ESTIMATING THE INSTABILITY OF A COMPOSITE CLOCK

C. A. Greenhall*
Jet Propulsion Laboratory, California Institute of Technology
4800 Oak Grove Dr., MS 298-100, Pasadena, CA 91109 USA
E-mail: cgreen@mail.jpl.nasa.gov

November 9, 2004

Abstract

A composite clock created from a local clock ensemble is known by its time offsets from the ensemble clocks. By a geometrical argument, estimates for the instability of the composite clock are calculated from the instabilities of the ensemble clocks, individually and against the composite clock. The method is illustrated by examples using simulated and real ensembles.

1 INTRODUCTION

Suppose that we have a local ensemble of uncorrelated clocks $A_1, ..., A_n$, called the base clocks. Their instabilities are assumed to be known, where instability is defined by any of the deviations (Allan, modified Allan, Hadamard) customarily used for clocks. In addition, there is another clock X whose offset from each A_i is known as a function of time. In particular, X could be a composite clock that is formed from some timescale algorithm operating on measurements of the differences $A_i - A_j$ of the base clocks. Having estimated the instabilities of the base clocks A_i and of the offsets $X - A_i$, how much can we find out about the instability of X itself? There has to be some information about it, but is it good enough to be useful?

By a geometrical argument, we will show how to compute the range of possible instabilities of X without using any outside comparisons and without knowing anything about the algorithm that produces X. For one or two clocks, the geometry can be shown by a simple diagram; for any number of clocks, the argument is carried out by algebra, and the results are given by simple formulas. There is a minimum instability, a maximum instability, and an intermediate "mid" estimate.

After developing the formulas we show three examples, two from simulations and one from real clocks, and discuss the limitations of the method that are inherent in our imperfect knowledge of the required data.

2 A GEOMETRICAL ARGUMENT

Any RMS instability measure, for a single averaging time, can be regarded as a norm generated by an inner product on a vector space of random phase residuals. To be specific, consider Allan

*This work was performed by the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautics and Space Administration.

deviation $\sigma_y(\tau)$, where τ is fixed in this discussion. If A and B are two clocks with phase deviations $x_A(t)$ and $x_B(t)$, define the inner product

$$\langle A, B\rangle_\tau = \frac{1}{2\tau^2}\,\mathrm{E}\left[\Delta_\tau^2 x_A(t)\right]\left[\Delta_\tau^2 x_B(t)\right], \tag{1}$$

where it is assumed that the expectation does not depend on t. Then $\sigma_y(\tau)$ for clock A is the norm $\|A\|_\tau$ generated by this inner product:

$$\sigma_y(\tau) = \|A\|_\tau = \sqrt{\langle A, A\rangle_\tau}. \tag{2}$$

Each τ determines a different norm[1], or metric. The instability of A and B relative to each other is $\|A - B\|_\tau$, the distance between the two points in this metric space. The clocks are orthogonal in this inner product, $\langle A, B\rangle_\tau = 0$, if the clocks are uncorrelated and $\left[\mathrm{E}\,\Delta_\tau^2 x_A(t)\right]\left[\mathrm{E}\,\Delta_\tau^2 x_B(t)\right] = 0$.

We may now use the algebraic machinery of inner product spaces to find our way about. Because we are dealing with a finite number of "vectors", this algebra is the same as the algebra of Euclidean spaces and can be interpreted geometrically. In the following arguments, the base subspace is defined as the space of linear combinations of the base clocks, which are assumed to be orthogonal.

2.1 One base clock

Silly as it may seem, we can begin with just one base clock A and the other clock X. Let $a = \|A\|_\tau$, $x = \|X\|_\tau$, $d = \|X - A\|_\tau$. We know a and d and want to estimate x. In Figure 1, the point X is somewhere on a circle with radius d and center at a on the baseline, and x is the length of OX. The maximum and minimum of x are $a + d$ and $|a - d|$ (whether or not $a \geq d$). The intermediate estimate x_{mid} comes from the assumption that clock X equals clock A plus something orthogonal to A. Thus for one base clock we express the results as

$$x_{\max} = a + d, \quad x_{\min} = |a - d|, \quad x_{\text{mid}} = \sqrt{a^2 + d^2}. \tag{3}$$

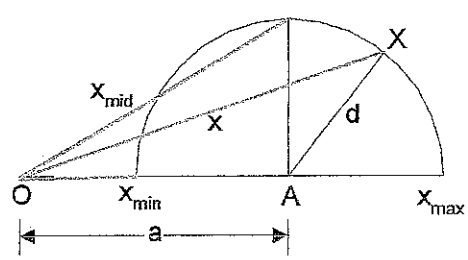

Figure 1: One base clock.

2.2 Two base clocks

For two base clocks, three dimensions are needed. Figure 2 (which lacks the third dimension) shows the geometry. The orthogonal base clocks A_1 and A_2 have norms a_1 and a_2. We also know the

[1] Strictly speaking, this is a seminorm because $\|A\| = 0$ if the clock has constant frequency. Or we can identify clocks that differ only by a constant frequency.

distances d_1 and d_2 of X from A_1 and A_2. The possible values of $x = \|X\|_\tau$ are realized by rotating X out of the plane about the axis A_1A_2, keeping the distances d_1 and d_2 fixed. The picture shows $x_{\min}$ and $x_{\max}$, which are the values of x when X is in the plane, that is, in the base subspace. (In the picture, X is at the max position.) When X is directly above or below a point on A_1A_2, then X is as far from the plane as it can be, and X is a weighted average of the base clocks plus a vector $X_\perp$ orthogonal to the base subspace. We define x_{mid} to be the length of either of these two X's. (In an extreme case, one of the two weights can be negative, though their sum is 1.) The author feels intuitively that this solution has special value for estimating the norm of a composite clock, simply because timescale algorithms express the composite clock phase or its increment as a weighted average of the detrended base clocks. The author has no further argument to support this intuition, however.

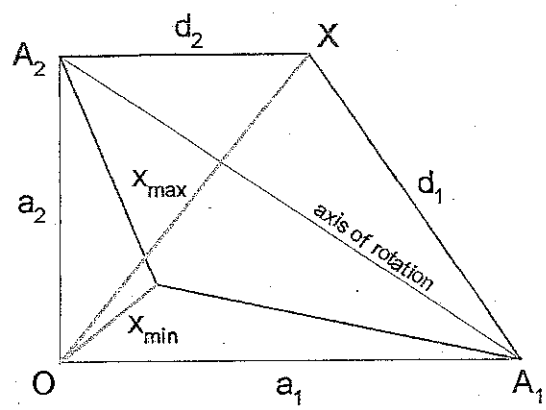

Figure 2: Two base clocks. The point X is rotated out of the plane.

2.3 Any number of base clocks

For n base clocks the argument can be carried out algebraically. The clocks A_i are assumed to be orthogonal. Given $a_i = \|A_i\|_\tau > 0$ and $d_i = \|X - A_i\|_\tau$ for all i, we wish to estimate $x = \|X\|_\tau$. We can represent X by the orthogonal expansion

$$X = \left(\sum_{i=1}^{n} x_i \frac{A_i}{a_i}\right) + X_\perp, \tag{4}$$

where $X_\perp$ is a vector that is orthogonal to the base subspace. Let $x_\perp = \|X_\perp\|_\tau$. Then

$$x^2 = \left(\sum_{i=1}^{n} x_i^2\right) + x_\perp^2. \tag{5}$$

where

$$x_i = \frac{\langle X, A_i\rangle_\tau}{a_i}. \tag{6}$$

Now

$$d_i^2 = \|X - A_i\|_\tau^2 = x^2 - 2\langle X, A_i\rangle_\tau + a_i^2 = x^2 - 2a_i x_i + a_i^2,$$

$$2x_i = \frac{x^2 + a_i^2 - d_i^2}{a_i}. \tag{7}$$

Square both sides of (7) and sum over i. By (5),

$$4\left(x^2 - x_\perp^2\right) = \sum_{i=1}^{n} \frac{\left(x^2 + a_i^2 - d_i^2\right)^2}{a_i^2}.$$

Expand the terms in this sum and collect powers of x. Setting

$$\varepsilon_i = 1 - \frac{d_i^2}{a_i^2} \tag{8}$$

we get

$$x^4 \sum a_i^{-2} - 2x^2\left(2 - \sum \varepsilon_i\right) + \sum a_i^2 \varepsilon_i^2 + 4x_\perp^2 = 0.$$

To simplify this equation further, set

$$y = x^2 \sum a_i^{-2}, \quad y_\perp = x_\perp^2 \sum a_i^{-2}. \tag{9}$$

Then

$$y^2 - 2By + C + 4y_\perp = 0, \tag{10}$$

where

$$B = 2 - \sum \varepsilon_i, \quad C = \left(\sum a_i^{-2}\right)\left(\sum a_i^2 \varepsilon_i^2\right). \tag{11}$$

The solutions of (10) are

$$y = B \pm \sqrt{B^2 - C - 4y_\perp}, \tag{12}$$

with corresponding solutions for x according to (9).

A necessary and sufficient condition for meaningful solutions to exist is

$$B \geq \sqrt{C}. \tag{13}$$

If (13) holds and $0 \leq y_\perp \leq \frac{1}{4}\left(B^2 - C\right)$ then there are nonnegative y solutions corresponding to the points X that are the same distance $x_\perp$ above and below the base subspace. When $y_\perp = 0$ we have the extremal solutions

$$y_{\min} = B - \sqrt{B^2 - C}, \quad y_{\max} = B + \sqrt{B^2 - C}, \tag{14}$$

for which X is in the base subspace. When $y_\perp$ takes its maximum value $\frac{1}{4}\left(B^2 - C\right)$, we have the intermediate estimate

$$y_{\text{mid}} = B. \tag{15}$$

For this special solution, $\sum \frac{x_i}{a_i} = 1$. In fact, if $y = B$ then $x^2 = B / \sum a_j^{-2}$. By (7) and (8),

$$\frac{x_i}{a_i} = \frac{B}{2a_i^2 \sum a_j^{-2}} + \frac{\varepsilon_i}{2},$$

and $\sum \frac{x_i}{a_i} = 1$ follows from (11). Then (4) says that X is a weighted average of the A_i plus a vector orthogonal to the base subspace. (Again, some of the weights can be negative.)

If $B < \sqrt{C}$ there are no meaningful solutions; there is no point X at the given distances d_i from orthogonal vectors A_i having given lengths a_i. In Figure 2, the two pivoting rods fastened to A_1 and A_2 are too short or too mismatched for their free ends to meet at a common point X.

The reader is reminded that this calculation has to be carried out anew for each averaging time τ.

3 EXAMPLES

The first example (Figure 3) is from an eight-clock simulation [1]. The duration of the simulated run is 1.44×10^8 s. The clocks are all simulated with white FM plus random walk FM. The odd-numbered clocks are statistically identical to clock 1, the even-numbered clocks to clock 2. The dotted curves show the measured Allan deviation (Adev) of clocks 1 and 2. The black curve with the asterisks is the measured Adev of the actual composite clock (CC) phase, which was produced with a modified Kalman filter algorithm. Not shown is the Adev of the base clocks against the CC, d_i in the previous section. The blue curves with the plus signs, labeled "True", show the min, mid, and max CC estimates using the measured Adev of the true simulated clocks for a_i. These curves end when the condition (13) is violated. Because stability measurements of the individual base clocks would not be directly available in reality, Barnes's n-cornered hat formulas [2][3] were also used to generate values for a_i from the measured Adevs of all base clock pairs. The resulting CC estimates are shown by the green curves with the open circles, labeled "NCH". These curves end when some of the variance estimates from the n-cornered hat are negative. The NCH mid and max estimates agree well with the True estimates where they both exist. The mid estimate is at most 1.25 times the actual CC Adev; the max estimates are less than 1.8 times the actual CC Adev. The min estimates are too low and variable to be useful.

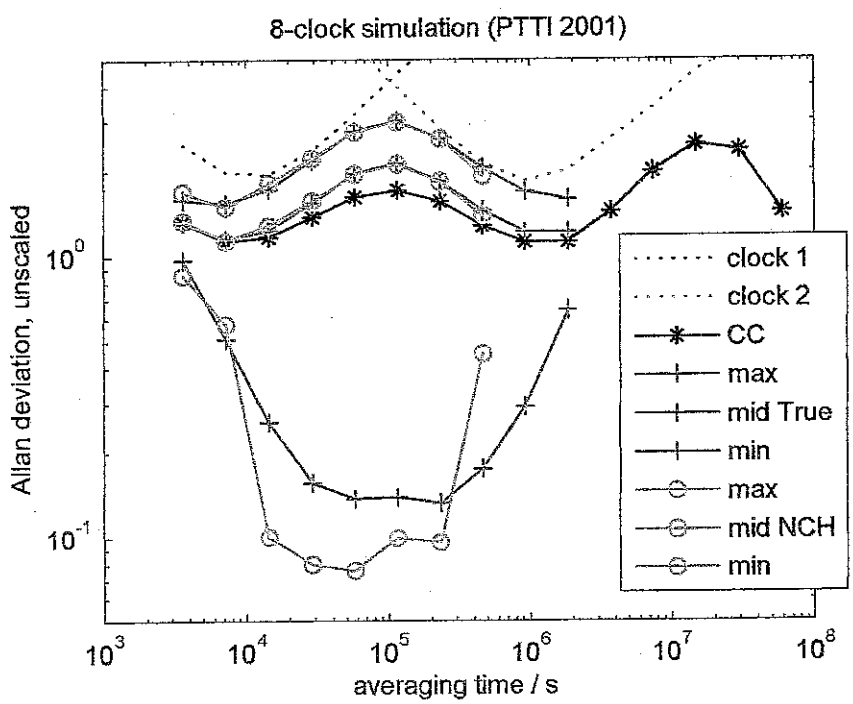

Figure 3: An eight-clock simulation

The second example (Figure 4) is from a four-clock simulation [4], in which the CC phase was also generated by a modified Kalman algorithm. The simulated duration is 10^6 s. The ratio of mid estimate to actual CC Adev is between 0.95 and 1.12.

The third example (Figure 5) uses actual data from a comparison of three hydrogen masers at National Physical Laboratory, UK, between MJD 53005 and MJD 53160 [5]. To eliminate the effect of linear frequency drift, Hadamard deviation (Hdev) is used for the instability measure. The

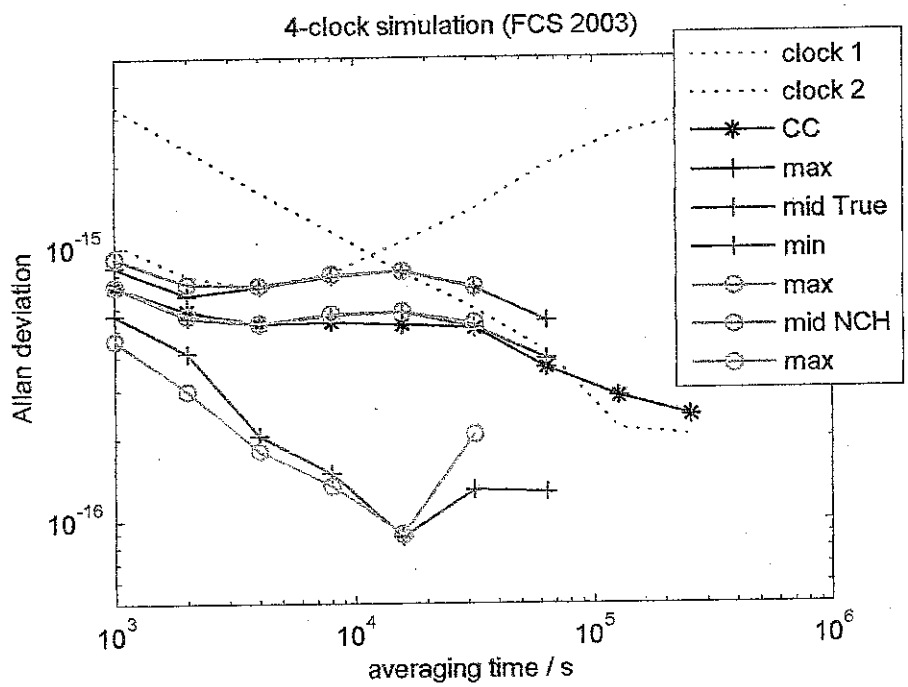

Figure 4: A four-clock simulation

dotted curves are the three-cornered hat Hdev estimates of the base clocks. The CC phase (relative to the base clocks) is generated by a novel Kalman algorithm, in which the frequency state of each clock is represented as the sum of independent stationary Markov processes. Here we do not know the actual CC Hdev, but the black curve shows a theoretical computation of its Hdev, obtained from an auxiliary Kalman filter [5]. The mid estimate agrees remarkably well with the theoretical CC Hdev. Furthermore, the min and max estimates are relatively tight. Geometrically, the point X has to stay close to the plane of weighted averages of the three clocks. For two clocks (Figure 2), X would be close to the line A_1A_2; this would happen if $d_1 + d_2$ were only slightly greater than $\sqrt{a_1^2 + a_2^2}$.

4 DISCUSSION

We have shown a method for calculating estimates for the instability of a composite clock from knowledge of the instability of the base clocks and of their offsets from the composite clock. Like the n-cornered hat, this method needs a long data interval for a given averaging time to give useful results. The theory behind the calculation assumes that we know the exact expected values of the required instabilities, and that the base clocks are uncorrelated stochastic processes. Of course, the values have to be estimated from finite data; moreover, the base clock instabilities a_i themselves often come from an n-cornered hat calculation. The base clocks may also be physically correlated. (In any case, their finite samples almost always have nonzero correlations.) Nevertheless, in the examples that were tried, the method gives practical max and mid estimates whenever they can be calculated at all, and the mid estimate tends to be slightly conservative.

The author's experience with this method is limited to a few examples in which the composite clock was generated by a modified Kalman algorithm. Users of other timescale algorithms should

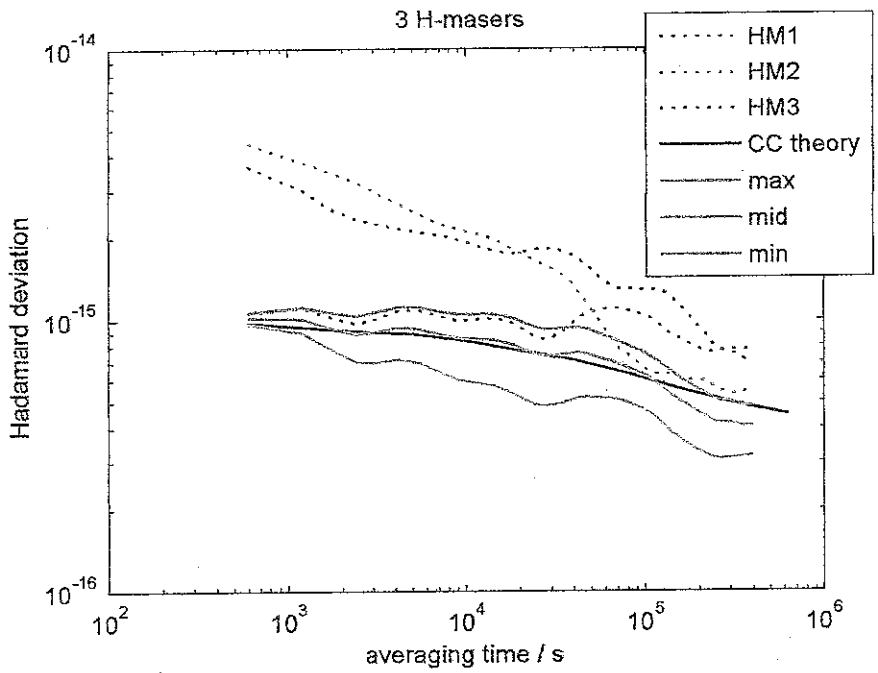

Figure 5: An ensemble of three hydrogen masers

try the method on simulated data before trusting it to give useful results for real clocks.

References

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[2] J. Barnes, 1982, Notes from NBS 2nd Symposium Atomic Time Scale Algorithms, Boulder, Colorado, USA (National Bureau of Standards).

[3] J. Barnes, undated, **The Analysis of Frequency and Time Data** (Austron, Inc.).

[4] C. Greenhall, 2003, *"An optimal modification of a Kalman filter for time scales,"* in Proceedings of the 17th European Frequency & Time Forum and 2003 IEEE International Frequency Control Symposium, 5-8 May 2003, Tampa, Florida, USA, 228–232.

[5] J. Davis, C. Greenhall, and P. Stacey, 2004, *"A Kalman filter clock algorithm for use in the presence of flicker frequency modulation noise,"* **Metrologia** (accepted for publication).

Estimating the instability of a composite clock

Charles Greenhall

Jet Propulsion Laboratory

California Institute of Technology

This work was performed at the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautics and Space Administration.

Statement of problem

Situation

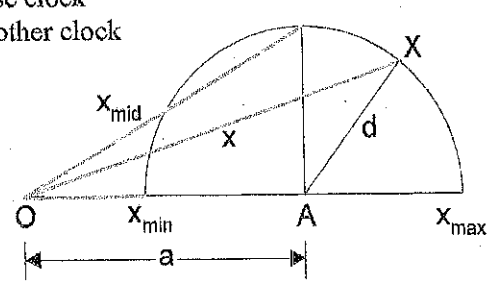
Local ensemble of clocks $A_1, \ldots, A_n$ (base clocks)
Assume they are uncorrelated.
Know stabilities for interval τ.
Measure relative phases of $A_i - A_j$
Calculate composite clock (timescale) X.
Know stability of $X - A_i$ for τ.

Problem

Estimate stability of X for τ.

One base clock

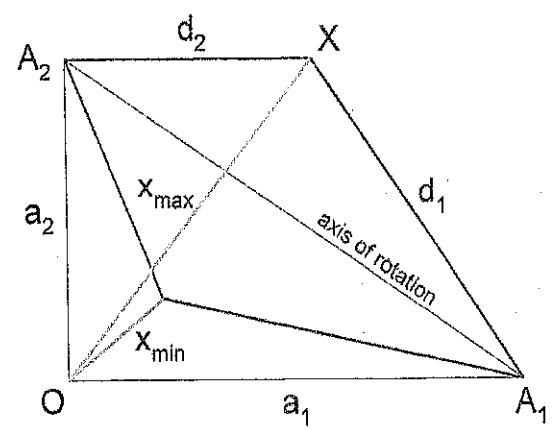
A = base clock
X = another clock



$$a = \sigma_{y,A}(\tau), \quad d = \sigma_{y,X-A}(\tau), \quad x = \sigma_{y,X}(\tau)$$

$$x_{\min} = |a - d|, \quad x_{\max} = a + d, \quad x_{\text{mid}} = \sqrt{a^2 + d^2}$$

Two base clocks



Geometrical approach

An RMS stability measure for a given τ is the norm from an inner product. Take Allan deviation:

$$\sigma_{y,A}(\tau) = \|A\|_\tau = \langle A, A\rangle_\tau^{1/2}$$

where the inner product of two clocks A and B is

$$\langle A, B\rangle_\tau = \frac{1}{2\tau^2}\mathrm{E}\left[\Delta_\tau^2 x_A(t)\right]\left[\Delta_\tau^2 x_B(t)\right]$$

Then

$$\sigma_{y,A-B}^2(\tau) = \|A-B\|_\tau^2 = \|A\|_\tau^2 - 2\langle A, B\rangle_\tau + \|B\|_\tau^2$$

Algebra = Geometry

n base clocks

Base clocks: $A_1, \ldots, A_n$. Assume $\langle A_i, A_j\rangle_\tau = 0$ for $i \neq j$

Composite clock: X

Given $a_i = \|A_i\|_\tau$, $d_i = \|X - A_i\|_\tau$

Estimate $x = \|X\|_\tau$

We will obtain

A lower bound $x_{\min}$

An upper bound $x_{\max}$

An intermediate estimate x_{mid}

Solution

Let

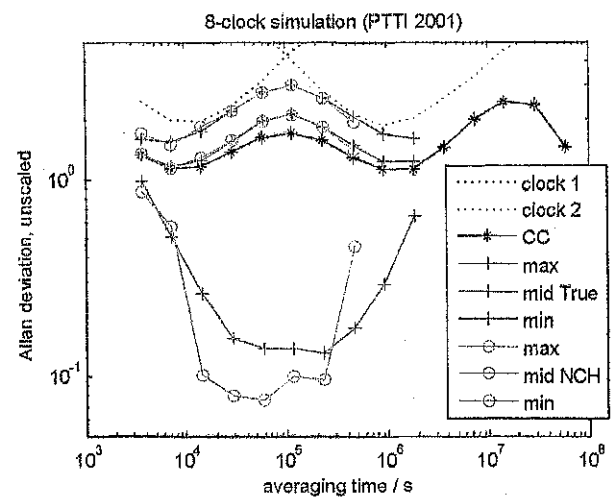
$$\varepsilon_i = 1 - \frac{d_i^2}{a_i^2}, \quad B = 2 - \sum \varepsilon_i \quad C = \left(\sum a_i^{-2}\right)\left(\sum a_i^2 \varepsilon_i^2\right)$$

Condition: $B \geq \sqrt{C}$

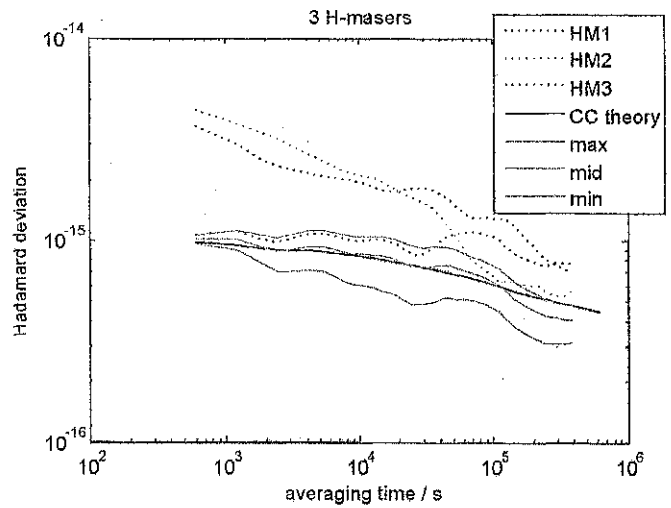
Then

$$x_{\min}^2 = \frac{B - \sqrt{B^2 - C}}{\sum a_i^{-2}}, \quad x_{\max}^2 = \frac{B + \sqrt{B^2 - C}}{\sum a_i^{-2}}, \quad x_{\mathrm{mid}}^2 = \frac{B}{\sum a_i^{-2}}$$

Example from simulation



Example with real clocks



Limitations

- Base clocks might be physically correlated; not orthogonal.
- Don't know $\|A_i\|_\tau$ or $\|X - A_i\|_\tau$ exactly; $\|A_i\|_\tau$ maybe from n-cornered hat.
- Need a lot of data for each τ.
- Try on simulated data before using on actual clocks.