The Computation of the k Factor for Lossy Materials Around Resonance

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Abstract — An important property of a piezoelectric material for practical applications is its ability to generate and to detect stress waves, i.e. to convert electrical energy into mechanical energy and vice versa. As it is well known, the electromechanical coupling factor k fully characterize this energy conversion under static conditions. In a previous work we demonstrated that, like in the static conditions, it is possible to define the k factor in dynamic situations as ratio of energies (k_w); we also showed that k_w is proportional to the static material coupling factor (k_m) of the considered 1–D vibration mode, and the proportionality coefficient does not depend on the mode. In this work we show that the definition of k_w as ratio of energies can also be extended to lossy materials ($\boldsymbol{k}_w$); the losses are accounted by considering complex quantities for the elastic, dielectric and piezoelectric material constants. The obtained result is that $\boldsymbol{k}_w$ is proportional to the material coupling factor $\boldsymbol{k}_m$, which in this case is a complex parameter, and the proportionality coefficient is the same of the case without losses.

I. Introduction

In the current IEEE Standard on Piezoelectricity [1], section 5.4 defines the quasistatic material coupling factors (k_m) for a loss–less material in terms of energy ratio of a piezoelectric material subjected to specific boundary conditions. In the derivation of these factors a mechanical stress (or an electric field) is applied to an extreme value under a specific electrical (or mechanical) boundary condition and the energy density is calculated. At the extreme value the boundary condition is changed and the stress (or field) is removed and the energy density is again calculated for the decreasing applied stress (or field). The difference in the energy densities of the sample is then related to the conversion of mechanical to electrical (or electrical to mechanical) energy that occurred in the sample and delivered to an ideal electrical (or mechanical) load. This approach was initially developed in order to give a physical significance to these coupling factors, rather than just convenient constants that appear in various derivations. In a previous work we demonstrated that, like in static conditions, it is possible to define the k factor as ratio of energies also in dynamic situations [2]. Indeed for a loss less piezoelectric element in free oscillation, mechanically and electrically insulated, the k factor can be defined as the square root of the ratio of the converted electrical energy to the total energy involved in a transformation cycle, i.e. the kinetic energy. By means of 1–D distributed models, we showed that these results can be applied to piezoelectric elements vibrating both in a longitudinal (for example length extensional) and a transverse (for example length transverse) mode, and that the obtained results are proportional to the appropriate k_m [3]. In this work we extend these definitions to lossy materials: we compute $\boldsymbol{k}_w$ for a piezoelectric element vibrating in the length longitudinal mode and for an element vibrating in the length transverse mode.

II. The Dynamic Coupling Factor for Longitudinal Modes

In order to compute the electromechanical coupling coefficient when the piezoceramic element is in oscillation, let us consider, a piezoelectric bar of length l along z (or 3 direction), with its end faces electroded and with its cross section A of small radius, compared with the length (see Figure 1). Due to the 1–D geometry, the piezoceramic material axial isot-

ropy and the electrical and mechanical boundary conditions, the element can be described by a one dimensional stress–strain system and therefore by two scalar constitutive equations [4]:

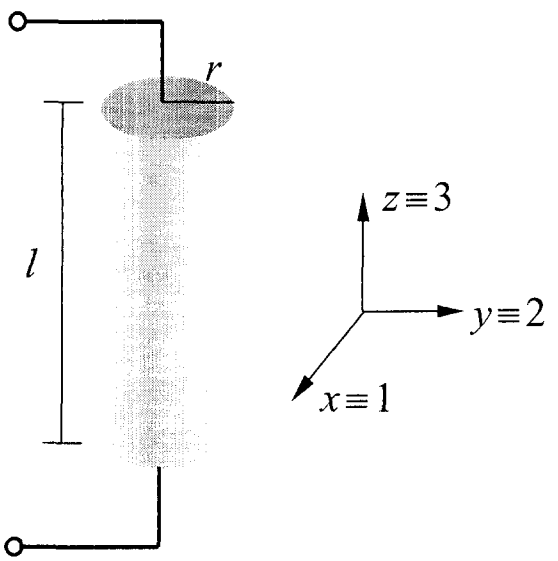

Fig. 1. Geometry of a piezoelectric bar vibrating in the length extensional mode.

$$T_3 = \frac{1}{s_{33}^D} S_3 - \frac{g_{33}}{s_{33}^D} D_3 \tag{1}$$

$$E_3 = -\frac{g_{33}}{s_{33}^D} S_3 + \frac{1}{\varepsilon_{33}^S} D_3 . \tag{2}$$

As it is well known, in lossy systems D and E, like S and T, are out of phase, therefore, in order to account losses, a complex notation can be introduced for the elastic, dielectric and piezoelectric constants [5, 6]:

$$s_{33}^D = s_{33r}^D + i\, s_{33i}^D = s_{33r}^D\left[1 + i\tan\left(\delta^D\right)\right] = \left|s_{33}^D\right| e^{i\delta^D} \tag{3}$$

$$\varepsilon_{33}^S = \varepsilon_{33r}^S + i\varepsilon_{33i}^S = \varepsilon_{33r}^S\left[1 + i\tan\left(\delta^S\right)\right] = \left|\varepsilon_{33}^S\right| e^{i\delta^S} \tag{4}$$

$$g_{33} = g_{33r} + i\, g_{33i} = g_{33r}\left[1 + i\tan\left(\delta^p\right)\right] = \left|g_{33}\right| e^{i\delta^p} . \tag{5}$$

For this propagation mode the wave equation can be written as [4]:

$$\rho \frac{\partial^2 \xi_3}{\partial t^2} = \frac{1}{s_{33}^D} \frac{\partial^2 \xi_3}{\partial z^2} , \tag{6}$$

where $\xi_3(z, t)$ is the particle displacement in the z direction. The propagation velocity is:

$$v_3 = \sqrt{\frac{1}{s_{33}^D \rho}} . \tag{7}$$

In order to solve the differential equation (6) we operate in the Laplace (s) domain; in this domain (6) becomes:

$$s^2 \Xi_3 = v_3^2 \frac{\partial^2 \Xi_3}{\partial z^2} , \tag{8}$$

where $\Xi_3(s, t)$ is the Laplace transformed of $\xi_3(z, t)$. Imposing stress–free conditions on the two terminal faces ($T_3(0) = T_3(l) = 0$) and supposing a constant (in the Laplace domain) excitation ($D_3 = D_0$), the solution of the wave equation (8) is:

$$\Xi_3(s, z) = \frac{g_{33} v_3}{s(1 + e^{\vartheta_3})} \left[e^{\frac{sz}{v_3}} - e^{\left(\vartheta_3 - \frac{sz}{v_3}\right)} \right] D_0 , \tag{9}$$

where $\vartheta_3 = (s\,l)/v_3$. From (9) the particle velocity and the strain in the z direction can easily be computed:

$$u_3(s, z) = s\,\Xi_3(s, z) = \frac{g_{33} v_3}{1 + e^{\vartheta_3}} \left[e^{\frac{sz}{v_3}} - e^{\left(\vartheta_3 - \frac{sz}{v_3}\right)} \right] D_0 , \tag{10}$$

$$S_3(s, z) = \frac{\partial \Xi_3(s, z)}{\partial z} = \frac{g_{33}}{1 + e^{\vartheta_3}} \left[e^{\frac{sz}{v_3}} - e^{\left(\vartheta_3 - \frac{sz}{v_3}\right)} \right] D_0 . \tag{11}$$

The electric field E_3 can be evaluated by the constitutive equation (2):

$$E_3(s, z) = \left\{ \frac{1}{\varepsilon_{33}^S} - \frac{g_{33}^2}{s_{33}^D \left(1 + e^{\vartheta_3}\right)} \left[e^{\frac{sz}{v_3}} - e^{\left(\vartheta_3 - \frac{sz}{v_3}\right)} \right] \right\} D_0 . \tag{12}$$

By integrating the electric field along z between 0 and l, we obtain the voltage between the two electroded surfaces of the bar:

$$V(s) = l\left[\frac{1}{\varepsilon_{33}^{S}} - \frac{g_{33}^{2}}{s_{33}^{D}}\frac{2}{\vartheta_3}\tanh\left(\frac{\vartheta_3}{2}\right)\right]D_0 . \qquad (13)$$

In a previous work [2] we demonstrated that, for a loss–less piezoelectric element in free oscillation, mechanically and electrically insulated, the coupling factor in dynamic conditions can be defined as the square root of the ratio between the converted electrical energy density (w_e) and the total energy density involved in a transformation cycle, i.e. the kinetic energy density (w_k):

$$k_w = \sqrt{\frac{w_e}{w_k}} . \qquad (14)$$

In the present case the energy densities w_k and w_e are complex quantities and can be computed by using (10) and (13):

$$w_k(s) = \frac{1}{2}\frac{\rho}{l}\int_0^l |u|^2 dz = \frac{1}{4}\frac{g_{33}^{2}}{s_{33}^{D}}\frac{\vartheta_3 - \sinh\vartheta_3}{\vartheta_3\cosh^2\frac{\vartheta_3}{2}}D_0^2 \qquad (15)$$

$$w_e(s) = \frac{1}{2Al}C_0|V|^2 = \frac{\varepsilon_{33}^{S}}{2}\left[\frac{1}{\varepsilon_{33}^{S}} - \frac{g_{33}^{2}}{s_{33}^{D}}\frac{2}{\vartheta_3}\tanh\left(\frac{\vartheta_3}{2}\right)\right]^2 D_0^2 , \qquad (16)$$

where $C_0 = \frac{\varepsilon_{33}^{S} A}{l}$ is the so called "clamped" capacitance of the element. According to IEEE Standard on piezoelectricity [1], in order to compute the electromechanical coupling coefficient, the element must be mechanically and electrically insulated from the surrounding, therefore the ratio w_e / w_k must be computed in this conditions. In the present analysis the element is mechanically insulated because we imposed stress free conditions; because the electrical boundary conditions ($D_3 = D_0$) impose an exciting current, the element can be considered electrically insulated when the input current goes to zero, or, equivalently, when the electrical input impedance goes to infinity. The electrical input impedance is given by:

$$Z_i(s) = \frac{1}{sC_0}\left[1 - \frac{g_{33}^{2}\varepsilon_{33}^{S}}{s_{33}^{D}}\frac{2}{\vartheta_3}\tanh\left(\frac{\vartheta_3}{2}\right)\right]. \qquad (17)$$

From previous equation we can be observe that the poles of the input impedance are:

$$s_{p0} = 0 \; ; \; s_{pn} = i\,n\,\pi\frac{v_3}{l} \qquad (18)$$

with n integer and positive. We are obviously interested to solutions s_{pn} and we can compute the electromechanical coupling factor as:

$$k_w^2 = \lim_{s\to s_{pn}}\frac{w_e}{w_k} = \frac{8}{\pi^2}\frac{g_{33}^{2}\varepsilon_{33}^{S}}{s_{33}^{D}} . \qquad (19)$$

This result has the same expression of that obtained in the case without losses (see [2]):

$$k_w^2 = \frac{8}{\pi^2}\frac{g_{33}^{2}\varepsilon_{33}^{S}}{s_{33}^{D}} ; \qquad (20)$$

in eqn. (19) are present the complex expressions of the material constants taking losses into account. It must be noted that the dynamic coupling factor is proportional to the material coupling factor, but it is smaller of about 10 % because in the dynamic case not all the involved energy is electrically (or mechanically) coupled, due to the sinusoidal electrical and mechanical variables distribution. The mechanical and dielectric losses are usually given in the material data sheets by piezoceramic manufacturers in terms of the mechanical quality factor Q_m and the dielectric loss $\tan\delta$; these parameters can be easily related to losses defined in (3) and (4):

$$\delta^D = \arctan(-\frac{1}{Q_m}) \qquad (21)$$

$$\delta^S = \arctan(-\tan\delta) \qquad (22)$$

and k_w can be expressed as a function of Q_m and $\tan\delta$:

$$\boldsymbol{k}_w^2 = k_w^2 \, e^{i\left[\arctan\left(\frac{1}{Q_m}\right) - \arctan(\tan\delta)\right]}. \qquad (23)$$

Figure 2 shows the behavior of the electromechanical coupling factor, normalized to the value without losses, as a function of mechanical losses, when $\tan\delta = 0$; we report $\boldsymbol{k}_w$ only for small Q_m values because only for $Q_m < 5$ there is an appreciable influence of the mechanical losses on the coupling factor.

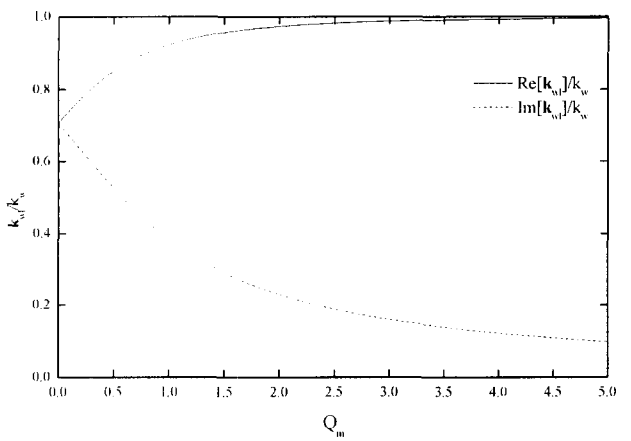

Fig. 2: Behavior of $\boldsymbol{k}_w$ (normalized to the value without losses) as a function of mechanical losses, when $\tan\delta = 0$.

It can be observed that the imaginary part of $\boldsymbol{k}_w$ is not negligible only for $Q_m < 2.5$ and therefore the coupling factor is practically a real quantity also when the mechanical losses are considered. Figure 3 shows the behavior of the electromechanical coupling factor, normalized to k_w, as a function of dielectric losses when $Q_m = \infty$; as it can be seen, also in this case the imaginary part of $\boldsymbol{k}_w$ is not negligible only for $\tan\delta$ values never shown by real piezoelectric materials and therefore the coupling factor can be considered a real quantity also when electric losses are accounted. As it is well known, in real piezoelectric materials are present both mechanical and electric losses and therefore it is interesting to study the behavior of the electromechanical coupling factor considering both these quantities. Figure 4 shows the behavior of $\boldsymbol{k}_w$, also in this case normalized to k_w, as a function of Q_m, with various values of $\tan\delta$. As it can be seen, the differences between the real part of $\boldsymbol{k}_w$ and k_w are not negligible only for Q_m values less than 5 and $\tan\delta$ values greater than 0.2; it must be also noted that the two losses compensate each other. In order to put in a more evidence the influence of the material losses on the coupling factor we computed $\Delta k = |\mathrm{Re}[\boldsymbol{k}_w] - k_w| \bullet 100$; Figure 5 shows the obtained result. As it can be seen, the electrical losses have a small influence on $\boldsymbol{k}_w$: for $Q_m > 3.2$ Δk varies less than 1 % when $\tan\delta$ goes from 0 to 0.2; further, Δk becomes more than the 10 % only for $Q_m < 0.9$. In Figure 5 it is more evident that the two losses compensate each other: for $Q_m < 5$, if we keep Q_m constant the increase of $\tan\delta$ produces a decrease of Δk.

II. The Dynamic Coupling Factor for Transverse Modes

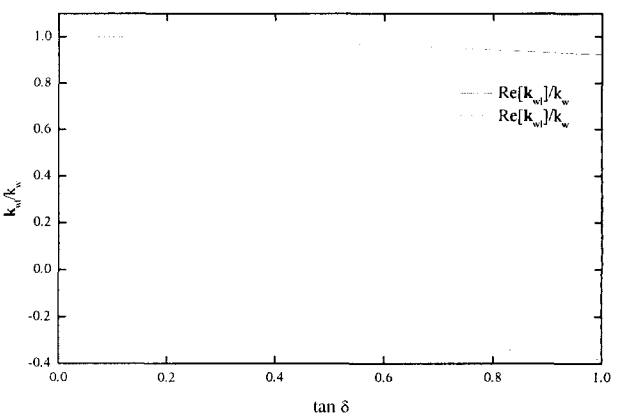

Fig. 3: Behavior of $\boldsymbol{k}_w$ (normalized to the value without losses) as a function of dielectric losses, when $Q_m = \infty$.

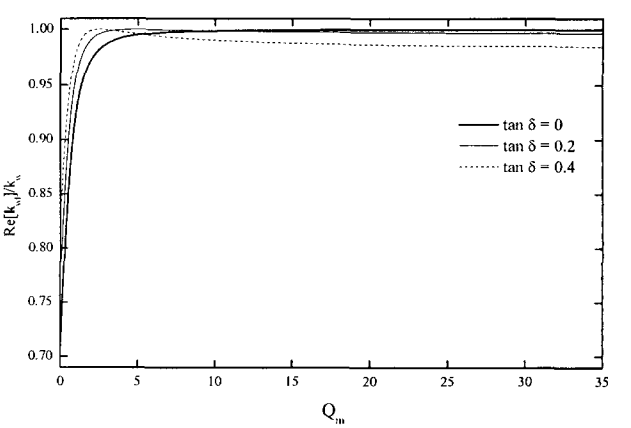

Fig. 4: Behavior of $\boldsymbol{k}_w$ real part (normalized to the value without losses) as a function of mechanical losses, computed with various values of $\tan\delta$.

In order to show that the previous results are independent on the wave propagation direction, we computed the dynamic coupling factor for a piezoelectric element vibrating in the length transverse mode. Let us consider a piezoelectric bar with its length d along the x direction, with the electroded surfaces normal to the z direction (the polarization direction) and with both cross–sectional dimensions a and b small compared with d (see Figure 6); applying the same approach as in the previous section and taking into account that the electrical boundary conditions impose a constant (in the Laplace domain) electric field ($\boldsymbol{E}_3 = \boldsymbol{E}_0$), the (total) kinetic energy density is:

$$w_k(s) = \frac{1}{4}\frac{d_{31}^2}{s_{11}^E}\frac{\vartheta_1 - \sinh\vartheta_1}{\vartheta_1 \cosh^2\frac{\vartheta_1}{2}} E_0^2, \tag{24}$$

while the (converted) potential electrical energy density is:

$$w_e(s) = \frac{\varepsilon_{33}^T}{2}\left[1 - k_{31}^2 + k_{31}^2 \frac{2}{\vartheta_1}\tanh\left(\frac{\vartheta_1}{2}\right)\right]^2 E_0^2, \tag{25}$$

where

$$s_{11}^E = s_{11r}^E + i\, s_{11i}^E = s_{11r}^E\left[1 + i\tan\left(\delta^E\right)\right] = \left|s_{11}^E\right| e^{i\delta^E} \tag{26}$$

$$d_{31} = d_{31r} + i d_{31i} = d_{31r}\left[1 + i\tan\left(\delta_1^P\right)\right] = \left|d_{31}\right| e^{i\delta_1^P} \tag{27}$$

$$\vartheta_1 = \frac{s\,d}{v_1}\ ;\ v_1 = \sqrt{\frac{1}{s_{11}^E\,\rho}}\,. \tag{28}$$

$$k_{31} = \frac{d_{31}}{\sqrt{\varepsilon_{33}^T\, s_{11}^E}} \tag{29}$$

is the static (material) coupling factor for this vibration mode, computed taking the losses into account, in fact it has the same expression given in [4], but with the complex material constants. In this case the electrical boundary conditions impose an exciting voltage and therefore the element can be considered insulated when the input voltage goes to zero, or

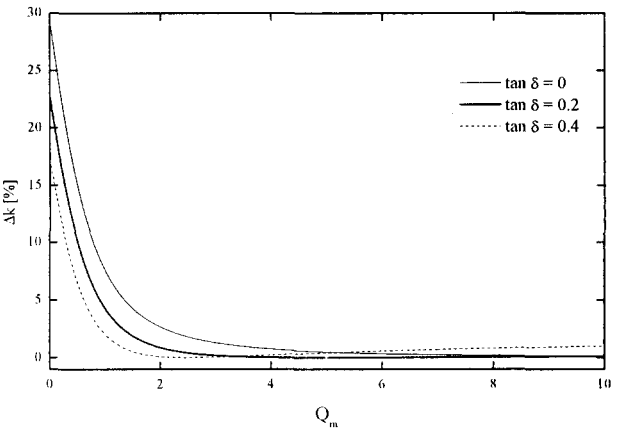

Fig. 5: Percent variation of the k_w real part in respect to k_w as a function of mechanical losses, computed with various values of the tanδ.

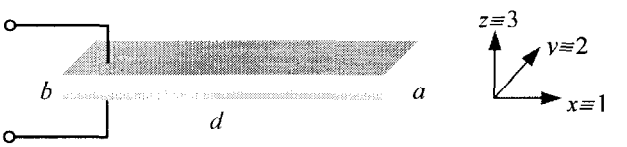

Fig. 6: Geometry of a piezoceramic element vibrating in the length transverse mode.

, equivalently, when the electrical input admittance $\boldsymbol{Y}_i(s)$ approaches infinity; the poles of $\boldsymbol{Y}_i(s)$ are:

$$s_{p0} = 0\ ;\ s_{pn} = i\,n\,\pi\frac{v_1}{l}\,. \tag{30}$$

The electromechanical coupling factor is given by:

$$k_w^2 = \lim_{s\to s_{pn}} \frac{w_e}{w_k} = \frac{8}{\pi^2} k_{31}^2\,. \tag{31}$$

We can conclude that, for one dimensional vibration modes, longitudinal or transverse, the dynamic coupling factor is related to the appropriate complex material coupling factor by the proportionality coefficient $\sqrt{8}/\pi$.

IV. Conclusions

In this work the definition of the electromechanical coupling factor (k) as ratio of energies, is extended to

the dynamic case and to lossy materials: we showed that the coupling factor can be computed as the square root of the ratio between the converted (electrical) and the total (kinetic) energy involved in a transformation cycle. We have showed that the k factor computed in dynamic conditions $\boldsymbol{k}_w$ is related to the static (material) parameter $\boldsymbol{k}_m$ by a proportionality coefficient:

$$\boldsymbol{k}_w = \frac{\sqrt{8}}{\pi} \boldsymbol{k}_m \,. \tag{32}$$

The proportionality coefficient takes into account that in the dynamic case not all the involved energy is coupled, due to the sinusoidal variables distribution. Both $\boldsymbol{k}_w$ and $\boldsymbol{k}_m$ are complex quantities related to the complex material parameters taking the losses into account. We computed both the material and the dynamic coupling factors as functions of mechanical and electrical losses; as expected from a physical point of view, both these parameters decrease increasing losses and the imaginary part of both $\boldsymbol{k}_w$ and $\boldsymbol{k}_m$ is comparable with the real part only for high loss materials ($Q_m < 5$ and $\tan\delta > 0.2$). The resumed results were obtained by means of the 1–D distributed models describing the vibration of a piezoelectric element in the length extensional and the length transverse mode, showing that (32) is independent on the vibration mode.

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