Dynamics of Earth Orbiting Formations

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Abstract

In this paper the equations of motion of a formation consisting of n spacecraft in Earth orbit are derived via Lagrange's equations. The equations of motion of the formation are developed with respect to both (1) a bound Keplerian reference orbit, and (2) a specific spacecraft in the formation. The major orbital perturbations acting on a formation in low Earth orbit are also included in the analysis. In contrast to the traditional approach based on the balance of linear momentum, the use of Lagrange's equations leads to a high-level matrix derivation of the formation equations of motion. The matrix form of the nonlinear motion equations is then linearized about a bound Keplerian reference orbit. Next, it is demonstrated that under the assumption of a circular reference orbit, the linearized equations of motion reduce to the well-known Hill-Clohessy-Wiltshire equations. The resulting linear and nonlinear dynamic equations lead to maximal physical insight into the structure of formation dynamics, and are ideally suited for use in the design and validation of formation guidance and control laws.

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1 Introduction

The ability to accurately model the dynamic behavior of separated spacecraft formations in orbit around a central body is critical to the success of many planned and future NASA missions. For example, the development and validation of high-precision formation guidance and control laws will require a spectrum of dynamic models ranging from linearized models to models that include all significant formation nonlinearities. Accurate modeling of the ambient disturbance environment, especially for formations in the presence of significant orbital dynamics, is also of paramount concern in order to predict system performance under stringent pointing and maneuvering constraints.

In this paper the complete nonlinear equations of motion of a formation consisting of n point-mass spacecraft about a closed Keplerian reference orbit are derived. The nonlinear relative equations of motion of the formation about a given spacecraft are also developed. Further, analytical models describing the major perturbing forces acting on an Earth orbiting formation are also given. Specifically, the equations of motion of the formation are derived using a Lagrangian (i.e., energy-based) approach rather than the standard derivation based on the balance of linear momentum. This approach results in a *high-level, matrix-based derivation* of the equations of motion that provides insight into the structure of the dynamic behavior of a formation. The resulting non-linear differential equations are then linearized about a bound Keplerian (i.e., elliptical) reference orbit. Finally, we demonstrate that under the assumption of a circular reference orbit, the linearized equations of motion reduce to the Hill-Clohessy-Wiltshire (HCW) equations [4].

The resulting nonlinear and linearized forms of the formation equations of motion are useful in the design and validation of precision formation guidance and control laws. For example, the linearized relative equations of motion of the formation are well-suited for use as a *control design model.* Typically, the disturbance forces acting on the formation are ignored in the control design model. Once an appropriate linear control design model has been established, the full arsenal of modern control design techniques such as LQR

(Linear Quadratic Regulator), LQG (Linear Quadratic Gaussian), or H_∞ loopshaping can then be utilized. However, once the control law has been designed, it must be validated (in simulation) with a *different, more accurate, model* of the formation dynamics. The model used for controller validation is called the *truth model* of the formation. The truth model is typically a set of *non-linear* differential equations that include nonlinear kinematic effects as well as all significant perturbations acting on the formation (e.g., central body oblateness effect, aerodynamic drag). Moreover, the structure of the nonlinear equations of motion can also be used directly to develop nonlinear formation control laws. For example, utilizing the nonlinear structure of the equations of motion in the control design can lead to fuel savings in certain formation flying applications [17]. Further, the design of optimal (e.g., minimum fuel, fuel balancing) guidance laws for collision-free formation reconfigurations will also require nonlinear formation models.

Although there has been a significant amount of research in the area of formation flying control [17], the area of formation dynamic modeling has received less attention. Much of the work to date in formation flying dynamics has concentrated on the development of 3 degree-of-freedom (3 DOF) translational equations of motion by utilizing the balance of linear momentum. For example, a derivation of the linearized translational dynamics of one spacecraft relative to another spacecraft in a circular orbit (commonly called the Hill-Clohessy-Wiltshire equations) based on Newton's Laws has been addressed by many researchers; see e.g., [4], [15]. The assumption of a circular reference orbit in the derivation has been relaxed in a number of papers involving formation flying and satellite rendezvous; see [1],[2],[3], [6], [9], and [21]. A common characteristic of these papers is that emphasis is placed on developing solutions to the differential equations governing relative spacecraft motion, rather than exposing the internal structure of the equations of motion for guidance and control law design and validation. An exception is the paper [18] where the translational dynamics of formations in deep space are studied in order to develop insight into the validity of utilizing linear dynamic models ("double integrator models") for control law design.

However, the application of methods of analytical mechanics [7] to the area of formation flying modeling has not been extensively studied. In [16] the equations of motion of a formation containing flexible tethers are developed using Lagrange's equations. In [12] the effect of J_2 oblateness on formation motion is studied using Routhian reduction. Finally, in [13], the dynamic properties of the linear and nonlinear relative motion equations are investigated using the Hamiltonian structure of the problem.

The remainder of this paper is organized as follows. First, some preliminary material from rotational kinematics is reviewed. Next, the geometry of an Earth orbiting formation is established. A complete derivation of the nonlinear equations of motion of the formation about a bound Keplerian reference orbit based on Lagrange's equations is then given. The equations of motion of the formation relative to a specific spacecraft are also derived. Next, the various orbital disturbances acting on a formation in Earth orbit are described. Analytical models of the primary disturbances acting on formations in LEO (Low Earth Orbit) are emphasized. The nonlinear equations of motion of the formation are then linearized with respect to the reference orbit. The use of the linearized dynamic model for formation controller design is then discussed. In the final section, some conclusions and directions for further research are presented.

2 Kinematic Preliminaries

In this section some basic concepts and notation from rotational kinematics are reviewed; see [8] for a more detailed discussion. In the sequel, geometric (or Gibbsian) vectors will play an important role. Recall that a geometric vector $\vec{Q}$ is a quantity possessing magnitude, direction, and obeying the parallelogram law of addition in three dimensional Euclidean point space, denoted E_3. A geometric vector is visualized as an arrow or directed line segment in E_3. The distinction between *geometric* vectors and column matrices (i.e., 3-tuples of real numbers) is critical in developing the equations of motion of separated spacecraft formations.

Specifically, a vector $\vec{Q}$ is a geometric object that exists independently of any particular basis chosen for E_3 while a column matrix of numbers $Q = [Q_1 Q_2 Q_3]^T \in \Re^{3\times1}$ is the representation of a geometric vector with respect to a particular basis.

A reference frame is a set of three mutually orthonormal vectors (basis vectors) located at an arbitrary point in E_3. The basis vectors associated with a reference frame can be easily manipulated by defining a *vectrix* as follows $\vec{\mathcal{F}}_A \equiv [\vec{a}_1\ \vec{a}_2\ \vec{a}_3]^T$. In other words, the elements of the vectrix $\vec{\mathcal{F}}_A$ are simply the basis vectors characterizing the given frame of reference. In the sequel a reference frame will be denoted as $\mathcal{F}_A$ and the vectrix associated with the frame as $\vec{\mathcal{F}}_A$.

Once a reference frame $\mathcal{F}_A$ has been established, a geometric vector can be represented uniquely as

$$\vec{Q} = Q_1\vec{a}_1 + Q_2\vec{a}_2 + Q_3\vec{a}_3 \tag{1}$$

$$= [\vec{a}_1\ \vec{a}_2\ \vec{a}_3] \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} \tag{2}$$

$$= \vec{\mathcal{F}}_A^T Q_A \tag{3}$$

In the sequel we will require the vectrix operator $\bullet$ which is defined such that $Q_A = \vec{\mathcal{F}}_A \bullet \vec{Q}$. That is, the symbolic notation $Q_A = \vec{\mathcal{F}}_A \bullet \vec{Q}$ should be read "Q_A is the column matrix whose entries are the components of $\vec{Q}$ in $\mathcal{F}_A$". See [8] for further information.

A fundamental result relating the time rates of change of a geometric vector relative to observers attached to different rotating reference frames, denoted $\mathcal{F}_A$ and $\mathcal{F}_B$, is the *Transport Theorem*:

$$\overset{A}{\dot{\vec{Q}}} = \overset{B}{\dot{\vec{Q}}} + {}^A\vec{\omega}^B \times \vec{Q} \tag{4}$$

Here $\vec{Q}$ denotes an arbitrary geometric vector, ${}^A\vec{\omega}^B$ denotes the angular velocity of $\mathcal{F}_B$ in

$\mathcal{F}_A$, and

$$\overset{A}{\vec{Q}} \equiv \dot{Q}_1\vec{a}_1 + \dot{Q}_2\vec{a}_2 + \dot{Q}_3\vec{a}_3 \tag{5}$$

$$\overset{B}{\vec{Q}} \equiv \dot{Q}'_1\vec{b}_1 + \dot{Q}'_2\vec{b}_2 + \dot{Q}'_3\vec{b}_3 \tag{6}$$

The notation $\overset{A}{\vec{Q}}$ (resp. $\overset{B}{\vec{Q}}$) can be interpreted physically as the rate of change of $\vec{Q}$ as seen by an observer rigidly fixed to $\mathcal{F}_A$ (resp. $\mathcal{F}_B$). As a consequence, if $\vec{Q}$ is a vector fixed in $\mathcal{F}_A$ (resp. $\mathcal{F}_B$) then $\overset{A}{\vec{Q}}= \vec{0}$ (resp. $\overset{B}{\vec{Q}}= \vec{0}$). The notation $\overset{A}{\vec{Q}}$ is not entirely standard: See [10] for a discussion on alternate notation.

The transport formula (4) can also be expressed in the equivalent form

$$\overset{A}{\vec{Q}}=\overset{B}{\vec{Q}} +[{}^A\vec{\omega}^B]\vec{Q} \tag{7}$$

where $[\vec{a}]$ denotes the skew-symmetric cross-product operator $[\vec{a}]\vec{b} = \vec{a} \times \vec{b}$. It can be shown that the cross-product operator takes the form of a skew-symmetric matrix in a specific coordinate system. Specifically, the skew-symmetric matrix $[a] \in \Re^{3\times 3}$ associated with the column vector $[a_1\ a_2\ a_3]^T \in \Re^{3\times 1}$ is given by

$$[a] = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{8}$$

3 Formation Dynamics: Nonlinear Equations of Motion

3.1 Orbital Geometry

In this section we consider a formation of n spacecraft located in Earth orbit where each spacecraft is modeled as a point-mass. An inertial frame of reference $\mathcal{F}_N$, called the geocentric inertial frame, is attached to the center of the Earth and described by the vectrix $\vec{\mathcal{F}}_{\mathcal{N}} = [\vec{n}_1 \; \vec{n}_2 \; \vec{n}_3]^T$. The unit vector $\vec{n}_1$ points toward the vernal equinox, $\vec{n}_3$ points toward the geographic North Pole, and $\vec{n}_2$ completes the right-handed triad. The motion of the formation is described with respect to a bound, pure-Keplerian reference orbit (See Figure 1). In particular, the reference orbit is a solution of the following differential equation

$$\ddot{\vec{R}}_O = -\frac{\mu \vec{R}_O}{\|\vec{R}_0\|^3} \tag{9}$$

where $\mu = 3.986 \times 10^5$ $[\frac{km^3}{s^2}]$ denotes the gravitational parameter of the Earth, and $\|\cdot\|$ denotes the Euclidian norm on E_3. Alternately, the reference orbit can be described by the orbital elements a (semi-major axis), e (eccentricity), i (inclination), $\tilde{\Omega}$ (longitude of the ascending node), $\tilde{\omega}$ (argument of perigee), ν (true anomaly), and T (time of perigee passage). See Figure 1.

The origin of the reference orbit frame $\mathcal{F}_O$ can be chosen to be any point of interest with respect to the formation. Here we will assume that the origin of the reference orbit corresponds to the formation center-of-mass.

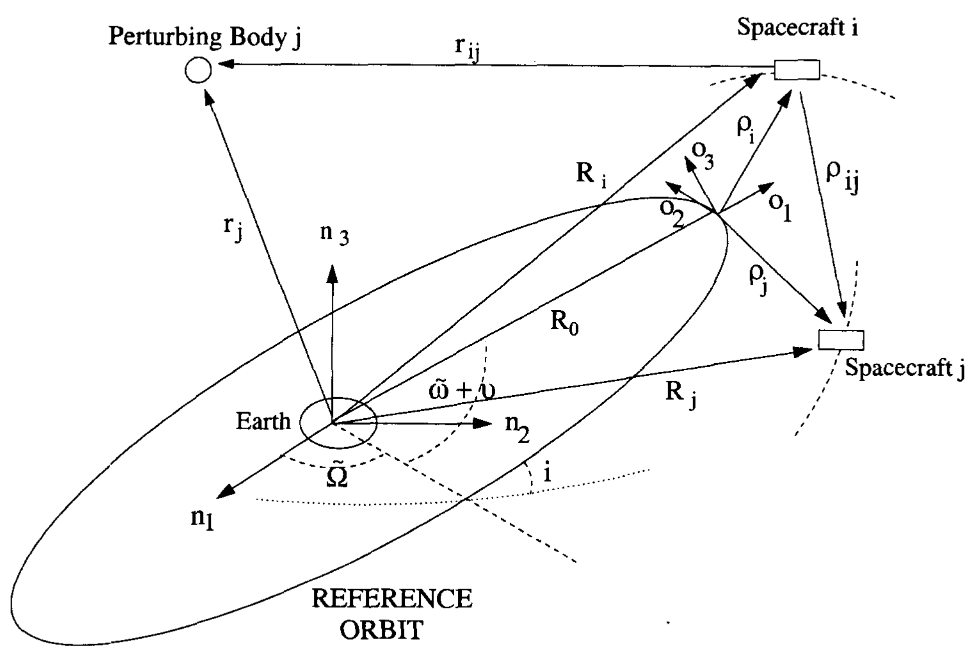

Figure 1: Orbital Geometry

The reference orbit defines an orbiting reference frame $\mathcal{F}_O$ as shown in Figure 1. The orbital frame serves as the primary frame to analyze the dynamics of the formation. The unit vector $\vec{o}_1$ points anti-nadir, the unit vector $\vec{o}_3$ points in the direction of the orbit normal, and $\vec{o}_2$ completes the right-handed triad. Note that $\vec{o}_2$ is not in general tangent to the reference orbit. The orientation of the orbital frame $\mathcal{F}_O$ relative to the inertial frame $\mathcal{F}_N$ is described by the direction-cosine matrix

$$L_{NO} = [o_1 \; o_2 \; o_3] \in \Re^{3\times 3} \tag{10}$$

where the column matrices $o_i \equiv \vec{\mathcal{F}}_N \bullet \vec{o}_i \in \Re^{3\times 1}$ and

$$\begin{aligned} \vec{o}_1 &= \frac{\vec{R}_o}{\|\vec{R}_o\|} && (11) \\ \vec{o}_3 &= \frac{\vec{R}_o \times \vec{V}_o}{\|\vec{R}_o \times \vec{V}_o\|} && (12) \\ \vec{o}_2 &= \vec{o}_3 \times \vec{o}_1 && (13) \end{aligned}$$

See Figure 1. Here $\vec{V}_o = \overset{N}{\dot{\vec{R}}_o}$ denotes the absolute velocity of the origin of $\mathcal{F}_O$. Note that in terms of orbital elements the direction-cosine matrix L_{NO} can also be expressed as

$$L_{NO} = L_z(\Omega)L_y(i)L_z(\omega + \nu) \tag{14}$$

where $L_z(\cdot)$ and $L_y(\cdot)$ denote principal rotation matrices about the direction indicated by the subscript. In the sequel the reference orbit will be treated as a prescribed motion.

3.2 Derivation of the Equations of Motion

From Figure 1, the absolute position of the i^{th} spacecraft is given by

$$\vec{R}_i = \vec{R}_o + \vec{\rho}_i \tag{15}$$

Differentiating, the absolute velocity of the i^{th} spacecraft is

$$\overset{N}{\vec{R}_i} = \overset{N}{\vec{R}_o} + \overset{N}{\vec{\rho}_i} \tag{16}$$

Applying the transport formula between $\mathcal{F}_O$ and $\mathcal{F}_N$ yields

$$\overset{N}{\vec{R}_i} = \overset{O}{\vec{R}_o} + [\vec{\omega}]\vec{R}_o + \overset{O}{\vec{\rho}_i} + [\vec{\omega}]\vec{\rho}_i \tag{17}$$

where $\vec{\omega} \equiv^N \vec{\omega}^B$ denotes the angular velocity of the orbital frame in N. As a result, the inertial velocity (17) resolved in $\mathcal{F}_O$ is given by:

$$\dot{R}_i \equiv \vec{\mathcal{F}}_O \bullet \overset{N}{\vec{R}_i} \tag{18}$$

$$= \vec{\mathcal{F}}_O \bullet (\overset{O}{\vec{R}_o} + [\vec{\omega}]\vec{R}_o) + \vec{\mathcal{F}}_O \bullet (\overset{O}{\vec{\rho}_i} + [\vec{\omega}]\vec{\rho}_i) \tag{19}$$

From Figure 1 it follows that the geometric vectors appearing in (19) have the following representation in $\mathcal{F}_O$:

$$\vec{R}_o = R_o\vec{o}_1 \tag{20}$$

$$\overset{O}{\vec{R}_o} = \dot{R}_o\vec{o}_1 \tag{21}$$

$$\vec{\omega} = \omega_o\vec{o}_3 \tag{22}$$

$$\vec{\rho}_i = x_i\vec{o}_1 + y_i\vec{o}_2 + z_i\vec{o}_3 \tag{23}$$

$$\overset{O}{\vec{\rho}_i} = \dot{x}_i\vec{o}_1 + \dot{y}_i\vec{o}_2 + \dot{z}_i\vec{o}_3 \tag{24}$$

Also, note that due to the assumption of pure Keplerian motion of the reference orbit, the angular velocity of the orbital frame $\vec{\omega}$ points along the orbit normal $\vec{o}_3$. However, the magnitude of $\vec{\omega}$, denoted ω_o, is time-varying. Substituting (20)-(24) into (19) and expanding yields

$$\dot{R}_i = \dot{\rho}_i + [\omega]\rho_i + b \tag{25}$$

where

$$\rho_i = [x_i \; y_i \; z_i]^T \in \Re^{3\times 1} \tag{26}$$

$$\dot{\rho}_i = [\dot{x}_i \; \dot{y}_i \; \dot{z}_i]^T \in \Re^{3\times 1} \tag{27}$$

$$\omega = [0 \; 0 \; \omega_o]^T \in \Re^{3\times 1} \tag{28}$$

and

$$b = [\dot{R}_o \; R_o\omega_o \; 0]^T \in \Re^{3\times 1} \tag{29}$$

Here $[\omega] \in \Re^{3\times 3}$ denotes the skew-symmetric matrix associated with the column vector ω given in (28).

In order to derive the equations of motion of the formation via Lagrange's equations, a set of admissible generalized coordinates are required. The generalized coordinates for the i^{th} spacecraft are chosen as

$$q_i = \rho_i \tag{30}$$

$$= [x_i \; y_i \; z_i]^T \in \Re^{3\times 1} \tag{31}$$

Note that q_i are the components of $\vec{\rho}_i$ in the *rotating frame* $\mathcal{F}_O$, i.e., $q_i \equiv \vec{\mathcal{F}}_O \bullet \vec{\rho}_i$. As a result, the relationship (25) takes the following form

$$\dot{R}_i = \dot{q}_i + [\omega]q_i + b \tag{32}$$

$$= \dot{q}_i + c_i \tag{33}$$

where $c_i = [\omega]q_i + b$.

In order to develop the equations of motion of the formation the following quantities are required:

$$
\begin{aligned}
\dot{R} &= \text{col}[\dot{R}_1,\ \dot{R}_2, \cdots, \dot{R}_n]^T \ \in \Re^{3n \times 1} && (34)\\
q &= \text{col}[q_1, q_2, \cdots, q_n] \ \in \Re^{3n \times 1} && (35)\\
\dot{q} &= \text{col}[\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n] \ \in \Re^{3n \times 1} && (36)\\
\beta &= \text{col}[b,\ b, \cdots, b] \in \Re^{3n \times 1} && (37)\\
\Omega &= \text{diag}[[\omega], [\omega], \cdots, [\omega]] \in \Re^{3n \times 3n} && (38)\\
M &= \text{diag}[m_1 I,\ m_2 I,\ \cdots, m_n I] \in \Re^{3n \times 3n} && (39)
\end{aligned}
$$

where m_i denotes the mass of the i^{th} spacecraft, diag[$\cdot$] denotes a block diagonal matrix structure, and I denotes the 3×3 identity matrix. Further, the following operator has been defined

$$
\text{col}[v_1,\ v_2,\ v_3,\ \cdots, v_p] \equiv \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_p \end{bmatrix} \in \Re^{m \times 1} \tag{40}
$$

where $v_i \in \Re^{n_i \times 1}$ and $m = \sum_{i=1}^{p} n_i$.

The kinetic energy of the formation is given by the following quadratic form

$$
T = \frac{1}{2} \dot{R}^T M \dot{R} \tag{41}
$$

Note that M is a constant matrix.

Upon stacking the velocities of each spacecraft (33) and utilizing equations (34)-(39) it follows that

$$
\dot{R} = \dot{q} + c \tag{42}
$$

where

$$c = \Omega q + \beta \tag{43}$$

Here $c = \text{col}[c_1,\ c_2,\ \cdots, c_n] \in \Re^{3n\times 1}$.

Substituting (42) into the kinetic energy (41) and expanding yields

$$T(q,\dot{q}) = \frac{1}{2}\dot{q}^T M \dot{q} + c^T M \dot{q} + \frac{1}{2}c^T M c \tag{44}$$

where $c = c(q)$ is given by (43).

The potential energy of the formation is given by

$$U(q) = \sum_{i=1}^{n} U_i(q_i) \tag{45}$$

where

$$U_i(q_i) = -\frac{\mu m_i}{R_i} \tag{46}$$

and $R_i \equiv \|\vec{R}_i\| = ((x_i + R_o)^2 + y_i^2 + z_i^2)^{\frac{1}{2}}$.

The equations of motion of the Earth orbiting formation will now be developed via Lagrange's equations:

$$\frac{d}{dt}T_{\dot{q}} - T_q = Q - U_q \tag{47}$$

where

$$T_{\dot{q}} = \text{col}[\frac{\partial T}{\partial \dot{q}_1},\ \frac{\partial T}{\partial \dot{q}_2}, \cdots, \frac{\partial T}{\partial \dot{q}_n}] \in \Re^{3n\times 1} \tag{48}$$

$$T_q = \text{col}[\frac{\partial T}{\partial q_1},\ \frac{\partial T}{\partial q_2}, \cdots, \frac{\partial T}{\partial q_n}] \in \Re^{3n\times 1} \tag{49}$$

$$U_q = \text{col}[\frac{\partial U}{\partial q_1},\ \frac{\partial U}{\partial q_2}, \cdots, \frac{\partial U}{\partial q_n}] \in \Re^{3n\times 1} \tag{50}$$

The vector of generalized forces is given by

$$Q = \mathrm{col}[Q_1,\ Q_2, \cdots, Q_n] \in \Re^{3n\times 1} \tag{51}$$

where $Q_i = [Q_{x_i}\ Q_{y_i}\ Q_{z_i}]^T \in \Re^{3\times 1}$ and

$$Q_{x_i} = \sum_{j=1}^{n} \vec{F}_j \cdot \frac{\partial \vec{R}_j}{\partial x_i} \tag{52}$$

$$Q_{y_i} = \sum_{j=1}^{n} \vec{F}_j \cdot \frac{\partial \vec{R}_j}{\partial y_i} \tag{53}$$

$$Q_{z_i} = \sum_{j=1}^{n} \vec{F}_j \cdot \frac{\partial \vec{R}_j}{\partial z_i} \tag{54}$$

Here $\cdot$ denotes the standard dot product on E_3. The vector $\vec{F}_j$ in (52)-(54) denotes the resultant force acting on the j^{th} spacecraft of the formation. From the definitions of $\vec{R}_i$ and $\vec{\rho}_i$ given in equations (15) and (23) respectively, it follows that

$$Q_{x_i} = \vec{F}_i \cdot \vec{o}_1 \tag{55}$$

$$Q_{y_i} = \vec{F}_i \cdot \vec{o}_2 \tag{56}$$

$$Q_{z_i} = \vec{F}_i \cdot \vec{o}_3 \tag{57}$$

We now explicitly calculate the gradients appearing in Lagrage's equations (47). To this end, the gradient of the kinetic energy (44) with respect to the generalized velocities is

$$T_{\dot{q}} = M(\dot{q} + c) \tag{58}$$

Further,

$$\frac{d}{dt} T_{\dot{q}} = M(\ddot{q} + \dot{c}) \tag{59}$$

The gradient of T with respect to the generalized coordinates is

$$T_q = (\frac{\partial c}{\partial q})^T M\dot{q} + (\frac{\partial c}{\partial q})^T Mc \tag{60}$$

where the Jacobian matrix $\frac{\partial c}{\partial q} \in \Re^{3n\times 3n}$ is given by $[\frac{\partial c}{\partial q}]_{ij} = \frac{\partial c_i}{\partial q_j}$. From (43) it follows that

$$\frac{\partial c}{\partial q} = \Omega \tag{61}$$

As a result,

$$\begin{aligned} T_q &= \Omega^T M\dot{q} + \Omega^T Mc & (62)\\ &= -M\Omega(\dot{q} + c) & (63) \end{aligned}$$

Note that $\Omega = -\Omega^T$ and the fact that Ω and M commute, i.e., $\Omega M = M\Omega$, have been used in the above derivation.

Upon substituting (59) and (63) into Lagrange's equations (47) and recalling the definition of c given in (43), the equations of motion of the formation relative to the reference orbit are

$$M\ddot{q} + 2M\Omega\dot{q} + (M\Omega^2 + M\dot{\Omega})q + M(\dot{\beta} + \Omega\beta) = Q - U_q \tag{64}$$

where the potential energy $U(q)$ is given by (45).

It immediately follows from expanding (64) that the nonlinear equations of motion of the i^{th} spacecraft relative to the reference orbit are given by

$$m_i\ddot{q}_i + 2m_i[\omega]\dot{q}_i + m_i([\omega]^2 + [\dot{\omega}])q_i + m_i(\dot{b} + [\omega]b) = Q_i + U_{q_i} \tag{65}$$

The equations of motion (65) are non-linear due to the gradient of the potential and the state dependence of the generalized force vector. Note that the gyroscopic coupling terms (arising from expressing the equations of motion of the formation relative to the rotating frame $\mathcal{F}_O$) are characterized in explicit matrix form in equation (65). The generalized force vector Q_i contains all disturbance forces acting on the formation and will be discussed in detail in the next section.

Alternately, the nonlinear equations of motion of the formation can be expressed in terms of the position of spacecraft j relative to spacecraft i. See Figure 1. To this end, we define the vector $\vec{\rho}_{ij} = \vec{\rho}_j - \vec{\rho}_i$ from spacecraft i to spacecraft j. Resolving $\vec{\rho}_{ij}$ in $\mathcal{F}_o$ yields

$$\begin{aligned} \rho_{ij} &\equiv \vec{\mathcal{F}}_O \bullet \vec{\rho}_{ij} && (66) \\ &= \vec{\mathcal{F}}_O \bullet \vec{\rho}_j - \vec{\mathcal{F}}_O \bullet \vec{\rho}_i && (67) \\ &= \rho_j - \rho_i && (68) \end{aligned}$$

Subtracting (65) from the equations of motion of spacecraft j,

$$m_j\ddot{q}_j + 2m_j[\omega]\dot{q}_j + m_j([\omega]^2 + [\dot{\omega}])q_j + m_j(\dot{b} + [\omega]b) = Q_j + U_{q_j} \tag{69}$$

and rearranging yields the *equation of motion of spacecraft j relative to spacecraft i*:

$$\ddot{\rho}_{ij} + 2[\omega]\dot{\rho}_{ij} + ([\omega]^2 + [\dot{\omega}])\rho_{ij} = (\frac{Q_j}{m_j} - \frac{Q_i}{m_i}) + (\frac{U_{q_i}}{m_i} - \frac{U_{q_j}}{m_j}) \tag{70}$$

Up to this stage of the analysis we have not made any assumptions regarding the relative magnitudes of the vectors $\vec{R}_i$ and $\vec{\rho}_i$. As a result, (65) provides the full non-linear dynamics of the formation relative to a closed Keplerian reference orbit, and (70) completely describes the dynamics of the formation relative to the i^{th} spacecraft. Equations (65) or (70) provide a set of nonlinear differential equations for designing nonlinear control laws and developing guidance laws for formation reconfigurations. Moreover, (65) or (70) are also useful as *truth models* for dynamical studies of open-loop formation behavior and control law validation. By the truth model we mean a dynamic model that captures the *full non-linear open-loop behavior* of the formation. A truth model has the property that the dynamic response of the system predicted by the truth model will be in close agreement with the *actual* on-orbit response of the system over a specified time horizon. Equations (65) and (70) are one of the main results of this paper.

4 Formation Dynamics: Orbital Disturbances

In this section we discuss the disturbances acting on a formation in LEO. The ability to accurately model the orbital perturbations is critical in precision formation flying guidance and control applications. However, predicting the effect of orbital disturbances on a formation is a challenging problem since that disturbances act differently on different parts of the formation. For a comprehensive discussion of the ambient disturbance environment in Earth orbit see [14] or [20].

A significant disturbance acting on formations in LEO is due to higher-order harmonics of the Earth's gravitational field. Specifically, the Earth is not a spherically symmetric body, but is bulged at the equator and flattened at the poles. For our purposes, the potential field external to an *oblate* Earth can be modeled as:

$$U(R,\theta) = -\frac{\mu m_i}{R}[1 - \sum_{k=2}^{\infty} (\frac{R_e}{R})^k J_k P_k^0(\cos\theta)] \tag{71}$$

where $R_e = 6378.12[km]$ is the equatorial radius of the Earth, R denotes the magnitude of the position vector $\vec{R}$ corresponding to an arbitrary point in space external to the Earth, θ denotes the polar angle between the $\vec{n}_3$ axis of $\mathcal{F}_N$ and $\vec{R}$, J_k is the k^{th} zonal harmonic of the Earth, and $P_k^0(\cdot)$ denotes a Legendre polynomial. Note that the first term in the series expansion is given by equation (46). It is well known that J_2 is the dominant harmonic in the expansion (71). As a result, it can be shown after some manipulation that the force on the i^{th} spacecraft due to J_2 oblateness is

$$\vec{F}_i^o = \frac{-3\mu m_i J_2 R_e^2}{2R_i^4}(c_{1i}\vec{e}_i + c_{2i}\vec{n}_3) \tag{72}$$

where

$$c_{1i} = 1 - 5(\vec{e}_i \cdot \vec{n}_3)^2 \tag{73}$$

$$c_{2i} = 2(\vec{e}_i \cdot \vec{n}_3) \tag{74}$$

Here $J_2 = 1082.63 \times 10^{-6}$ is the first zonal harmonic of the Earth, R_i is the magnitude of $\vec{R}_i$, and $\vec{e}_i = \frac{\vec{R}_i}{R_i}$. Equation (72) can also be modified to include higher-order zonal harmonics as well as tesseral and sectorial harmonics. See [14] or [19] for additional information.

For formations in LEO, the interaction between the spacecraft and the particles of the atmosphere results in a significant deviation from pure Keplerian motion. The aerodynamic forces acting on the spacecraft are extremely complicated and depend on the geometry and motion of the spacecraft, as well as on the density, temperature, and composition of the atmosphere. As a result, high-fidelity modeling of the applied aerodynamic loads is a difficult task. However, the following formula, based on empirical observations, provides a useful approximation to the aerodynamic drag acting on the i^{th} spacecraft

$$\vec{F}_i^a = -\frac{1}{2} m_i C_{di} A_i \rho V_{ri} \vec{V}_{ri} \tag{75}$$

Here C_{di} is the drag coefficient[1] associated with the i^{th} spacecraft, A_i denotes the effective cross-sectional area of the the i^{th} spacecraft, and $\rho = \rho(R_i, t)$ is the atmospheric density in the vicinity of spacecraft i. Further, $\vec{V}_{ri}$ denotes of the velocity of the spacecraft relative to the atmosphere of the Earth and is given by

$$\vec{V}_{ri} = \vec{V}_i - \vec{\omega}_p \times \vec{R}_i, \tag{76}$$

V_{ri} denotes the magnitude of $\vec{V}_{ri}$, $\vec{V}_i = \overset{N}{\vec{R}_i}$ is the absolute velocity of spacecraft i, and $\vec{\omega}_p = 0.7292 \times 10^{-4}[\frac{rad}{s}]\vec{n}_3$ denotes the angular velocity of the Earth. Equation (76) assumes that

[1]Typical values of C_d for non-spherical, convex-shaped spacecraft range from 2.0 to 2.3.

the atmosphere does not have any velocity relative to the rotating Earth.

The gravitational perturbations due to the Moon and Sun (modeled as point masses) are also important to characterize. To this end, the perturbation on the i^{th} spacecraft of the formation due to $j = 1, 2, \ldots, N$ gravitationally interacting bodies (See Figure 1) is given by

$$\vec{F}_i^g = \sum_{j=1}^{N} m_i \mu_j \left(\frac{\vec{r}_{ij}}{r_{ij}^3} - \frac{\vec{r}_j}{r_j^3}\right) \tag{77}$$

where μ_j denotes the gravitational parameter of the j^{th} perturbing body, $\vec{r}_{ij}$ denotes the vector *from* spacecraft i *to* perturbing body j, and $\vec{r}_j$ is the vector *from* the Earth *to* perturbing body j. Note from Figure 1 that $\vec{r}_{ij} = \vec{r}_j - \vec{R}_i$.

Collecting the above results, the resultant force acting on the i^{th} spacecraft of the formation in LEO is given by

$$\vec{F}_i = \vec{F}_i^o + \vec{F}_i^a + \vec{F}_i^g + \vec{F}_i^c + \vec{F}_i^r \tag{78}$$

where $\vec{F}_i^o, \vec{F}_i^a$, and $\vec{F}_i^g$ are as defined above, $\vec{F}_i^c$ denotes the applied control forces due to thrusters on the I^{th} spacecraft, and $\vec{F}_i^r$ denotes the resultant of all other perturbations acting on spacecraft i such as disturbances due to direct solar radiation pressure, disturbances due to indirect solar radiation pressure, higher order zonal harmonics of the Earth's potential field, the Earth's geomagnetic field, tidal effects, thermal inputs, and relativistic corrections.

Once the resultant force $\vec{F}_i$ has been determined, the generalized force vector appearing on the right-hand side of the equations of motion (65) or (70) is computed by projecting $\vec{F}_i$ into $\mathcal{F}_O$ as shown in equations (55)-(57). Together, equations (45), (46), (51), (55)-(57), (65), (70), (78), and supporting equations are a description of the full nonlinear dynamics of LEO formations. In the next section we develop simplified equations of motion that can be used for formation guidance and control law design.

5 Formation Dynamics: Linearized Equations of Motion

In this section the equations of motion derived in the previous sections will be linearized about the bound Keplerian reference orbit. Our goal is to develop linear models that are suitable for the design of formation linear control or guidance laws. The linearized model used for formation control design is called the *control design model.* Typically, the control design model does not include orbital perturbations. Specifically, we will assume that the only external force retained in the control design model is the force $\vec{F}_i^c$ due to thrusters.

5.1 Reference Orbit Relative Equations of Motion

In order to develop a suitable linear model for *control design* the potential function $U(q)$ appearing in equation (64) will be linearized about the reference orbit. Expanding the potential function in a Taylor series about the reference orbit $q = 0 \in \Re^{3n\times 1}$ yields

$$U(q) = U(0) + U_q(0)q + \frac{1}{2}q^T U_{qq}(0)q + \eta(q) \tag{79}$$

where $\frac{\eta(q)}{\|q\|_2^2} \longrightarrow 0$ as $\|q\|_2 \longrightarrow 0$ and $\|\cdot\|_2$ denotes the standard 2-norm. Here

$$\begin{aligned} U_q(0) &= \mathrm{col}[U_{q_1}(0),\ U_{q_2}(0), \cdots, U_{q_n}(0)] \in \Re^{3n\times 1} \quad (80)\\ U_{qq}(0) &= \mathrm{diag}[U_{q_1q_1}(0),\ U_{q_2q_2}(0), \cdots, U_{q_nq_n}(0)] \in \Re^{3n\times 3n} \quad (81)\end{aligned}$$

where $U_{q_i}(0)$ and $U_{q_iq_i}(0)$ denote the gradient and gravity-gradient matrix of the potential function (with respect to $q_i \in \Re^{3\times 1}$) evaluated at $q_i = [0\ 0\ 0]^T$, respectively.

Taking the gradient of (79) and dropping quadratic and higher-order terms yields

$$U_q(q) \approx U_q(0) + U_{qq}(0)q \tag{82}$$

Note that (82) is a valid approximation under the condition that $||q||_2$ is small. Substituting (82) into (64) and rearranging we obtain the equations of motion of the formation *linearized about the reference orbit*:

$$\ddot{q} + 2\Omega\dot{q} + (M^{-1}U_{qq}(0) + \Omega^2 + \dot{\Omega})q + \left(\dot{\beta} + \Omega\beta + M^{-1}U_q(0)\right) = M^{-1}Q \tag{83}$$

where

$$M^{-1} = \text{diag}[\frac{1}{m_1}I,\ \frac{1}{m_2}I, \cdots, \frac{1}{m_n}I] \in \Re^{3n\times 3n} \tag{84}$$

It immediately follows from expanding (83) that the linearized equations of motion of the i^{th} spacecraft relative to the reference orbit are

$$\ddot{q}_i + 2[\omega]\dot{q}_i + \left(\frac{U_{q_iq_i}(0)}{m_i} + [\omega]^2 + [\dot{\omega}]\right) q_i + \left(\dot{b} + [\omega]b + \frac{U_{q_i}(0)}{m_i}\right) = \frac{Q_i}{m_i} \tag{85}$$

Recall that $q_i = [x_i\ y_i\ z_i]^T \in \Re^{3\times 1}$ and $Q_i = [Q_{x_i}\ Q_{y_i}\ Q_{z_i}]^T \in \Re^{3\times 1}$. The term $[\omega]^2$ in equation (85) can be alternately expressed using the matrix identity

$$[\omega]^2 = \omega\omega^T - (\omega^T\omega)I \tag{86}$$

The terms involving $\dot{\omega}$ and $\dot{b}$ are found from differentiating equations (28) and (29)

$$\dot{\omega} = [0\ 0\ \dot{\omega}_o]^T \in \Re^{3\times 1} \tag{87}$$

and

$$\dot{b} = [\ddot{R}_o\ \dot{R}_o\omega_o + R_o\dot{\omega}_o\ 0]^T \in \Re^{3\times 1} \tag{88}$$

The gradient of the potential function is given by

$$U_{q_i} = \left[\frac{\partial U}{\partial x_i}\ \frac{\partial U}{\partial y_i}\ \frac{\partial U}{\partial z_i}\right]^T \in \Re^{3\times 1} \tag{89}$$

where from (46):

$$\frac{\partial U}{\partial x_i} = \frac{\mu m_i (x_i + R_o)}{{R_i}^3} \tag{90}$$

$$\frac{\partial U}{\partial y_i} = \frac{\mu m_i y_i}{{R_i}^3} \tag{91}$$

$$\frac{\partial U}{\partial z_i} = \frac{\mu m_i z_i}{{R_i}^3} \tag{92}$$

Upon evaluating U_{q_i} at $q_i = [0\ 0\ 0]^T$ we find

$$U_{q_i}(0) = \begin{bmatrix} \frac{\mu m_i}{{R_o}^2} \\ 0 \\ 0 \end{bmatrix} \tag{93}$$

where $R_o = \|\vec{R}_o\|$.

The gravity-gradient matrix associated with the potential function is given by

$$U_{q_i q_i} = \begin{bmatrix} \frac{\partial^2 U}{\partial x_i^2} & \frac{\partial^2 U}{\partial y_i \partial x_i} & \frac{\partial^2 U}{\partial z_i \partial x_i} \\ \frac{\partial^2 U}{\partial x_i \partial y_i} & \frac{\partial^2 U}{\partial y_i^2} & \frac{\partial^2 U}{\partial z_i \partial y_i} \\ \frac{\partial^2 U}{\partial x_i \partial z_i} & \frac{\partial^2 U}{\partial y_i \partial z_i} & \frac{\partial^2 U}{\partial z_i^2} \end{bmatrix} \tag{94}$$

where

$$\frac{\partial^2 U}{\partial x_i^2} = \frac{\mu m_i}{{R_i}^3} - \frac{3\mu m_i (x_i + R_o)^2}{{R_i}^5} \tag{95}$$

$$\frac{\partial^2 U}{\partial y_i^2} = \frac{\mu m_i}{{R_i}^3} - \frac{3\mu m_i y_i}{{R_i}^5} \tag{96}$$

$$\frac{\partial^2 U}{\partial z_i^2} = \frac{\mu m_i}{{R_i}^3} - \frac{3\mu m_i z_i}{{R_i}^5} \tag{97}$$

and

$$\frac{\partial^2 U}{\partial y_i \partial x_i} = \frac{\partial^2 U}{\partial x_i \partial y_i} \tag{98}$$

$$= -\frac{3\mu m_i(x_i+R_o)y_i}{{R_i}^5} \tag{99}$$

$$\frac{\partial^2 U}{\partial z_i \partial x_i} = \frac{\partial^2 U}{\partial x_i \partial z_i} \tag{100}$$

$$= -\frac{3\mu m_i(x_i+R_o)z_i}{{R_i}^5} \tag{101}$$

$$\frac{\partial^2 U}{\partial z_i \partial y_i} = \frac{\partial^2 U}{\partial y_i \partial z_i} \tag{102}$$

$$= -\frac{3\mu m_i y_i z_i}{{R_i}^5} \tag{103}$$

Note that due to the equality of mixed partial derivatives $U_{q_iq_i}$ is a symmetric matrix.

Evaluating $U_{q_iq_i}$ at $q_i = [0\ 0\ 0]^T$ yields

$$U_{q_iq_i}(0) = \begin{bmatrix} \frac{-2\mu m_i}{{R_o}^3} & 0 & 0 \\ 0 & \frac{\mu m_i}{{R_o}^3} & 0 \\ 0 & 0 & \frac{\mu m_i}{{R_o}^3} \end{bmatrix} \tag{104}$$

We now express the reference orbit relative equations of motion of the formation explicitly in terms of scalar components. Upon expanding (85) it can be shown that

$$\ddot{x}_i - 2\omega_o\dot{y}_i - \frac{2\mu x_i}{R_o^3} - {\omega_o}^2 x_i - \dot{\omega}_o y_i + \ddot{R}_o - R_o{\omega_o}^2 + \frac{\mu}{R_o^2} = \frac{Q_{x_i}}{m_i} \tag{105}$$

$$\ddot{y}_i + 2\omega_o\dot{x}_i + \frac{\mu}{R_o^3}y_i - {\omega_o}^2 y_i + 2\omega_o\dot{R}_o + \dot{\omega}_o R_o = \frac{Q_{y_i}}{m_i} \tag{106}$$

$$\ddot{z}_i + \frac{\mu}{R_o^3}z_i = \frac{Q_{z_i}}{m_i} \tag{107}$$

Note that R_o and ω_o are considered prescribed time-varying functions in the above equations.

A useful control design model for many Earth orbiting formation flying applications results from assuming that the reference orbit in (105)-(107) is *circular.* In the case of a circular reference orbit, the angular-rate of the orbital frame (called the *mean motion*) is *constant* and satisfies the relationship

$$\omega_o^2 = \frac{\mu}{R_o^3} \tag{108}$$

Further $\dot{R}_o = 0$, and $\ddot{R}_o = 0$.

As a result, under the circular reference orbit assumption the linearized equations of motion are given by

$$\begin{aligned} \ddot{x}_i - 2\omega_o\dot{y}_i - 3{\omega_o}^2 x_i &= \frac{Q_{x_i}}{m_i} &(109)\\ \ddot{y}_i + 2\omega_o\dot{x}_i &= \frac{Q_{y_i}}{m_i} &(110)\\ \ddot{z}_i + \omega_o^2 z_i &= \frac{Q_{z_i}}{m_i} &(111) \end{aligned}$$

Equations (109)-(111) are commonly called the Hill-Clohessy-Wiltshire (HCW) equations [4].

5.2 Spacecraft Relative Equations of Motion

The linearized equations of motion of the formation can also be expressed in terms of the position of spacecraft j relative to spacecraft i. The linearized equations of motion for spacecraft j about the reference orbit are given by

$$\ddot{q}_j + 2[\omega]\dot{q}_j + \left(\frac{U_{q_j q_j}(0)}{m_j} + [\omega]^2 + [\dot{\omega}]\right) q_j + \left(\dot{b} + [\omega]b + \frac{U_{q_j}(0)}{m_j}\right) = \frac{Q_j}{m_j} \tag{112}$$

Subtracting the equations of motion of spacecraft i from the equation of motion of spacecraft j and rearranging results in

$$\ddot{\rho}_{ij} + 2[\omega]\dot{\rho}_{ij} + ([\omega]^2 + [\dot{\omega}] + \mathcal{G})\rho_{ij} = (\frac{Q_j}{m_j} - \frac{Q_i}{m_i}) \tag{113}$$

Under the circular reference orbit assumption the gravity-gradient matrix is given by

$$\mathcal{G} = \frac{U_{q_j q_j}(0)}{m_j} - \frac{U_{q_i q_i}(0)}{m_i} \tag{114}$$

$$= \omega_o^2 \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{115}$$

Although the linearized equations of motion (109)-(111) and (113) are utilized in a variety of formation flying scenarios [17], it is important to note that formation control laws based on linear control design models also have limitations. For example, in formation flying mission with long durations, large inter-spacecraft separations (e.g., long baseline optical interferometry), or significant reference orbital eccentricity, controllers designed via the linearized HCW equations may result in excessive fuel consumption [9],[17]. For longer-duration missions with non-circular orbital geometry, the use of non-linear control-design models becomes increasingly important.

6 Conclusions

In this paper, a complete matrix-based derivation via Lagrange's equations of the nonlinear equations of motion of a formation in LEO was given. Analytical models of the major disturbances acting on formations in LEO were also developed. The nonlinear equations of motion of the formation were then linearized with respect to an elliptical reference orbit. The resulting linear and nonlinear equations of motion are well-suited for use in the design and validation of formation guidance and control laws.

In future research, the impact of various curvilinear coordinate systems (e.g. orbital elements, spherical coordinates) on the structure of the open and closed-loop equations of motion will be addressed. The goal is then to select generalized coordinates that are optimal (e.g., as dictated by the formation geometry, sensor geometry/topology, control performance requirements, etc.) for various formation flying scenarios, or various phases within a single formation flying mission. For example, the development of advanced control architectures and nonlinear formation control laws will depend critically on both the (1) choice of generalized coordinates, and (2) the corresponding analytical structure of the equations of motion.

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