Adjoint sensitivity analysis of orbital mechanics: Application to computations of observables' partials with respect to harmonics of the planetary gravity fields

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An approach is presented to the inversion of gravity fields based on evaluation of partials of observables with respect to gravity harmonics using the solution of adjoint problem of orbital dynamics of the spacecraft. Corresponding adjoint operator is derived directly from the linear operator of the linearized forward problem of orbital dynamics. The resulting adjoint problem is similar to the forward problem and can be solved by the same methods. For given highest degree N of gravity harmonics desired, this method involves integration of N adjoint solutions as compared to integration of N^2 partials of the forward solution with respect to gravity harmonics in the conventional approach. Thus, for higher resolution gravity models, this approach becomes increasingly more effective in terms of computer resources as compared to the approach based on the solution of the forward problem of orbital dynamics. The presented theoretical framework is illustrated by results of validation numerical experiments.

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1. Introduction

Inversion of gravity fields from data of orbital tracking of planetary spacecraft (see, e.g., [1]) provides important data for:

- Geophysical studies of planet's interior
- Accurate prediction of evolution of the orbits of spacecraft

From the general viewpoint, this is a typical problem of remote sensing, where observable data of orbital tracking of the spacecraft (usually, range and Doppler shift) are interpreted in terms of harmonics of the planetary gravity field. As usual, this problem reduces to two sub-problems:

(1) Evaluation of partials, i.e., sensitivities of observables with respect to data to be retrieved, in this case – gravity harmonics for a current gravity model
(2) Update of the current (gravity) model based on discrepancies between predicted and actual observables (residuals)

This presentation deals with the first sub-problem. There are two alternative approaches to its solution. The widely-used conventional approach is based on the solutions of the linearized forward problem of orbital dynamics (Section 3) computed all the sensitivities with respect to individual gravity harmonics. The approach presented here is based in the solution of the corresponding adjoint problem (Sections 4, 5). This approach is being successfully used in numerous areas of remote sensing, but, to the best of authors' knowledge, was never used in the area explored here.

2. Physical background

Initial, nonlinear 3D problem:

$$\frac{d^2\mathbf{r}}{dt^2} = \nabla U$$
$$\left.\frac{d\mathbf{r}}{dt}\right|_{t_0} = \mathbf{v}_0$$
$$\mathbf{r}|_{t_0} = \mathbf{r}_0$$

Introducing the state vector $\mathbf{X}(t)$:

$$\mathbf{X}(t) = \begin{bmatrix} \mathbf{r}(t) \\ \mathbf{v}(t) \end{bmatrix}$$

it can be rewritten in the form of a 1st order matrix differential equation with an initial condition:

$$\frac{d\mathbf{X}}{dt} + \mathsf{M}\,\mathbf{X}(t) = \mathbf{B}(\mathbf{X})$$
$$\mathbf{X}|_{t_0} = \mathbf{X}_0$$

where the matrix coefficient

$$\mathsf{M} = \begin{bmatrix} 0 & -\mathsf{I} \\ 0 & 0 \end{bmatrix}$$

is a block matrix built of four 3×3 matrices, and I is an identity matrix. The right–hand vector term has the form

$$\mathbf{B}(\mathbf{X}) = \begin{bmatrix} \mathbf{0} \\ \nabla U(\mathbf{r}) \end{bmatrix}$$

In spherical coordinates, the gravity potential U can be written in the form of expansion (see, e.g., [2]):

$$U(r,\varphi,\lambda) = \frac{\mu}{r} - \frac{\mu}{r}\sum_{n=1}^{N}\left(\frac{a_p}{r}\right)^n J_n P_n(\sin\varphi)$$
$$+\frac{\mu}{r}\sum_{n=1}^{N}\left(\frac{a_p}{r}\right)^n \sum_{m=1}^{n} P_n^m(\sin\varphi)\left(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right)$$

and its gradient ∇U obtains the form:

$$\nabla U = -\frac{\mu}{r^2}\begin{bmatrix}1\\0\\0\end{bmatrix} + \frac{\mu}{r^2}\sum_{n=1}^{N}\left(\frac{a_p}{r}\right)^n J_n \begin{bmatrix}(n+1)P_n\\0\\-\cos\varphi P_n'\end{bmatrix}$$
$$+\frac{\mu}{r^2}\sum_{n=1}^{N}\left(\frac{a_p}{r}\right)^n \sum_{m=1}^{n}\begin{bmatrix}-(n+1)P_n^m\left(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right)\\ m\sec\varphi P_n^m\left(-C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right)\\ \cos\varphi P_n^{m'}\left(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right)\end{bmatrix}$$

To simplify the presentation here, we assume that the values of the vector of observables $\mathbf{R}(t)$ at the instants t are obtained, using some known procedure, from instant values of six components of the state vector $\mathbf{X}(t)$. The matrix $\mathsf{W}(t) = (\partial\mathbf{R}(t)/\partial\mathbf{X}(t))^{\mathrm{T}}$ is, thus, also known.

The goal is to evaluate the Jacobian of sensitivities $\mathbf{K} = \partial\mathbf{R}/\partial\mathbf{H}$ of observables $\mathbf{R}$ to the vector of gravity harmonics $\mathbf{H}$ consisting of the coefficients of expansion of the gravity potential U.

The discrepancies $\mathbf{R}'$ between actual and modelled observables can be reduced in terms of updates $\mathbf{H}'$ of gravity field harmonics solving corresponding inverse problem

$$\mathbf{K}\,\mathbf{H}' = \mathbf{R}'$$

by an appropriate least square routine.

The conventional and adjoint approaches to evaluation of $\partial\mathbf{R}/\partial\mathbf{H}$ are presented below.

3. Conventional forward approach

Initial non-linear forward problem is linearized around the baseline solution:

$$\frac{d}{dt}\begin{bmatrix}\mathbf{r}'\\ \mathbf{v}'\end{bmatrix} + \begin{bmatrix}0 & -\mathsf{I}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{r}'\\ \mathbf{v}'\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ (\nabla U)'\end{bmatrix}$$

Here, we have:

$$(U)' = U' + \nabla^{\mathrm{T}} U \mathbf{r}', \quad (\nabla U)' = \nabla U' + \nabla\nabla^{\mathrm{T}} U \mathbf{r}'$$

and resulting linearized forward problem has the form:

$$\left(\frac{d}{dt} + \mathsf{C}(t)\right)\mathbf{X}'(t) = \mathbf{B}'(t)$$

$$\mathbf{X}'|_{t_0} = \mathbf{X}'_0$$

Here:

$$\mathsf{C}(t) = \begin{bmatrix}0 & -\mathsf{I}\\ \nabla\nabla^{\mathrm{T}} U & 0\end{bmatrix}, \quad \mathbf{B}'(t) = \begin{bmatrix}\mathbf{0}\\ \nabla U'\end{bmatrix}$$

The matrix $\mathsf{C}(t)$ contains div grad of the unperturbed potential computed along the unperturbed trajectory:

$$\nabla\nabla^{\mathrm{T}} U = \left[\frac{\partial \nabla U}{\partial r}, \frac{1}{r\cos\varphi}\frac{\partial \nabla U}{\partial \lambda}, \frac{1}{r}\frac{\partial \nabla U}{\partial \varphi}\right]$$

where (see [2]):

$$\frac{\partial \nabla U}{\partial r} = \frac{3\mu}{r^3}\begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{\mu}{r^3}\sum_{n=1}^{N}\left(\frac{a_p}{r}\right)^n J_n\,(n+2)\begin{bmatrix}(n+1)P_n(\sin\varphi)\\ 0\\ -\cos\varphi P_n'\end{bmatrix}$$

$$-\frac{\mu}{r^3}\sum_{n=1}^{N}\left(\frac{a_p}{r}\right)^n (n+2)\sum_{m=1}^{n}\begin{bmatrix} -(n+1)P_n^m\left(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right) \\ m\sec\varphi P_n^m\left(-C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right) \\ \cos\varphi P_n^{m'}\left(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right) \end{bmatrix}$$

$$\frac{1}{r\cos\varphi}\frac{\partial\nabla U}{\partial\lambda} =$$

$$-\frac{\mu}{r^3}\sum_{n=1}^{N}\left(\frac{a_p}{r}\right)^n \sum_{m=1}^{n} m\begin{bmatrix} (n+1)P_n^m\left(C_{nm}\cos m\lambda - S_{nm}\sin m\lambda\right) \\ m\sec^2\varphi P_n^m\left(-C_{nm}\cos m\lambda - S_{nm}\sin m\lambda\right) \\ \cos\varphi P_n^{m'}\left(-C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right) \end{bmatrix}$$

$$\frac{1}{r}\frac{\partial\nabla U}{\partial\varphi} = -\frac{\mu}{r^3}\sum_{n=1}^{N}\left(\frac{a_p}{r}\right)^n J_n \begin{bmatrix} (n+1)\cos\varphi P_n' \\ 0 \\ \sin\varphi P_n' - \cos\varphi P_n'' \end{bmatrix}$$

$$-\frac{\mu}{r^3}\sum_{n=1}^{N}\left(\frac{a_p}{r}\right)^n \sum_{m=1}^{n}\begin{bmatrix} -(n+1)\cos\varphi P_n^{m''}\left(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right) \\ m\left(\sin\varphi\sec^2\varphi P_n^m + P_n^{m'}\right)\left(-C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right) \\ \left(\cos^2\varphi P_n^{m''} - \sin\varphi P_n^{m'}\right)\left(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right) \end{bmatrix}$$

Perturbations of gravity harmonics, J_n', C_{nm}', and S_{nm}' are confined in the right-hand term:

$$\nabla U' = -\frac{\mu}{r^2}\begin{bmatrix}1\\0\\0\end{bmatrix} + \frac{\mu}{r^2}\sum_{n=1}^{N}\left(\frac{a_p}{r}\right)^n J_n' \begin{bmatrix} (n+1)P_n \\ 0 \\ -\cos\varphi P_n' \end{bmatrix}$$

$$+\frac{\mu}{r^2}\sum_{n=1}^{N}\left(\frac{a_p}{r}\right)^n \sum_{m=1}^{n}\begin{bmatrix} -(n+1)P_n^m\left(C_{nm}'\cos m\lambda + S_{nm}'\sin m\lambda\right) \\ m\sec\varphi P_n^m\left(-C_{nm}'\cos m\lambda + S_{nm}'\sin m\lambda\right) \\ \cos\varphi P_n^{m'}\left(C_{nm}'\cos m\lambda + S_{nm}'\sin m\lambda\right) \end{bmatrix}$$

The conventional forward approach is based on a direct applicability of the linearized forward problem to evaluation of partials $\partial\mathbf{X}/\partial\mathbf{H}$:

$$\frac{d}{dt}\frac{\partial\mathbf{X}(t)}{\partial\mathbf{H}} + \begin{bmatrix}0 & -\mathrm{I}\\ 0 & 0\end{bmatrix}\frac{\partial\mathbf{X}(t)}{\partial\mathbf{H}} = \begin{bmatrix}\mathbf{0}\\ \partial\nabla U'/\partial\mathbf{H}\end{bmatrix}$$

where the partials wrt gravity harmonics are as follows:

$$-\frac{\partial \nabla U'}{\partial J_n} = \frac{\mu}{r^2}\left(\frac{a_p}{r}\right)^n \begin{bmatrix} -(n+1)P_n \\ 0 \\ \cos\varphi P_n' \end{bmatrix},$$

$$-\frac{\partial \nabla U'}{\partial C_{nm}} = \frac{\mu}{r^2}\left(\frac{a_p}{r}\right)^n \begin{bmatrix} (n+1)P_n^m \\ m\sec\varphi P_n^m \\ -\cos\varphi P_n^{m'} \end{bmatrix} \cos m\lambda,$$

$$-\frac{\partial \nabla U'}{\partial S_{nm}} = \frac{\mu}{r^2}\left(\frac{a_p}{r}\right)^n \begin{bmatrix} (n+1)P_n^m \\ -m\sec\varphi P_n^m \\ -\cos\varphi P_n^{m'} \end{bmatrix} \sin m\lambda,$$

Then the jacobian of sensitivities $\mathbf{K}$ at given instant, t is obtained simply as

$$\mathbf{K}(t) = \frac{\partial \mathbf{R}(t)}{\partial \mathbf{H}} = \left(\frac{\partial \mathbf{R}(t)}{\partial \mathbf{X}(t)}\right)^{\mathrm{T}} \frac{\partial \mathbf{X}(t)}{\partial \mathbf{H}} = \mathsf{W}(t)\, \frac{\partial \mathbf{X}(t)}{\partial \mathbf{H}}$$

It is important to point out that the specific expressions for $\nabla\nabla^{\mathrm{T}} U$, and $\partial \nabla U'/\partial \mathbf{H}$ displayed above, as well as the implementation computer code, are re-used *virtually unchanged* in the adjoint approach described below. Same refers to the specific implementation of the matrix W.

4. Straightforward adjoint approach

This approach is based on the solution of the adjoint problem which is uniquely defined by the linearized forward problem and by matrix W (see, e.g., [3]):

$$\left(-\frac{d}{dt} + \mathsf{C}^{\mathrm{T}}(t)\right) \mathsf{X}^*(t) = \mathsf{W}(t)$$

$$\mathsf{X}^*|_{t_0} = \mathsf{W}(t_0)$$

This is a *final* value problem where the adjoint matrix solution X^* is defined by its value at *final* instant of integration t_0.

Once the adjoint solution $\mathsf{X}^*(t)$ is obtained, the jacobian matrix of sensitivities is evaluated by doing the convolution of $\mathsf{X}^*(t)$ with the right-hand term of the linearized forward problem above:

$$\frac{\partial \mathbf{R}}{\partial \mathbf{H}} = \int_{t_0}^{t_1} (\mathsf{X}^*)^{\mathrm{T}} \begin{bmatrix} \mathbf{0} \\ \partial \nabla U' / \partial \mathbf{H} \end{bmatrix} dt$$

An advantage of the adjoint approach here, as compared to the forward approach, is in the number of elements of the solutions. The solution of the linearized forward problem, $\partial \mathbf{X}/\partial \mathbf{H}$ has $6 \times N_h$ elements, where N_h is a total number of gravity harmonics in the model ($\sim 10^4$ or more). The adjoint solution has up to $6^2 = 36$ elements.

A disadvantage of the adjoint approach in the form presented above is that one has to perform the integration over the time interval involved for each specific set of observables. The version of the adjoint approach presented below lifts this problem.

5. Using an adjoint propagator

We define the adjoint propagator as a matrix solution $\mathsf{P}(t)$ of the following adjoint problem [4]:

$$-\frac{d\mathsf{P}}{dt} + \mathsf{C}^{\mathrm{T}}(t)\,\mathsf{P}(t) = 0$$
$$\mathsf{P}|_{t_1} = \mathsf{I}$$

where I is a 6×6 identity matrix.

It can be shown that the jacobian of sensitivities $\mathbf{K}$ at given instant t can be obtained as follows:

$$\frac{\partial \mathbf{R}(t)}{\partial \mathbf{H}} = \mathsf{W}^{\mathrm{T}}(t)(\mathsf{P}^{-1}(t))^{\mathrm{T}} \int_{t_0}^{t} \mathsf{P}(t') \begin{bmatrix} \mathbf{0} \\ \partial \nabla U'/\partial \mathbf{H} \end{bmatrix} dt'$$

In practical applications, the integration needs to be done once, from t_0 to t_1, with saving of values at a reasonable grid of values of the intermediate upper limit t and interpolation to the instants corresponding to actual observations.

6. Results of numerical experiments

We have performed numerical experiments to verify the theoretical background illustrated above. We simulated the gravity field of the Moon using 18 order by 18 degree truncated gravity model. We have simulated the Doppler-only data (Is that correct? What units?) of tracking of an orbiter in a low Moon orbit (polar circular, 100 km)

The results of computation of partials using conventional and adjoint approach are presented in Appendix A. As it can be seen, the values are in good agreement.

7. Discussion and conclusion

As any method, the adjoint approach has its advantages and disadvantages. The obvious disadvantage of this approach is that it is profoundly counter-intuitive. Indeed, we have to run a non-physical, adjoint problem with the source term defined by the *procedure of obtaining observables.* The gravity potential, or rather, its partials with respect to gravity harmonics enter into play *after* the adjoint solution is obtained.

The advantages of this approach are due to the fact that it results into a much less computer-intensive code. The 6×6 matrix adjoint solution is independent on the number of gravity harmonics to be retrieved. Moreover, if the adjoint propagator is used, corresponding adjoint solution is independent on the nature of observables either. Also, the algorithm is suitable for parallel computations, which promises substantial additional benefits in terms of computer time and memory requirements.

References

[1] Yuan DN, Sjogren WL, Konopliv AS, Kucinskas AB. Gravity field of Mars: A 75th degree and order model. JGR-Planets, v.106, No.E10, 23377-23401, 2001.

[2] Moyer TD. Formulation for observed and computed values of Deep Space Network data types for navigation. JPL Publication 00-7, JPL, 2000.

[3] Ustinov EA. Adjoint sensitivity analysis of atmospheric dynamics: Application to the case of multiple observables. J. Atmos. Sci., v.58, No.21, 3340-3348, 2001.

[4] Ustinov EA, Sunseri RF. Adjoint sensitivity analysis of orbital mechanics: Applications to computation of observable partials with respect to gravity harmonics. (To be published)

Appendix A. Observables and partials computations

OBSERVABLE:

Forward 8.67498757077283 Adjoint 8.67498757077283

Jn PARTIALS

n	Forward	Adjoint	Fractional Difference
1	5.453591669962016	5.453591384639059	5.231835724561868E-08
2	-6.795751273662416	-6.795750829371058	6.537781335407339E-08
3	5.700247891622721	5.700246866763052	1.797921226808025E-07
4	-3.163396623737186	-3.163394644646909	6.256219226609346E-07
5	6.897422387826320E-01	6.897406250755110E-01	2.339579962185463E-06
6	6.625082512384907E-01	6.625061616031904E-01	3.154127207221041E-06
7	-7.195618649493446E-01	-7.195534231479559E-01	1.173186323499453E-05
8	-2.948538570282195E-04	-3.077367649468618E-04	4.186340205684169E-02
9	8.033990412677993E-01	8.034090317910492E-01	1.243516422466790E-05
10	-1.242379736384515	-1.242378343860642	1.120852048771251E-06
11	1.2213446037162	1.221328862228658	1.288865361541417E-05
12	-8.503012115194003E-01	-8.502772046438857E-01	2.823337799522630E-05
13	2.612635124574022E-01	2.612433220045656E-01	7.728003289370354E-05
14	4.598681541238140E-01	4.598740924975571E-01	1.291304259134080E-05
15	-1.228263732317993	-1.2282531037978	8.653288306873201E-06
16	1.907028541883789	1.907007830628617	1.086048515618874E-05
17	-2.330852208047	-2.330831759705185	8.772903637699720E-06
18	2.396828969106402	2.396816742466822	5.101173148975880E-06

Cnm PARTIALS

n	m	Forward	Adjoint	Fractional Difference
1	1	4.950813898145181	4.950814520157865	1.256384543378814E-07
2	1	-8.576906511167515	-8.576906541078914	3.487434477239394E-09
2	2	-2.579591614232964	-2.579591977559725	1.408466006570116E-07
3	1	1.074465385149275E+01	1.074465086710600E+01	2.777555039083374E-07
3	2	4.425529346381476	4.425532182742316	6.409084201424583E-07
3	3	2.972939776534321	2.972936647832846	1.052393156405409E-06
4	1	-1.154362648116282E+01	-1.154361997037809E+01	5.640155401945730E-07
4	2	-5.03532514251934	-5.035331270198357	1.216936620197765E-06
4	3	-6.222597122479237	-6.222594819792393	3.700523750044245E-07
4	4	-2.052358190555029	-2.052356147514796	9.954598777332886E-07
5	1	1.127385019226109E+01	1.127384359162877E+01	5.854816417874586E-07
5	2	4.339539448862144	4.339547038816411	1.749019932027833E-06
5	3	8.883515225805619	8.883518363483045	3.532021095629761E-07
5	4	4.864404657344388	4.864404024608694	1.300746419412945E-07
5	5	-2.567852349625732E-01	-2.567890974982640E-01	1.504166543036839E-05
6	1	-1.019099977758261E+01	-1.019099848269281E+01	1.270620970224022E-07
6	2	-2.973118242851088	-2.973123947632453	1.918783564360280E-06
6	3	-1.002956307526577E+01	-1.002957275899568E+01	9.655176888474106E-07
6	4	-7.834467964277665	-7.834473228637693	6.719481800841643E-07
6	5	6.587772571491534E-01	6.587844782148322E-01	1.096119583512429E-05
6	6	7.645765574441028E-01	7.645760594915011E-01	6.512789293102726E-07
7	1	8.591645166506604	8.591652056458113	8.019355839372730E-07
7	2	1.743790882154454	1.743792417468301	8.804453049247476E-07
7	3	9.639877658749899	9.639889314477786	1.209114286131471E-06
7	4	9.866122247662714	9.866134337658645	1.225403538695153E-06
7	5	-1.358972724783543	-1.358979959873882	5.323912458055114E-06
7	6	-1.73443975797942	-1.734440048877337	1.677186350259258E-07
7	7	-1.210398730034779	-1.210395536251082	2.638621156843422E-06
8	1	-6.938181432691885	-6.938194264697298	1.849473353348136E-06
8	2	-1.141039199335807	-1.141036854490062	2.055008930887717E-06

8	3	-8.496030233511517	-8.496036854242988	7.792729226818960E-07
8	4	-1.025786507956218E+01	-1.025787779464057E+01	1.239542783004764E-06
8	5	2.492606708148765	2.49260885367499	8.607552775112719E-07
8	6	2.662300554377722	2.66230293854403	8.955278055929132E-07
8	7	3.123677778045532	3.123673434807209	1.390424567108079E-06
8	8	9.243457207633073E-01	9.243413240224473E-01	4.756597841364857E-06
9	1	5.776736482841052	5.776749639167943	2.277461844949090E-06
9	2	1.168473363268854	1.168469224802063	3.541772471117047E-06
9	3	7.578645132538051	7.578643237525068	2.500464068082192E-07
9	4	9.061418867491499	9.061421540889574	2.950307590586492E-07
9	5	-4.057090309742451	-4.057084360223166	1.466449802939857E-06
9	6	-3.156483306178684	-3.156488036076551	1.498468491909691E-06
9	7	-5.557868697596962	-5.557867537213083	2.087821684711394E-07
9	8	-2.491123257621768	-2.491116093752191	2.875758778950403E-06
9	9	-4.019585700799544E-01	-4.019539846907908E-01	1.140761636885875E-05
10	1	-5.469525971351578	-5.469534652289701	1.587143820246713E-06
10	2	-1.499008558779082	-1.499004538302151	2.682090710814402E-06
10	3	-7.443242512783205	-7.443235218756921	9.799527923176164E-07
10	4	-6.940067213872605	-6.940052867934546	2.067118028791954E-06
10	5	5.786805043553446	5.786792702553249	2.132610327266081E-06
10	6	3.010093173211545	3.010099247569238	2.017992495973615E-06
10	7	7.868109042998946	7.868115192270349	7.815431335247134E-07
10	8	4.597023658998713	4.597019243776396	9.604523807247047E-07
10	9	1.107969361381834	1.107959657023352	8.758688480164325E-06
10	10	8.918559331873081E-03	8.916701723637075E-03	2.082856845911424E-04
11	1	5.967927163043376	5.967929941701132	4.655982532426338E-07
11	2	1.760842793134552	1.76083946055898	1.892602556536881E-06
11	3	7.979687348889854	7.979681547934634	7.269652263992546E-07
11	4	4.696180696787921	4.696153182083067	5.858953611578380E-06
11	5	-7.196581574854176	-7.196567954740355	1.892581037136939E-06
11	6	-2.309595298110094	-2.309601652142653	2.751137865408330E-06
11	7	-9.416680923777468	-9.416694167256633	1.406383060726373E-06
11	8	-6.677962596919747	-6.677968473115269	8.799375955335051E-07
11	9	-2.057208099770621	-2.057195808208998	5.974875183779666E-06
11	10	-4.326911320913177E-02	-4.326481713959710E-02	9.928721011451044E-05
11	11	2.592908089893002E-01	2.592902829402329E-01	2.028799513956791E-06
12	1	-6.827221750812288	-6.827219546468702	3.228756390731480E-07
12	2	-1.749348052132318	-1.749345139786977	1.664817551636281E-06
12	3	-8.63200758808588	-8.632007753691974	1.918511879237316E-08
12	4	-2.834449216470458	-2.83442265644936	9.370434631225623E-06
12	5	7.806196400627451	7.806185892472347	1.346129992762968E-06
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12	8	8.062403470600451	8.06242334465003	2.465021833982364E-06
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12	10	1.097130218284953E-01	1.097070938890958E-01	5.403132008120395E-05
12	11	-7.731760814131280E-01	-7.731756416910268E-01	5.687218109123350E-07
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13	3	8.842236495477568	8.842240785081911	4.851263890123144E-07
13	4	1.433087121321372	1.433077091308468	6.998885660893605E-06
13	5	-7.40595944860476	-7.405952715198775	9.091875308675734E-07
13	6	-5.452208467306151E-01	-5.452294921515386E-01	1.585648070755719E-05
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13	10	-1.898285936727079E-01	-1.898239506927215E-01	2.445880199903871E-05
13	11	1.603235329564401	1.603236457492719	7.035321036789506E-07
13	12	9.263728367332835E-01	9.263674888968871E-01	5.772876950109879E-06
13	13	2.250151713517504E-01	2.250120830168061E-01	1.372500763295952E-05
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14	3	-8.390391083875315	-8.390393263066908	2.597246070683346E-07
14	4	-3.251741680194232E-01	-3.251870937393617E-01	3.974856378771035E-05

14	5	6.180938871176646	6.180933879947154	8.075196334994713E-07
14	6	1.010951355017927E-01	1.011063591463530E-01	1.110082951754092E-04
14	7	8.671584691410347	8.67157953637572	5.944743446755666E-07
14	8	7.493508327601626	7.493534438371227	3.484439794789274E-06
14	9	2.897937840727126	2.897947054396388	3.179378052553218E-06
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14	12	-1.947097844811247	-1.947091235438864	3.394473678354551E-06
14	13	-7.047782794782257E-01	-7.047707695699617E-01	1.065570333636851E-05
14	14	-1.021407110054937E-01	-1.021384153872966E-01	2.247505597376366E-05
15	1	6.825417027726465	6.825401702425917	2.245328085645386E-06
15	2	7.210714297727951E-01	7.210738497970737E-01	3.356139290320485E-06
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15	4	-6.247784957919841E-01	-6.247495707819769E-01	4.629642377577145E-05
15	5	-4.611167286454727	-4.61116323982686	8.775712502779085E-07
15	6	-1.098210555062768E-01	-1.098332895827697E-01	1.113876907388976E-04
15	7	-7.885737744348556	-7.885722613030774	1.918820822148448E-06
15	8	-6.016831560445777	-6.016842200118544	1.768315075223306E-06
15	9	-1.724313573161651	-1.724326447679491	7.466403972940532E-06
15	10	-2.338656435861526E-01	-2.338728386277782E-01	3.076475946425981E-05
15	11	3.799609727153138	3.799618608237767	2.337362126237483E-06
15	12	3.246064543775929	3.246060968781176	1.101331999073822E-06
15	13	1.516445630423261	1.516434203471989	7.535351774470329E-06
15	14	3.286155455657769E-01	3.286096096211479E-01	1.806349306688670E-05
15	15	-6.571935893364874E-03	-6.572898283982428E-03	1.464179994842417E-04
16	1	-5.759277220422489	-5.759262214752089	2.605478053263245E-06
16	2	-5.026730445422214E-01	-5.026785908959095E-01	1.103359838384332E-05
16	3	-6.374268850615655	-6.374259008266657	1.544074972156878E-06
16	4	1.379573304435349	1.3795436116092	2.152319565327470E-05
16	5	3.221555982057809	3.221555936734264	1.406883666451203E-08
16	6	4.871764325569908E-01	4.871857778104660E-01	1.918211471028386E-05
16	7	7.390924377986273	7.3909076588894	2.262111749118129E-06
16	8	4.471291460545055	4.471282020181464	2.111328164267023E-06
16	9	1.280331406693907E-01	1.280412786475192E-01	6.355745752074656E-05
16	10	1.651262773797349E-01	1.651380836777605E-01	7.149349055435063E-05
16	11	-4.784850974481675	-4.784863028932891	2.519288669914113E-06
16	12	-4.537463268662006	-4.537467797121584	9.980147034570955E-07
16	13	-2.577405722778473	-2.577394290788003	4.435464066967135E-06
16	14	-7.242576372031894E-01	-7.242478820821718E-01	1.346913103353680E-05
16	15	2.409830494924649E-02	2.410111628570479E-02	1.166475620869101E-04
16	16	6.448109330478075E-02	6.448067872458743E-02	6.429484552233204E-06
17	1	4.643881028809711	4.643873745085179	1.568456316338048E-06
17	2	4.444902322512226E-01	4.444972175844122E-01	1.571513366859465E-05
17	3	5.485993833598603	5.485985727127199	1.477666882292169E-06
17	4	-1.797202420172616	-1.797185430158812	9.453589430886564E-06
17	5	-2.345588332354886	-2.345597351025565	3.844935566288647E-06
17	6	-1.058114418249168	-1.058116693631535	2.150407777637740E-06
17	7	-7.14719857849282	-7.147189264178986	1.303211843312322E-06
17	8	-3.263991629572193	-3.263968515940074	7.081400549276895E-06
17	9	1.383423755543625	1.383424972476558	8.796522814921830E-07
17	10	-8.811058378434761E-02	-8.812110409653098E-02	1.193847068898731E-04
17	11	5.480988179928969	5.48100020612605	2.194161034222989E-06
17	12	5.473640938178221	5.473655980661957	2.748160240450785E-06
17	13	3.651416643052402	3.65141129875791	1.463622208658850E-06
17	14	1.256673114583259	1.256662161807047	8.715692318604452E-06
17	15	-6.612932196415727E-02	-6.613476054790290E-02	8.223487467974388E-05
17	16	-2.186287334207635E-01	-2.186278078988088E-01	4.233304288202519E-06
17	17	-7.533845241318662E-02	-7.533721699058349E-02	1.639830078217816E-05
18	1	-3.755028417513008	-3.755034923309345	1.732552817771070E-06
18	2	-5.102970514583611E-01	-5.103026929994504E-01	1.105528378877522E-05
18	3	-4.895621427565487	-4.895622004102553	1.177658458118372E-07
18	4	1.804087347200244	1.804087493839908	8.128190260606065E-08
18	5	2.033364295095452	2.033384260318922	9.818716441945627E-06
18	6	1.628967298758667	1.628960650813059	4.081079842947745E-06
18	7	6.950548624084582	6.950550675205772	2.951019690489071E-07
18	8	2.477866972247128	2.47784323539913	9.579548968584323E-06
18	9	-2.382146528737093	-2.382153870813165	3.082116635035478E-06

18	10	7.318485732359437E-02	7.318763493366914E-02	3.795190372383558E-05
18	11	-5.850961366756536	-5.850969477974003	1.386303158324842E-06
18	12	-5.798351688577432	-5.798374461462191	3.927460171919871E-06
18	13	-4.424877806151954	-4.424883169174642	1.212014528346813E-06
18	14	-1.81000028492803	-1.809993325926291	3.844751736675329E-06
18	15	1.539907632937860E-01	1.539986723279716E-01	5.135780761007496E-05
18	16	5.143487047446789E-01	5.143475755830779E-01	2.195323115588431E-06
18	17	2.602492562965217E-01	2.602458816792638E-01	1.296686609573603E-05
18	18	5.484637297372189E-02	5.484500431050113E-02	2.495448917664072E-05

Snm PARTIALS

n	m	Forward	Adjoint	Fractional Difference
1	1	-2.614321395671419	-2.614320909348877	1.860224768668231E-07
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2	2	2.90075989108488	2.900759544661805	1.194249397130685E-07
3	1	-6.52393543437875	-6.523940889830651	8.362203143632298E-07
3	2	-6.513437191791092	-6.513437829984126	9.798098180855889E-08
3	3	1.859040624877813	1.859043130670689	1.347893889823964E-06
4	1	7.113596125481384	7.113604469479362	1.172963441204576E-06
4	2	9.658590621281753	9.658593104917694	2.571426204091748E-07
4	3	-3.950132285536507	-3.950136335854355	1.025361532719172E-06
4	4	-1.637706812687912	-1.637705980236446	5.083031098927909E-07
5	1	-7.251310803780889	-7.251317101951207	8.685553574846883E-07
5	2	-1.093302942987641E+01	-1.093303159753712E+01	1.982671227326757E-07
5	3	6.132887116374235	6.132889192686745	3.385537297566649E-07
5	4	3.308160675647037	3.308161572961831	2.712427352139334E-07
5	5	2.096520753890751	2.096516871758094	1.851702469395357E-06
6	1	7.615726232977821	7.615725650603257	7.646999731305468E-08
6	2	9.900075158248439	9.900071942451913	3.248254660133600E-07
6	3	-8.099677477364672	-8.09967350664258l	4.902321236594168E-07
6	4	-4.474724826194735	-4.474729791772442	1.109693308491808E-06
6	5	-4.926471537887344	-4.926467109771594	8.988412326303236E-07
6	6	-1.421566152383407	-1.421562256170671	2.740788903468405E-06
7	1	-8.499064422117817	-8.499056557840071	9.253109937470474E-07
7	2	-7.216964609159797	-7.216952581096275	1.666637454035411E-06
7	3	9.595314520579146	9.595303734749287	1.124072570663486E-06
7	4	4.60861122252238	4.608620347484118	1.979976880325470E-06
7	5	7.94237945450701	7.942379685762989	2.911671172225880E-08
7	6	3.58481481289295	3.584810235686839	1.276831956564312E-06
7	7	3.083522654138168E-01	3.083477006073666E-01	1.480386869881761E-05
8	1	9.601804038572746	9.601793434485716	1.104384862270303E-06
8	2	4.081282645997392	4.081265372834983	4.232287716381719E-06
8	3	-1.034473496063483E+01	-1.034472138415190E+01	1.312405100736229E-06
8	4	-3.74284087722462	-3.742851650872857	2.878459859379684E-06
8	5	-1.013872002564003E+01	-1.013872811799981E+01	7.981632104891317E-07
8	6	-6.173846708629177	-6.173847874879942	1.889017657917246E-07
8	7	-7.699547444075510E-01	-7.699456778875257E-01	1.177539341260756E-05
8	8	2.398376334200567E-01	2.398383874338541E-01	3.143841173727162E-06
9	1	-1.029165627036484E+01	-1.029164856487672E+01	7.487121522125425E-07
9	2	-1.497773308129428	-1.497761581845305	7.829144811360844E-06
9	3	1.012270738988267E+01	1.012269722819125E+01	1.003851147198809E-06
9	4	2.396965843280928	2.396975125257259	3.872370736287420E-06
9	5	1.090741002392816E+01	1.090742320209921E+01	1.208183712443452E-06
9	6	8.349509797214187	8.349522359477966	1.504548791858523E-06
9	7	1.220065422117479	1.220054963607023	8.572089878376789E-06
9	8	-5.857826199247397E-01	-5.857846355489871E-01	3.440896406422244E-06
9	9	-6.063772113339279E-01	-6.063753487318686E-01	3.071688751556995E-06
10	1	1.008539738558511E+01	1.008539488750684E+01	2.476925970248403E-07
10	2	-1.313812805803020E-01	-1.313763632467934E-01	3.742796147928216E-05
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10	5	-1.035372738197987E+01	-1.035373785907336E+01	1.011914115682167E-06
10	6	-9.366266616219793	-9.366289240936482	2.415547513703127E-06
10	7	-1.306939203550094	-1.306932972609276	4.767582762282833E-06
10	8	9.792291882556585E-01	9.792322459867112E-01	3.122580026569973E-06
10	9	1.69082801137417	1.690825308959642	1.598278778325215E-06
10	10	5.539608995333228E-01	5.539572044901946E-01	6.670223713193146E-06
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11	3	7.13414513763338	7.134149811307052	6.551129139140855E-07
11	4	6.485836204539238E-01	6.485873148802482E-01	5.696112519716581E-06
11	5	9.144672061934997	9.144672852532773	8.645446244384804E-08
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11	7	7.176657627206058E-01	7.176678847241943E-01	2.956804440715876E-06
11	8	-1.286777121323069	-1.286779792318182	2.075720437540933E-06
11	9	-3.263794514573736	-3.263793754675815	2.328265206767756E-07
11	10	-1.592191192854238	-1.592184043670856	4.490153829683250E-06
11	11	-3.294935599466829E-01	-3.294895485527884E-01	1.217442275716459E-05
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12	4	-7.262871640009606E-01	-7.262891104298956E-01	2.679964365536128E-06
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12	6	-7.54490829377741	-7.544917199508212	1.180361635019788E-06
12	7	5.978895502923423E-01	5.978800077546130E-01	1.596036880818145E-05
12	8	1.396683514582143	1.396683996263655	3.448750855346081E-07
12	9	5.035519634175555	5.035524239787712	9.146241658486949E-07
12	10	3.142990999487768	3.142983818169685	2.284867530192385E-06
12	11	9.735755328088560E-01	9.735664271809176E-01	9.352769899796210E-06
12	12	9.467161723558810E-02	9.466925767631923E-02	2.492361848010843E-05
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13	2	1.258464762482544	1.258441423242577	1.854580331793809E-05
13	3	4.02638080769029	4.026392971843998	3.021104445710859E-06
13	4	1.212979345555499	1.212979126775668	1.803656693628837E-07
13	5	7.508714491234207	7.508702397977973	1.610562799897809E-06
13	6	5.725356121567664	5.725344031084059	2.111743505325849E-06
13	7	-2.334262652313323	-2.334251268268656	4.876933902728582E-06
13	8	-1.269251743587839	-1.269249651664684	1.648154643292338E-06
13	9	-6.596871592038764	-6.596882814529966	1.701180924025790E-06
13	10	-4.901372183975781	-4.901371492875773	1.410013321786326E-07
13	11	-1.964049484054099	-1.964036817547506	6.449178952176554E-06
13	12	-2.927396779037407E-01	-2.927339482586579E-01	1.957249226959552E-05
13	13	8.362609276254018E-02	8.362654104493360E-02	5.360527744098078E-06
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14	6	-4.1072356085699	-4.107207329493458	6.885184863344950E-06
14	7	3.94953802591898	3.949530463707567	1.914707837585802E-06
14	8	9.442540815699111E-01	9.442516824947610E-01	2.540709324873602E-06
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14	11	3.102564791330761	3.102553841935927	3.529143006123426E-06
14	12	6.149839028625056E-01	6.149752896946900E-01	1.400551750305991E-05
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15	3	3.757094581266781	3.757109537343079	3.980740020809678E-06
15	4	2.130642296442559	2.130633931605123	3.925969858746443E-06
15	5	7.622896585680013	7.622901508110099	6.457423175957659E-07
15	6	2.915415515018239	2.915385596766087	1.026208854195485E-05
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15	15	1.367562488756922E-01	1.367541531000190E-01	1.532489879224906E-05
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16	4	-2.192013491031133	-2.192002255348299	5.125736168790393E-06
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16	7	5.040529765696295	5.040526819724111	5.844568569894246E-07
16	8	1.004431637826468E-01	1.004518587039420E-01	8.655809267665513E-05
16	9	7.838300472241629	7.838304035375698	4.545797219148082E-07
16	10	6.943530959304708	6.943559974265374	4.178686548922028E-06
16	11	4.391489247787602	4.391497536508375	1.887447437678290E-06
16	12	1.35149006807375	1.351487271771137	2.069051545373929E-06
16	13	-1.069047075137947	-1.069053068715603	5.606436042034185E-06
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16	15	-4.509223888988113E-01	-4.509169630637463E-01	1.203274709471688E-05
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17	3	4.701937474327742	4.701938450516575	2.076141240794836E-07
17	4	1.984492338468175	1.984482137072265	5.140556963522872E-06
17	5	7.148819205353959	7.148829467615298	1.435516315639963E-06
17	6	1.193817665541801	1.193820465522479	2.345395106588007E-06
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18	13	-2.317580677736468	-2.317589602594676	3.850922612887982E-06
18	14	-2.857142950444923	-2.8571444138728	5.121994777555638E-07
18	15	-1.850881223564832	-1.850871139689297	5.448148377575382E-06
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18	18	2.124860030312060E-02	2.124871611963790E-02	5.450518358288069E-06