A Formulation of Stability for Spacecraft Formations

A. Behçet Açıkmeşe, Fred Y. Hadaegh, Daniel P. Scharf, and Scott R. Ploen

***Abstract*— A formulation of stability for a set of spacecraft in formation flight is presented. First, a formation is defined in a precise mathematical form in terms of control interactions. Then, stability is formulated based on input-to-output stability with respect to a partitioning of the formation dynamics. This formulation of stability is shown to be useful in characterizing disturbance propagation in the formation as a function of the partition interconnection topology, and also in analyzing the robustness of sensing, communication, and control topologies. Stability results are presented for hierarchical, cyclic, and disturbance attenuating formations.**

I. INTRODUCTION

In this paper we develop a formulation of stability for a set of spacecraft coupled via control [1]. This research is motivated by NASA's formation flying missions (e.g., [2], [1]) in which several spacecraft operate in a coordinated manner to achieve a common objective.

Spacecraft formations have two fundamental properties: (i) a set of spacecraft with completely decoupled dynamics, and (ii) a control law coupling their dynamics. These two properties are used to define a formation and derive a condition for spacecraft to be in formation. Once a set of spacecraft is determined to be a formation, stability notions can be applied to the overall, closed-loop formation dynamics.

We formulate a definition of *formation stability* based on a specific form of input-to-output stability [3], [4], [5]. In standard application of a stability notion such as input-to-state or input-to-output, a set of stability conditions are required to hold for the entire formation dynamics. Formation stability could simply be defined as the input-to-output stability of the entire formation. We additionally require the input-to-output stability of components of the formation dynamics. These components result from partitioning the overall formation dynamics, and this partitioning will be defined in the sequel. As a result, formation stability is defined with respect to a partitioning of the coupled formation dynamics. When we say "a formation is stable," we mean "a formation is stable *with respect to the partition*."

In earlier related works [6], [7], [8], [9], the partitioning was prescribed based on a specific decentralized control architecture [10], and stability was assumed for the components of the partition. Next, conditions were derived guaranteeing overall formation stability in a particular sense. The decentralized control architecture used for a given set of spacecraft is the main system theoretic characteristic of a formation, and it has implications in terms of disturbance attenuation, and robustness of the coupled sensing, communication, and control systems. Consequently, the specific decentralized formation control architecture must be part of the stability definition adopted for formations. To do so, we introduce the concept of *formation stability with respect to a partition* and utilize the notion of *input-to-output stability* described in [3]. This approach to formation stability is shown to be useful in characterizing both disturbance propagation within a formation and the robustness of coupled sensing, communication, and control architectures. It also leads to several results that have connections to earlier results in the theory of interconnected systems [11], [12].

The following notation is used: $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^n$ is the space of n dimensional vectors with real components, I denotes the identity matrix of appropriate dimensions, $Q = Q^T > 0$ denotes a symmetric and positive definite matrix, $\|x\|$ denotes the standard 2-norm of a vector x, $M \succeq 0$ ($M \succ 0$) implies that all entries of a matrix M are non-negative (positive), $M \preceq 0$ ($M \prec 0$) implies the opposite, and $w = \max\{u, v\}$ implies that $w_k = \max\{u_k, v_k\}$, $k = 1, \ldots, n$, for $u, v \in \mathbb{R}^n$.

II. SPACECRAFT IN FORMATION

In this section, we give a precise definition of a formation of spacecraft. Consider a set of spacecraft $\mathcal{S}$ with dynamics given by

$$\mathcal{S}: \quad \begin{cases} S_k: \ \dot{x}_k = F_k(t, x_k, u_k, d_k) \\ \qquad\ y_k = H_k(t, x_k, u_k, d_k) \end{cases} \quad k = 1, \ldots, N, \quad (1)$$

where for each spacecraft x_k is the state, u_k is the control input, y_k is the output, and d_k is an exogenous input such as a disturbance. From (1) it can be seen that the spacecraft have decoupled dynamics in the absence of control inputs. The state vector can contain translational states (position and velocity), rotational states (quaternion and angular velocity), and other possible states relevant to the application (e.g., actuator dynamics).

A dynamic feedback controller for spacecraft in $\mathcal{S}$ can be expressed in the following form,

$$\dot{x}_c = F_c(t, x_c, u_c, d_c) \quad (2)$$

$$y_c = H_c(t, x_c, u_c, d_c), \quad (3)$$

where we omit indexing individual controllers for clarity, x_c is the controller state, y_c is the output of the controller, which is used to determine the control input for the spacecraft, u_c is the input to the controller, which is a function of the outputs of other spacecraft and controllers, and d_c is an exogenous input, which can contain disturbances, reference commands, and measurement errors.

Authors are with Jet Propulsion Laboratory, California Institute of Technology, 4800 Oak Grove Dr., M/S 198-326, Pasadena, CA 91109, USA. Email: behcet@jpl.nasa.gov

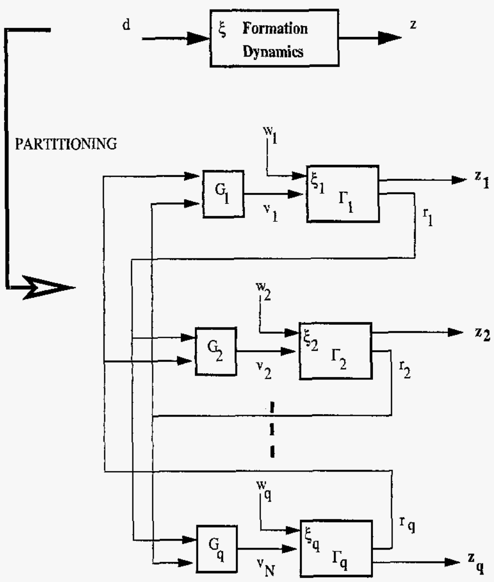

Fig. 2. A partitioning of the formation dynamics.

give an equivalent characterization of input-to-state stability (ISS). See [14] for a proof, and [15] for a definition of ISS and the statement of the equivalence. ◇

Remark 2: Ref. [3] introduces a variety of notions of IOS, and [4] discusses Lyapunov conditions for these different types of IOS. These Lyapunov conditions can be used to analyze a formation's stability in terms of the input-to-output stability notion given in Definition 3. ◇

Given a partition and the definition of IOS, we define formation stability with respect to this partition.

Definition 4 (Formation Stability w.r.t. a Partition): Formation $\mathcal{S}$ with dynamics (6) is *formation stable with respect to a partition (FSP)* of the form (8) if:

- There exist class $\mathcal{K}$ functions λ and ρ such that (6) is IOS from z to d with λ and ρ.
- For each $k = 1, \ldots, q$,, there exist class $\mathcal{K}$ functions λ_k, ρ_k such that (8) is IOS from the vector $[w_k^T, v_k^T]^T$ to z_k with λ_k and ρ_k. ◇

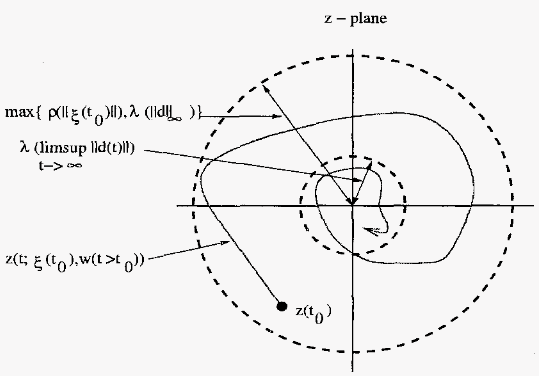

Fig. 3. An illustration of input-to-output stability.

In the definition of formation stability, the first condition requires the input-to-output stability of (6) independent of the partition. The second condition requires the input-to-output stability of each component of the partition, where a component's input consists of both exogenous inputs and signals from other components. Note that if a formation is stable with respect to a given partition it does not imply that the formation is stable with respect to another partition.

Examples of formation dynamics, partitions, and stability analysis are given in Section V.

IV. An Analysis of Formation Stability and Disturbance Propagation

The main result of this section is a sufficient condition for formation stability with respect to a partition given the stability properties of each partition and the characteristics of the interconnections. This result is used to characterize disturbance propagation within a formation. In the sequel the following condition is needed:

Condition 1 (Component Interconnection IOS): Given a partition (8) of the formation dynamics (6), each partition Γ_k, $k = 1, \ldots, q$, is IOS from (w_k, v_k) to r_k with $\tilde{\lambda}_k$ and $\tilde{\rho}_k$. □

Remark 3: Consider a partition Γ_k, $k = 1, \ldots, q$, that is IOS from (w_k, v_k) to z_k (or r_k) with λ_k and ρ_k. Since,

$$\begin{aligned} \|(w_k, v_k)\| &\le \|w_k\| + \|v_k\| \quad \text{and} \\ \lambda_k(a+b) &\le \max\{\lambda_k(2a), \lambda_k(2b)\} \text{ for all } a, b \ge 0, \end{aligned}$$

we can express IOS in the following equivalent form

$$\begin{aligned} &\|z_k\|_\infty \le \\ &\quad \max\{\rho_k(\|\xi_k(t_0)\|), 2\lambda_k(\|w_k\|_\infty), 2\lambda_k(\|v_k\|_\infty)\} \end{aligned} \tag{11}$$

$$\begin{aligned} &\limsup_{t\to\infty} \|z_k(t)\| \le \\ &\quad \max\{2\lambda_k(\limsup_{t\to\infty} \|w_k(t)\|), 2\lambda_k(\limsup_{t\to\infty} \|v_k(t)\|)\}. \end{aligned} \tag{12}$$

◇

We now present a general result on FSP where the gain functions are linear. In this theorem, we establish a stability criteria for the overall formation based upon knowledge on stability properties of individual portions in the partition and their interactions. This approach is also typical in the general theory of interconnected systems [11], [12].

Theorem 1 (FSP with Linear Gains): Consider a set of spacecraft in formation given by (6) with a partition as in (8). Suppose that (6) has well defined solutions for all initial conditions and exogenous inputs, and satisfies both Condition 1 and the second condition in Definition 4 with

$$\lambda_k(s) = a_k s/2, \quad \tilde{\lambda}_k(s) = b_k s/2, \qquad \text{for all } s \ge 0, \tag{13}$$

$$\|g_k(r_1, \ldots, r_q)\| \le \sum_{j=1}^{q} c_{kj}\|r_j\|, \qquad \text{for all } r_1, \ldots, r_q, \tag{14}$$

This characterization of the interconnections in the partitioned formation dynamics is used to define the following classes of formations, and establish specific stability results via Theorem 1.

Definition 5: Given a partitioning and corresponding adjacency matrix for a formation Σ, and:

- A formation is *hierarchical* if Σ is lower triangular.
- A formation is *cyclic* if there exists some i and j such that $\Sigma_{ij} > 0$ and $\Sigma_{ji} > 0$.

$\diamond$

A hierarchical control structure is also referred to as "leader/follower" control, and is commonly used for small-to-medium sized formations [1].

Corollary 1 (Hierarchical Formation): Consider a formation (6) and a partition (8) that is hierarchical. Suppose that all conditions of Theorem 1 are satisfied. Then, the formation is FSP, and the propagation of any disturbance can be expressed by the following relations,

$$\phi \preceq \max\{\theta,\, S\tilde{\theta},\, E\,\tilde{\theta}\} \tag{23}$$
$$\phi_\infty \preceq \max\{\theta_\infty,\, S\tilde{\theta}_\infty,\, E\theta_\infty\} \tag{24}$$

where $E = S\sum_{k=1}^{q-1} R^k$, and R is defined by (15). $\diamond$

Proof: Moreover, since all eigenvalues of R are zero,

$$(I-R^q)(I-R)^{-1} = \sum_{k=0}^{q-1} R^k .$$

Since $R \in \mathbb{R}^{q\times q}$ is lower triangular with zero diagonal entries, it is nilpotent with a largest possible degree of nilpotence q, i.e., $R^q = 0$. Consequently,

$$R(I-R)^{-1} = R\sum_{k=0}^{q-1} R^k = \sum_{k=1}^{q} R^k = \sum_{k=1}^{q-1} R^k .$$

By applying Theorem 1, we complete the proof. ■

Corollary 2 (Cyclic Formation): Consider a formation (6) and a partition (8) that is cyclic. Suppose that all conditions of Theorem 1 are satisfied, and $||R|| < 1$. Then, the formation is FSP, and the propagation of any disturbance can be expressed by the following relations,

$$\phi \preceq \max\{\theta,\, S\tilde{\theta},\, E\tilde{\theta}\} \tag{25}$$
$$\phi_\infty \preceq \max\{\theta_\infty,\, S\tilde{\theta}_\infty,\, E\theta_\infty\} \tag{26}$$

where $E = S(I-R)^{-1} - S$, and R is defined by (15). $\diamond$

Proof: Since $||R|| < 1$, we have

$$(I-R)^{-1} = \sum_{k=0}^{\infty} R^k .$$

This implies that

$$R(I-R)^{-1} = \sum_{k=1}^{\infty} R^k = \sum_{k=0}^{\infty} R^k - I = (I-R)^{-1} - I .$$

The proof is completed by applying Theorem 1. ■

The interconnections between components of a formation can have uncertainties, such unknown or time-varying system parameters. A natural question is to decide whether the formation remains stable under all variations of these interconnections. Under the hypothesis of Theorem 1, the interconnections are characterized by $c_{kj},\ k,j = 1,\ldots,q$. The following corollary gives a condition under which the formation remains stable as the interconnections vary.

Corollary 3: Consider a formation with dynamics (6) and a partition (8). Suppose that all hypothesis of Theorem 1 are satisfied with $c_{kj} = c_{kj}(\delta)$ where δ is an uncertain parameter. Furthermore, suppose that there exists $\tilde{c}_{kj} \geq 0$ such that

$$||c_{kj}(\delta)|| \leq \tilde{c}_{kj}\,, \quad \text{for all } \delta, \quad k,j = 1,\ldots,q. \tag{27}$$

Let $\tilde{R}$ be defined as

$$\tilde{R} = \begin{bmatrix} b_1\tilde{c}_{11} & \ldots & b_1\tilde{c}_{1q} \\ \vdots & \vdots & \vdots \\ b_q\tilde{c}_{q1} & \ldots & b_q\tilde{c}_{qq} \end{bmatrix} . \tag{28}$$

Then the formation is FSP if $I - \tilde{R}$ is an M-matrix. $\diamond$

Proof: $I - \tilde{R}$ is an M-matrix if and only if there exists some vector $x \succeq 0$ such that $(I-\tilde{R})x \succ 0$ [16]. Since $\tilde{R} - R \geq 0$, we have $(I-R)x \succ 0$. Consequently, $I - R$ is an M-matrix for all δ. The proof in now concluded by applying Theorem 1. ■

A. Disturbance Attenuation

Theorem 1 and its corollaries describe the propagation of disturbances through the formation for a given partition. In this section, we develop a technique to determine whether a disturbance is amplified or attenuated as it propagates in the formation. Disturbance attenuation can be formally described by the following definition.

Definition 6 (Disturbance Attenuating Formation): Formation S as given in (6) is *disturbance attenuating with respect to the partition of the formation given in (8)*, if:

- It is stable with respect to the partition (8).
- If, for any given integer $1 \leq j \leq q$, we have $\xi_k(t_0) = 0$ and $w_k = 0$ for all $k = 1,\ldots,q,\ k \neq j$, then the following conditions also hold for any solution of the system (6) and for all $k = 1,\ldots,q$,

$$||z_k||_\infty \leq ||z_j||_\infty\,, \tag{29}$$
$$\limsup_{t\to\infty} ||z_k(t)|| \leq \limsup_{t\to\infty} ||z_j(t)||\,. \tag{30}$$

$\diamond$

The following theorem gives a condition guaranteeing that a formation is disturbance attenuating.

Theorem 2: Consider a set of spacecraft in formation with dynamics (6) with a partition as in (8) satisfying items 1 and 2 in Definition 4, and Condition 1 with (13). Let matrices R and S given by (15), and matrix E given by

Then the formation dynamics with control implicitly given by $u_k(t) = g_k(t, \xi(t)$,

$$\begin{aligned} \dot{\xi}_1 &= A\xi_1 + B\left(g_2(t,\xi) - g_1(t,\xi) + d_2 - d_1\right), \\ \dot{\xi}_2 &= A\xi_2 + B\left(g_3(t,\xi) - g_1(t,\xi) + d_3 - d_1\right), \\ \dot{\xi}_3 &= A\xi_3 + B\left(g_3(t,\xi) - g_2(t,\xi) + d_3 - d_2\right), \\ z_k &= \|\xi_k\| - r_k, \quad k = 1, 2, 3, \end{aligned}$$

where r_1, r_2, r_3 are the desired distances between each spacecraft pair in the formation. Clearly, the formation objective can be satisfied when the formation output z is regulated around the origin, and the partitioning above is useful to exploit the interactions between the components of the formation output and the corresponding dynamics.

VI. Conclusions

In this paper, a mathematical definition of a formation of spacecraft is presented that is based on the characterization of the control interactions. A notion of formation stability is introduced based on a given partitioning of the formation dynamics, that establishes the standard conditions of stability of the overall formation dynamics, as well as stability conditions of each partition. We also generalized the concept of a disturbance attenuating formation in which disturbances attenuate as they propagate. This is a particularly desirable operational requirement for formations with large number of spacecraft.

Acknowledgements

We wish to gratefully acknowledge Dr. David S. Bayard and Dr. Marco Quadrelli of JPL, Prof. Paul K.C. Wang of UCLA, and Prof. Mario Garcia-Sanz of Universidad Publica de Navarra for their very valuable comments and suggestions. This research was performed at the Jet Propulsion Laboratory, California Institute of Technology, under the contract with National Aeronautics and Space Administration.

References

[1] D. Scharf, F. Hadaegh, and S. Ploen, "A survey of spacecraft formation flying guidance and control (Part II): Control," in *Preceedings of the 2004 American Control Conference*, vol. 4, June 2004, pp. 2976–2985.

[2] P. R. Lawson, "The terrestrial planet finder," *Proceedings of IEEE Aerospace Conference*, vol. 101, pp. 2005–2011, 2001.

[3] E. Sontag and Y. Wang, "Notions of input to output stability," *Systems and Control Letters*, vol. 38, pp. 235–248, 1999.

[4] ——, "Lyapunov characterizations of input to output stability," *SIAM Journal Control and Optimization*, vol. 39, pp. 226–249, 2001.

[5] ——, "A notion of input to output stability," *Proceedings of European Control Conference*, Brussels, July, 1997.

[6] A. Pant, P. Seiler, and K. Hedrick, "Mesh stability of look-ahead interconnected systems," *IEEE Transactions on Automatic Control*, vol. 47, no. 2, pp. 403–407, 2002.

[7] H. G. Tanner, G. J. Pappas, and V. Kumar, "Leader-to-formation stability," *IEEE Transactions on Robotics and Automation*, vol. 20, pp. 443–455, 2004.

[8] ——, "Input-to-state stability on formation graphs," *Proceedings of 41st IEEE Conference on Decision and Control*, vol. 3, pp. 2439–2444, 2002.

[9] P. Wang and F. Hadaegh, "Coordination and control of multiple microspacecraft moving in formation," *Journal of the Astronautical Sciences*, vol. 44, no. 3, pp. 315–355, 1996.

[10] D. D. Siljak, *Decentralized Control of Complex Systems*. Academic Press, 1991.

[11] A. N. Michel, "On the status of stability of interconnected systems," *IEEE Transactions on Automatic Control*, vol. 30, no. 6, pp. 325–340, 1983.

[12] W. Tang, A. N. Michel, and H. W. Hale, "On structure of interconnected dynamical systems," *IEEE Transactions on Automatic Control*, vol. 27, no. 5, pp. 391–405, 1980.

[13] N. Deo, *Graph Theory with Applications to Engineering and Computer Science*. Prentice-Hall, 1974.

[14] E. Sontag and Y. Wang, "New characterizations of the input-to-state stability property," *IEEE Transactions on Automatic Control*, vol. 41, pp. 1283–1294, 1996.

[15] A. Isidori, *Nonlinear Control Systems II*. Springer-Verlag, 1999.

[16] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*. Cambridge University Press, 1991.