consistent relations that include not only the electron drag but also the impact of the IAT on both electron heating and ion dynamics. Second, while it was necessary in previous modeling efforts to assume phenomenological scaling laws for the total wave energy, we have presented here—through the formalism of the plasma wave kinetic equation—an explicit 2D partial differential equation for how the wave energy evolves in the plasma.

With this in mind, one of the concerns that stems from the model for the wave energy that we have presented in Eq. 37 is that this expression has the potential to diverge if the driving terms on the right side are sufficiently large. This is not a problem encountered in previous modeling efforts as the waves were always assumed to be saturated. Fortunately, the self-consistent nature of our formulation presented here allows for a regulating process to prevent this type divergence. Indeed, as the wave energy increases, so too does the ion temperature (Eq. 47). This in turn leads to an enhanced ion Landau damping (dependent on T_e/T_i in Eq. 37) which ultimately quenches the growth of the wave energy. Convergent solutions thus in principle can be found through the feedback represented by the ion Landau damping, and as we show in our companion paper, this self-regulation does in fact lead to a stable result.

Although the ion heating primarily serves to help stabilize the wave growth in our simulations, the fact that this process can be modeled with the above formalism offers some potential insight into the formation of energetic ions in the cathode plume. Indeed, previous investigations have leveraged simple 1-D kinetic models to show how the growth of the IAT that occurs at high cathode discharge currents is correlated with the formation of an energetic ion tail.[15,16] This tail in turn is believed to be linked to anomalous keeper erosion by ion bombardment. The fluid-based formalism we have presented in the above study cannot capture tail formation explicitly, but it does approximate the effect of the waves on the bulk distribution of ions. Estimates for ion temperature that stem from this formulation therefore may serve as an approximate metric for determining high-erosion states for the hollow cathode keeper. Indeed, the simulations presented in our companion paper predict ion temperatures that only underestimate experimental measurements of ion temperature (which are skewed by tail formation) by a factor of two.[13,25]

IX. Conclusion

In this investigation, we have presented a self-consistent formulation for determining how the growth of ion acoustic turbulence in a hollow cathode plasma impacts the average fluid parameters of the ions and electrons. We have used plasma kinetic theory in conjunction with the random phase approximation to show how the waves lead to an effective drag and heating of the separate species. In both cases, the driving terms for the fluid parameters depend explicitly on the total wave energy in the spectrum. We thus have introduced an additional equation for how this new unknown evolves in the cathode. This governing relation for the wave energy stems from the plasma wave kinetic equation and takes the form of a two-dimensional partial differential equation.

The physical interpretation for the interaction of the waves with the plasma is that the turbulence is driven unstable at the expense of the energy and momentum in the electron drift. This process leads to an effective drag and a resistive, frictional heating of this species. Since the waves are carried by ion oscillatory motion and are damped by resonant interactions with the ions, the turbulent growth in turn increases the ion temperature. This heating occurs until the ion Landau damping is sufficiently high to quench the IAT. The growth of the waves therefore can reach a self-consistent and convergent steady-state without the need to invoke artificial saturation mechanisms.

In the companion paper (Ref. 13) the relations we have found in this investigation are evaluated explicitly

with OrCa2D, JPL's in house fluid-based code, and applied to a simulation of a 140 A LaB_6 hollow cathode. The promising agreement with experimental measurements helps validate the relations derived here and ultimately represents a promising step forward for improving the physics and accuracy of ab initio fluid simulations..

Acknowledgments

The research described in this paper was carried out at the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautics and Space Administration. The support of the joint NASA GRC and JPL development of HERMeS by NASAs Space Technology Mission Directorate through the Solar Electric Propulsion Technology Demonstration Mission (SEP TDM) project is gratefully acknowledged.

References

[1]Herman, D., Santiago, W., Kamhawi, H., Polk, J., Synder, J., Hofer, R., and Parker, J., "The Ion Propulsion System for the Solar Electric Propulsion Technology Demonstration Mission," *34th International Electric Propulsion Conference, Hyogo-Kobe, Japan, July 6-10, 2015. IEPC-2015-08.*.

[2]Mikellides, I. G., Katz, I., Goebel, D. M., and Polk, J. E., "Hollow cathode theory and experiment. II. A two-dimensional theoretical model of the emitter region," *Journal of Applied Physics*, Vol. 98, No. 11, 2005.

[3]Polk, J. E., Mikellides, I. G., Katz, I., and Capece, A. M., "Tungsten and barium transport in the internal plasma of hollow cathodes," *Journal of Applied Physics*, Vol. 105, No. 11, 2009.

[4]Polk, J. E., Mikellides, I. G., Capece, A. M., and Katz, I., "Barium depletion in hollow cathode emitters," *Journal of Applied Physics*, Vol. 119, No. 2, 2016.

[5]Brophy, J. and Garner, C., "Tests of high current hollow cathodes for ion engines," *24th AIAA Joint Propulsion Conference, Boston, MA, July 11-13, 1988. AIAA-88-2913*, 1988.

[6]Polk, J. E., Anderson, J. R., Brophy, J. R., Rawlin, V. K., Patterson, M. J., Sovey, J., and Hamley, J., *35th AIAA/ASME/SAE/ASEE Joint Propulsion Conference & Exhibit, Los Angeles, CA, June 20-24, 1999. AIAA-99-2446*.

[7]Sengupta, A., Brophy, J. R., Anderson, J. R., Garner, C., de Groh, K., Karniotis, T., and Banks, B., "An Overview of the Results from the 30,000 Hr Life Test of Deep Space 1 Flight Spare Ion Engine," *40th AIAA/ASME/SAE/ASEE Joint Propulsion Conference & Exhibit, Fort Lauderdale, FL, July 11-14, 2004. AIAA-2004-3608*.

[8]Mikellides, I. G., Katz, I., Goebel, D. M., Jameson, K. K., and Polk, J. E., "Wear Mechanisms in Electron Sources for Ion Propulsion, II: Discharge Hollow Cathode," *Journal of Propulsion and Power*, Vol. 24, No. 4, July 2008, pp. 866–879.

[9]Mikellides, I. G., Katz, I., Goebel, D. M., and Jameson, K. K., "Evidence of nonclassical plasma transport in hollow cathodes for electric propulsion," *Journal of Applied Physics*, Vol. 101, No. 6, 2007, pp. 063301.

[10]Mikellides, I. G., Goebel, D. M., Jorns, B. A., Polk, J. E., and Guerrero, P., "Numerical Simulations of the Partially Ionized Gas in a 100-A LaB6 Hollow Cathode," *IEEE Transactions on Plasma Science*, Vol. 43, No. 1, Jan 2015, pp. 173–184.

[11]Jorns, B. A., Mikellides, I. G., and Goebel, D. M., "Ion acoustic turbulence in a 100-A LaB_6 hollow cathode," *Phys. Rev. E*, Vol. 90, Dec 2014, pp. 063106.

[12]Sagdeev, R. and Galeev, A., *Nonlinear plasma theory*, Frontiers in physics, W. A. Benjamin, 1969.

[13]Ortega, A., Mikellides, I., and Jorns, B., "First-principles Modeling of IAT-driven Anomalous Resistivity in Hollow Cathode Discharges II: Numerical Simulations and Comparisons with Experiments," *52nd AIAA/ASME/SAE/ASEE Joint Propulsion Conference and Exhibit, July 25-27, 2016, Salt Lake City, UT*.

[14]Jorns, B., Hofer, R., and Mikellides, I., "Power Dependence of the Electron Mobility Profile in a Hall Thruster," *50th AIAA Joint Propulsion Conference, Cleveland, OH, July 28-30, 2014. AIAA-2014-3620.*.

[15]Jorns, B., Goebel, D. M., and Mikellides, I. G., "Investigation of Energetic Ions in a 100-A Hollow Cathode," *50th AIAA/ASME/SAE/ASEE Joint Propulsion Conference and Exhibit, Cleveland, OH, July 28-30, 2014. AIAA-2014-3826*.

[16]Jorns, B., Mikellides, I., Goebel, D., and Lopez Ortega, A., "Mitigation of energetic ions and keeper erosion in a high-current hollow cathode," *34th International Electric Propulsion Conference, Hyogo-Kobe, Japan, July 6-10, 2015. IEPC-2015-134.*.

[17]Dum, C. T., "Can We Find Useful Algorithms for Anomalous Transport?" *Cross-Scale Coupling in Space Plasmas*, American Geophysical Union, 1995, pp. 1–12.

[18]Davidson, R. and Krall, N., "Anomalous transport in high-temperature plasmas with applications to solenoidal fusion systems," *Nuclear Fusion*, Vol. 17, No. 6, 1977, pp. 1313.

[19]Braginskii, S. I., "Transport Processes in a Plasma," *Reviews of Plasma Physics*, Vol. 1, 1965, pp. 205.

[20]Stix, T., *Waves in Plasmas*, AIP, New York, 1992.

[21]Fenneman, D. B., Raether, M., and Yamada, M., "Ion acoustic instability in the positive column of a helium discharge," *Physics of Fluids*, Vol. 16, No. 6, 1973, pp. 871–878.

[22]Bychenkov, V., Silin, V., and Uryupin, S., "Ion-acoustic turbulence and anomalous transport," *Physics Reports*, Vol. 164, No. 3, 1988, pp. 119 – 215.

[23]de Kluiver, H., Perepelkin, N., and Hirose, A., "Experimental results on current-driven turbulence in plasmas - a survey," *Physics Reports*, Vol. 199, No. 6, 1991, pp. 281 – 381.

[24]Williams, G., Smith, T., Domonkos, M., Gallimore, A., and Drake, R., "Laser induced fluorescence characterization of ions emitted from hollow cathodes," *IEEE Transactions on Plasma Science*, Vol. 28, No. 5, 2000, pp. 1664–1675.

[25]Dodson, C., Perez-Grande, D., Jorns, B., Goebel, D., and Wirz, R., "Laser-induced Fluorescence Measurements of EnergeticIons in a 100-A LaB_6 Hollow Cathode," *52nd AIAA/ASME/SAE/ASEE Joint Propulsion Conference and Exhibit, July 25-27, 2016, Salt Lake City, UT*.

[26]Chu, E. and Goebel, D. M., "High-Current Lanthanum Hexaboride Hollow Cathode for 10-to-50-kW Hall Thrusters," *Plasma Science, IEEE Transactions on*, Vol. 40, No. 9, 2012, pp. 2133–2144.

[27]Jameson, K., Goebel, D., and Watkins, R., "Hollow Cathode and Keeper-Region Plasma Measurements," .

[28]Davidson, R., *Methods in Nonlinear Plasma Theory*, Academic Press, New York, NY, 1972.