Observers for A Class of Systems with Nonlinearities Satisfying an Incremental Quadratic Inequality

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Abstract— **We consider the problem of state estimation for nonlinear time-varying systems whose nonlinearities satisfy an incremental quadratic inequality. Observers are presented which guarantee that the state estimation error exponentially converges to zero. Observer design is reduced to solving linear matrix inequalities for the observer gain matrices. Results are illustrated by application to a simple model of an underwater vehicle.**

I. INTRODUCTION

A fundamental problem in system analysis and control design is that of determining the state of a system from its measured output. Many solutions to this problem use an asymptotic observer (or state estimator) that produces an estimate of the system state which asymptotically approaches the system state. Typical observers for linear systems consist of a copy of the system dynamics along with a linear correction term based on the output error, that is, the difference between the measured output and its estimate based on the estimated state [12], [6].

References [3] and [4] consider a class of nonlinear systems whose nonlinearities have bounded derivatives. They present asymptotic observers which involve a copy of the system dynamics and two correction terms based on the output error; one term is the usual linear correction term, while the other term (called the nonlinear injection term) enters into the nonlinear element in the observer. Some other results on observers for nonlinear systems can also be found in [10] and [13].

In this paper, we consider nonlinear time-varying systems whose nonlinearities satisfy an incremental quadratic inequality. This inequality is characterized by a set $\mathcal{M}$ of multiplier matrices. The nonlinearities considered include many of the common nonlinearities including those considered in [3] and [4]. Section V exhibits some of the nonlinearities under consideration along with their multiplier matrices. For these systems, we present observers whose structure is inspired by [3] and [4]. These observers are characterized by two gain matrices: the gain matrix L for the linear correction term and the gain matrix L_n for the nonlinear injection term. Initially, we consider L_n fixed (for example zero) and convert the problem of determining L into that of solving linear matrix inequalities. Such inequalities can be readily treated using the LMI toolbox in MATLAB [8].

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We also consider the problem of simultaneously computing L and L_n. By imposing a specific condition on the set $\mathcal{M}$ of multiplier matrices describing the nonlinearities, we also convert the problem of simultaneously determining L and L_n into that of solving linear matrix inequalities.

All of our results are based on analysis on the state estimation error dynamics using quadratic Lyapunov functions. To illustrate our results, we apply the proposed observers to estimate the state of a simple model of an underwater vehicle from [11].

II. SYSTEM DESCRIPTION

We consider nonlinear/time-varying systems described by

$$\begin{aligned} \dot{x} &= Ax+Bu+B_p p(t,x,u) \\ y &= Cx+Du+D_p p(t,x,u) \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^l$ is the measured output and $t \in \mathbb{R}$ is the time variable. All the nonlinear/time-varying elements in the system are lumped into the term $p(t,x,u) \in \mathbb{R}^{l_p}$. We suppose that

$$p(t,x,u) = \psi(t,z) \qquad \text{where} \qquad z = C_q x + D_q u \tag{2}$$

and ψ is a piecewise continuous function of t and a continuous function of $z \in \mathbb{R}^{l_q}$. The matrices A,B,B_p,C,D,D_p and C_q,D_q are constant and of appropriate dimensions.

Our characterization of ψ is based on a set $\mathcal{M}$ of symmetric matrices which we refer to as multiplier matrices. Specifically, for all $M \in \mathcal{M}$, the following incremental quadratic inequality holds for all $t \in \mathbb{R}$ and $z_1,z_2 \in \mathbb{R}^{l_q}$:

$$\begin{pmatrix} q(t,z_2)-q(t,z_1) \\ \psi(t,z_2)-\psi(t,z_1) \end{pmatrix}^T M \begin{pmatrix} q(t,z_2)-q(t,z_1) \\ \psi(t,z_2)-\psi(t,z_1) \end{pmatrix} \geq 0 \tag{3}$$

where

$$q(t,z) = z + D_{qp}\psi(t,z)\,. \tag{4}$$

Basically, the constant matrix D_{qp} and $\mathcal{M}$ provide a characterization of ψ in a incremental sense. Section V exhibits some of the nonlinearities under consideration along with their multiplier matrices.

III. OBSERVERS

We propose the following observers to provide an estimate $\hat{x}$ of the state x of the system described by (1):

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x}+Bu+B_p\hat{p}+L(\hat{y}-y) \\ \hat{y} &= C\hat{x}+Du+D_p\hat{p} \end{aligned} \tag{5}$$

where

$$\hat{p} = \psi(t,\, \hat{z}+L_n(\hat{y}-y)) \quad \text{and} \quad \hat{z} = C_q\hat{x}+D_q u\,. \tag{6}$$

Here, L is the gain for the linear correction term and L_n is the gain for the nonlinear injection term, which was also introduced in [4]. The nonlinear injection term results in additional flexibility in the design. As an example of the usefulness of this term, suppose $z = y$. Then, $L_n = -I$ yields $\hat{z}+L_n(\hat{y}-y) = z$ and $\hat{p} = \psi(t,z)$; hence we have an exact copy of the nonlinearity in the observer.

In the observer description above, we have

$$\hat{p} = \psi(t,\, \hat{z}+L_n(C\hat{x}+Du-y)+L_nD_p\hat{p})\,.$$

When $L_nD_p \neq 0$, this is an implicit equation for $\hat{p}$. So, we assume that there is a continuous function ϕ such that for all t and η, the equation

$$\hat{p} = \psi(t, \eta + L_nD_p\hat{p}) \tag{7}$$

is uniquely solved by $\hat{p} = \phi(t,\eta)$. Then, $\hat{p}$ is uniquely given by

$$\hat{p} = \phi(t,\hat{z}+L_n(C\hat{x}+Du-y))\,. \tag{8}$$

The next section provides some sufficient conditions which guarantee the existence of ϕ. Introducing the the state estimation error,

$$e := \hat{x}-x\,, \tag{9}$$

we have

$$\begin{aligned} \hat{p} &= \psi(t,z+(C_q+L_nC)e+L_nD_p(\hat{p}-p)) \\ &= \phi(t,z+(C_q+L_nC)e-L_nD_pp) \end{aligned} \tag{10}$$

and the error dynamics are described by

$$\dot{e} = (A+LC)e+(B_p+LD_p)\delta p(t,e) \tag{11}$$

where

$$\delta p(t,e) = \phi\left(t,z(t)+(C_q+L_nC)e-L_nD_p\psi(t,z(t))\right)-\psi(t,z(t))\,.$$

It follows from (10) that δp satisfies

$$\delta p = \psi(t,z+(C_q+L_nC)e+L_nD_p\delta p)-\psi(t,z)\,.$$

Let

$$\delta q(t,e) = (C_q+L_nC)e+(D_{qp}+L_nD_p)\delta p(t,e)\,.$$

Then, using (3) with $z_1 = z$ and $z_2 = z+(C+L_nC_q)e+L_nD_p\delta p$ we obtain that for all t and e,

$$\begin{pmatrix} \delta q(t,e) \\ \delta p(t,e) \end{pmatrix}^T M \begin{pmatrix} \delta q(t,e) \\ \delta p(t,e) \end{pmatrix} \geq 0 \quad \text{for all} \quad M \in \mathcal{M}\,. \tag{12}$$

The following result yields conditions for observer gains which result in exponentially decaying estimation errors.

Theorem 1: Consider a system described by (1)-(2) and satisfying (3) with a set $\mathcal{M}$ of matrices. Suppose that there exist matrices $P = P^T > 0$, L, L_n and $M \in \mathcal{M}$ satisfying matrix inequality (13) (on the next page) for some $\alpha > 0$. Also suppose that there is a continuous function ϕ such that $\hat{p} = \phi(t,\eta)$ uniquely solves equation (7). Consider any input $u(\cdot)$ and initial condition $x(t_0) = x_0$ such that system (1) has a solution for all $t \geq t_0$. Then, the state estimation error $e = \hat{x}-x$ corresponding to observer (5) decays exponentially to zero with rate α.

Proof: The error dynamics given in (11) can be described by

$$\begin{aligned} \dot{e} &= A_ce+B_c\delta p(t,e) \\ \delta q &= C_ce+D_c\delta p(t,e)\,, \end{aligned}$$

where

$$\begin{aligned} A_c &:= A+LC\,, & B_c &:= B_p+LD_p\,, \\ C_c &:= C_q+L_nC\,, & D_c &:= D_{qp}+L_nD_p\,, \end{aligned} \tag{14}$$

and δp satisfies inequality (12) for all t, e, and $M \in \mathcal{M}$. Inequality (13) now simplifies to

$$\begin{pmatrix} A_c^TP+PA_c+2\alpha P & PB_c \\ B_c^TP & 0 \end{pmatrix} + \begin{pmatrix} C_c & D_c \\ 0 & I \end{pmatrix}^T M \begin{pmatrix} C_c & D_c \\ 0 & I \end{pmatrix} \leq 0.$$

Pre- and post-multiplying both sides of the above inequality by $[e^T\,\delta p^T]$ and its transpose and using condition (12) we obtain that

$$e^TP(A_ce+B_c\delta p(t,e)) \leq -\alpha e^TPe \quad \text{for all } t, e.$$

This shows that the error dynamics are quadratically stable about zero with rate α; (see [7], [2] and/or [5] for a definition of quadratic stability). This implies that the error decays exponentially to zero with rate α. ∎

The following corollary yields an observer design procedure for a given L_n.

Corollary 1: Consider a system described by (1)-(2) and satisfying (3) with a set $\mathcal{M}$ of matrices. For a given L_n, suppose that there exist matrices $P = P^T > 0$, R and $M \in \mathcal{M}$ such that the following matrix inequality holds:

$$\begin{pmatrix} PA+A^TP+RC+C^TR^T+2\alpha P & PB_p+RD_p \\ B_p^TP+D_p^TR^T & 0 \end{pmatrix} + \begin{pmatrix} C_c & D_c \\ 0 & I \end{pmatrix}^T M \begin{pmatrix} C_c & D_c \\ 0 & I \end{pmatrix} \leq 0\,, \tag{15}$$

where C_c and D_c are given in (14) and let

$$L = P^{-1}R\,. \tag{16}$$

Also suppose that there is a continuous function ϕ such that $\hat{p} = \phi(t,\eta)$ solves equation (7). Consider any input $u(\cdot)$ and initial condition $x(t_0) = x_0$ such that system (1) has a solution for all $t \geq t_0$. Then the state estimation error for the observer (5) decays exponentially to zero with a rate α.

Remark 1: Note that, for a fixed α and L_n, inequality (15) is an LMI (linear matrix inequality) in the variables P, R, and M.

A. On the Existence of a Solution to Equation (7)

As mentioned in the previous section, if $L_nD_p \neq 0$, then we need to be able to solve equation (7) for $\hat{p}$ to implement the observer. This equation defines an implicit relation for $\hat{p}$ in terms of t and η. Here $\eta = \hat{z}+L_n(C\hat{x}+Du-y)$. The following lemma provides a sufficient condition which

$$\begin{pmatrix} PA+A^TP+PLC+C^TL^TP+2\alpha P & PB_p+PLD_p \\ B_p^TP+D_p^TL^TP & 0 \end{pmatrix}+\begin{pmatrix} C_q+L_nC & D_{qp}+L_nD_p \\ 0 & I \end{pmatrix}^T M\begin{pmatrix} C_q+L_nC & D_{qp}+L_nD_p \\ 0 & I \end{pmatrix}\leq 0. \quad (13)$$

guarantees that, for each t and η, equation (7) has a solution $\hat{p}=\phi(t,\eta)$, where ϕ is continuous. Since the proof of this lemma is rather long, it is not presented here. See [1] for the proof.

Lemma 1: Suppose that ψ satisfies (3) for all t, z_1, z_2 and $M\in\mathcal{M}$. Given L_n, suppose there are matrices $M\in\mathcal{M}$ and Q and a scalar $\beta>0$ such that

$$\begin{pmatrix} -Q-Q^T+\beta I & Q \\ Q^T & 0 \end{pmatrix}+\begin{pmatrix} L_nD_p & D_{qp} \\ 0 & I \end{pmatrix}^T M\begin{pmatrix} L_nD_p & D_{qp} \\ 0 & I \end{pmatrix}\leq 0. \quad (17)$$

Then, there is a continuous function ϕ such that $\hat{p}=\phi(t,\eta)$ solves equation (7).

Remark 2: When $L_nD_p\neq 0$, Lemma 1 suggests that we can design an observer for a given L_n by simultaneously solving the LMIs (15) and (17) for P, M, R, Q and β. Then $L=P^{-1}R$. Consequently we obtain a well defined observer to estimate the states, because equation (7) has a continuous solution.

IV. A Condition for Simultaneous Design of L and L_n via LMIs

The previous section contains an observer design procedure where the observer gain L is designed for a fixed L_n. However, the simultaneous design of L and L_n is not addressed. The following condition, which is satisfied by many common nonlinearities (see Section V), allows for the simultaneous design of L and L_n.

Condition 1: There exist a nonsingular matrix T and a set $\mathcal{N}$ of matrix pairs (X,Y) with $Y\in\mathbb{R}^{m_p\times m_p}$ such that $X^T=X>0, Y^T=Y\geq 0$ and the matrix

$$M=T^T\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}T \quad (18)$$

is in $\mathcal{M}$. In addition, $T_{22}+T_{21}D_{qp}$ is nonsingular where

$$T=\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \quad (19)$$

and $T_{22}\in\mathbb{R}^{m_p\times m_p}$.

A. A Transformation

Suppose Condition 1 holds and note that

$$T\begin{pmatrix} q \\ p \end{pmatrix}=\begin{pmatrix} T_{11}z+\Gamma_{12}p \\ T_{21}z+\Gamma_{22}p \end{pmatrix}$$

where

$$\Gamma_{12}=T_{12}+T_{11}D_{qp}, \qquad \text{and} \qquad \Gamma_{22}=T_{22}+T_{21}D_{qp}.$$

and $q=z+D_{qp}p$. Now introduce the transformed nonlinear term $\tilde{p}$ defined by

$$\tilde{p}:=T_{21}z+\Gamma_{22}p. \quad (20)$$

Since by assumption, Γ_{22} is nonsingular, we have

$$p=-\Gamma_{22}^{-1}T_{21}z+\Gamma_{22}^{-1}\tilde{p}; \quad (21)$$

hence $T_{11}z+\Gamma_{12}p=\tilde{z}+\tilde{D}_{qp}\tilde{p}$ where

$$\tilde{z}=\Sigma z, \quad \Sigma=T_{11}-\Gamma_{12}\Gamma_{22}^{-1}T_{21}, \quad \tilde{D}_{qp}=\Gamma_{12}\Gamma_{22}^{-1}. \quad (22)$$

We now show that that Σ is invertible. Note that

$$\begin{pmatrix} T_{11} & \Gamma_{12} \\ T_{21} & \Gamma_{22} \end{pmatrix}=\begin{pmatrix} T_{11} & T_{12}+T_{11}D_{qp} \\ T_{21} & T_{22}+T_{21}D_{qp} \end{pmatrix}=T\begin{pmatrix} I & D_{qp} \\ 0 & I \end{pmatrix}.$$

Since the two matrices on the righthandside of the second equality are invertible, the matrix on the lefthandside of the first equality is invertible. Since Γ_{22} is assumed to be invertible, by using the matrix inversion lemma [9], [14], the first matrix above is invertible if and only if the following Schur complement of the matrix is invertible:

$$T_{11}-\Gamma_{12}\Gamma_{22}^{-1}T_{21}=\Sigma.$$

This implies that Σ is invertible. Consequently, $z=\Sigma^{-1}\tilde{z}$ and

$$\tilde{p}(t,x,u)=\tilde{\psi}(t,\tilde{z}):=T_{21}\Sigma^{-1}\tilde{z}+\Gamma_{22}\psi(t,\Sigma^{-1}\tilde{z}). \quad (23)$$

Letting

$$\tilde{q}(t,\tilde{z})=\tilde{z}+\tilde{D}_{qp}\tilde{\psi}(t,\tilde{z})$$

we obtain that

$$T\begin{pmatrix} q(t,z_2)-q(t,z_1) \\ \psi(t,z_2)-\psi(t,z_1) \end{pmatrix}=\begin{pmatrix} \tilde{q}(t,\tilde{z}_2)-\tilde{q}(t,\tilde{z}_1) \\ \tilde{\psi}(t,\tilde{z}_2)-\tilde{\psi}(t,\tilde{z}_1) \end{pmatrix}.$$

Hence satisfaction of inequality (3) by ψ implies that the transformed nonlinear function $\tilde{\psi}$ satisfies

$$\begin{pmatrix} \tilde{q}(t,\tilde{z}_2)-\tilde{q}(t,\tilde{z}_1) \\ \tilde{\psi}(t,\tilde{z}_2)-\tilde{\psi}(t,\tilde{z}_1) \end{pmatrix}^T\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}\begin{pmatrix} \tilde{q}(t,\tilde{z}_2)-\tilde{q}(t,\tilde{z}_1) \\ \tilde{\psi}(t,\tilde{z}_2)-\tilde{\psi}(t,\tilde{z}_1) \end{pmatrix}\geq 0. \quad (24)$$

Now, using the transformed term $\tilde{\psi}$, system (1) is described by

$$\begin{aligned} \dot{x} &= \tilde{A}x+\tilde{B}u+\tilde{B}_p\tilde{\psi}(t,\tilde{z}) \\ y &= \tilde{C}x+\tilde{D}u+\tilde{D}_p\tilde{\psi}(t,\tilde{z}) \\ \tilde{z} &= \tilde{C}_qx+\tilde{D}_qu \end{aligned} \quad (25)$$

where $\tilde{\psi}$ satisfies (24) and

$$\begin{aligned} &\tilde{A}=A-\tilde{B}_pT_{21}C_q, \quad \tilde{B}=B-\tilde{B}_pT_{21}D_q, \quad \tilde{B}_p=B_p\Gamma_{22}^{-1} \\ &\tilde{C}=C-\tilde{D}_pT_{21}C_q, \quad \tilde{D}=D-\tilde{D}_pT_{21}D_q, \quad \tilde{D}_p=D_p\Gamma_{22}^{-1}, \\ &\tilde{C}_q=\Sigma C_q, \qquad \tilde{D}_q=\Sigma D_q. \end{aligned} \quad (26)$$

B. Observer for the Transformed System

Based upon the previous section, we propose the following observers for the transformed system (25):

$$\begin{aligned}\dot{\hat{x}} &= \tilde{A}\hat{x}+\tilde{B}u+\tilde{B}_p\hat{p}+L(\hat{y}-y) \\ \hat{y} &= \tilde{C}\hat{x}+\tilde{D}u+\tilde{D}_p\hat{p} \\ \hat{p} &= \tilde{\psi}(t,\,\hat{z}+L_n(\hat{y}-y)) \\ \hat{z} &= \tilde{C}_q\hat{x}+\tilde{D}_q u\end{aligned} \tag{27}$$

In the observer description we have,

$$\hat{p}=\tilde{\psi}(t,\,\hat{z}+L_n(\tilde{C}\hat{x}+\tilde{D}u-y)+L_n\tilde{D}_p\hat{p})\,.$$

So, when $L_n\tilde{D}_p\neq 0$, we again assume that there is a continuous function ϕ such that for all t and η, the equation

$$\hat{p}=\tilde{\psi}(t,\eta+L_n\tilde{D}_p\hat{p}) \tag{28}$$

is uniquely solved by $\hat{p}=\phi(t,\eta)$. Then,

$$\hat{p}=\phi(t,\hat{z}+L_n(\tilde{C}\hat{x}+\tilde{D}u-y))\,. \tag{29}$$

Now, we can present the main result of this section, which is a corollary to Theorem 1.

Corollary 2: Consider a system described by (1)-(2) and satisfying (3) with a set $\mathcal{M}$ of matrices which satisfy Condition 1. Suppose that there exist matrices $P=P^T>0$, R_1, R_2 and $(X,Y)\in\mathcal{N}$ which satisfy (30) (on the next page) and let

$$L=P^{-1}R_1\,,\qquad L_n=X^{-1}R_2\,. \tag{31}$$

Also suppose that there is a continuous function ϕ such that $\hat{p}=\phi(t,\eta)$ solves equation (28). Consider any input $u(\cdot)$ and initial condition $x(t_0)=x_0$ such that system (1) has a well defined solution for all $t\geq t_0$. Then, the state estimation error, $e=\hat{x}-x$, decays exponentially to zero with a rate of α.

Proof: Substitute (31) into inequality (30) and apply a Schur complement result [5] to obtain

$$\begin{pmatrix} P\tilde{A}+\tilde{A}^TP+PL\tilde{C}+\tilde{C}^TL^TP+2\alpha P & P\tilde{B}_p+PL\tilde{D}_p \\ \tilde{B}_p^TP+\tilde{D}_p^TL^TP & 0\end{pmatrix}+$$
$$\begin{pmatrix}\tilde{C}_c & \tilde{D}_c\\ 0 & I\end{pmatrix}^T\begin{pmatrix}X & 0\\ 0 & -Y\end{pmatrix}\begin{pmatrix}\tilde{C}_c & \tilde{D}_c\\ 0 & I\end{pmatrix}\ \leq\ 0.$$

where $\tilde{C}_c:=\tilde{C}_q+L_n\tilde{C}$ and $\tilde{D}_c:=\tilde{D}_{qp}+L_n\tilde{D}_p$. The result now follows by applying Theorem 1 to the transformed system. ■

Remark 3: Note that, for a fixed α, inequality (30) is an LMI (linear matrix inequality) in the variables P, R_1, R_2, X and Y.

When $L_n\tilde{D}_p\neq 0$, the following corollary of Lemma 1 presents an LMI which guarantees a continuous solution to equation (28) for $\hat{p}$ (see [1] for a proof).

Corollary 3: Suppose that $\tilde{\psi}$ satisfies (24) for all t, z_1, z_2 and $(X,Y)\in\mathcal{N}$. Given L_n, suppose that there are matrices $(X,Y)\in\mathcal{N}$, R_2, Q and and a scalar $\beta>0$ such that,

$$\begin{pmatrix} -Q-Q^T+\beta I & Q & \tilde{D}_p^TR_2^T \\ Q^T & -Y & \tilde{D}_{qp}^TX \\ R_2\tilde{D}_p & X\tilde{D}_{qp} & -X\end{pmatrix}\leq 0 \quad\text{and}\quad R_2=XL_n. \tag{32}$$

Then there is a continuous function ϕ such that $\hat{p}=\phi(t,\eta)$ satisfies (28).

Remark 4: When $L_n\tilde{D}_p\neq 0$, Corollary 3 tells that we can design the observer gains L and L_n by simultaneously solving LMI's (30) and (32) for P, R_1, R_2, X, Y, Q and β. Then $L=P^{-1}R_1$ and $L_n=X^{-1}R_2$.

V. Some Classes of Nonlinearities Satisfying (3)

In this section, we discuss some typical nonlinearities satisfying (3). We also present additional conditions under which these nonlinearities satisfy Condition 1.

A. Incrementally Sector Bounded Nonlinearities

Here we consider nonlinearities which, for all t_1, t_2, and z_1, z_2, satisfy

$$(\delta\psi-K_1\delta q)^TX(K_2\delta q-\delta\psi)\geq 0\quad\text{for all } X\in\mathcal{X}\,, \tag{33}$$

where

$$\delta\psi:=\psi(t_2,z_2)-\psi(t_1,z_1)\,,\,\delta q:=\delta z+D_{qp}\delta\psi,\,\delta z:=z_2-z_1\,, \tag{34}$$

$\mathcal{X}$ is a set of symmetric positive definite matrices and K_1, K_2 are fixed matrices. Here, without loss of generality, we assume that the set $\mathcal{X}$ is invariant under multiplication by a positive number. It readily follows from (33) that a set $\mathcal{M}$ of multiplier matrices for the nonlinearities under consideration is given by

$$\mathcal{M}=\left\{\begin{pmatrix} -K_1^TXK_2-K_2^TXK_1 & (K_1+K_2)^TX\\ X(K_1+K_2) & -2X\end{pmatrix}\,:\,X\in\mathcal{X}\right\}.$$

To satisfy Condition 1, suppose that there exists a positive scalar σ such that $S_1-\sigma S_2$ is nonsingular where $S_1:=K_2D_{qp}-I$ and $S_2:=I-K_1D_{qp}$. One can verify by substitution that the following equality holds

$$2\begin{pmatrix} -K_1^TXK_2-K_2^TXK_1 & (K_1+K_2)^TX\\ X(K_1+K_2) & -2X\end{pmatrix}=T^T\begin{pmatrix}X & 0\\ 0 & -\frac{1}{\sigma}X\end{pmatrix}T,$$

where

$$T=\begin{pmatrix}\frac{1}{\sqrt{\sigma}}K_2-\sqrt{\sigma}K_1 & -\frac{1}{\sqrt{\sigma}}I+\sqrt{\sigma}I\\ K_2+\sigma K_1 & -I-\sigma I\end{pmatrix}.$$

Here $\Gamma_{22}=S_1-\sigma S_2$ is nonsingular. Therefore, Condition 1 is satisfied with the matrix T defined above and

$$\mathcal{N}=\left\{(X,\,\frac{1}{\sigma}X)\,:\,X\in\mathcal{X}\right\}.$$

When q and p are scalars, one can always choose a positive scalar σ such that $S_1-\sigma S_2$ is nonzero. To prove this claim, note that if $S_1-\sigma S_2$ is zero for all $\sigma>0$ then, $S_1=S_2=0$. In this case, $K_1=K_2=1/D_{qp}$ and $\delta\psi=K\delta q$ where $K:=K_1=K_2$. Using $\delta q=\delta z+D_{qp}\delta\psi$ and $\delta\psi=K\delta q$, we have $\delta z=(1-D_{qp}K)\delta q=0$. However, δz should be arbitrary; hence we cannot have $S_1=S_2=0$. Consequently, Condition 1 is always satisfied by $\mathcal{M}$ in the scalar case.

As a specific example of a nonlinearity under consideration, consider a globally Lipschitz nonlinearity which satisfies $\|\delta\psi\|\leq\gamma\|\delta q\|$ for some $\gamma>0$. In this case, inequality (33) holds with $K_1=-\gamma I$, $K_2=\gamma I$ and $\mathcal{X}=\{\lambda I:\ \lambda>0\}$.

$$\begin{pmatrix} \tilde{A}^T P + P\tilde{A} + R_1\tilde{C} + \tilde{C}^T R_1^T + 2\alpha P & P\tilde{B}_p + R_1\tilde{D}_p & \tilde{C}_q^T X + \tilde{C}^T R_2^T \\ \tilde{B}_p^T P + \tilde{D}_p^T R_1^T & -Y & \tilde{D}_{qp}^T X + \tilde{D}_p^T R_2^T \\ X\tilde{C}_q + R_2\tilde{C} & X\tilde{D}_{qp} + R_2\tilde{D}_p & -X \end{pmatrix} \leq 0 . \tag{30}$$

B. Incrementally Positive Real Nonlinearities

This class of nonlinearities is described by a set $\mathcal{X}$ of symmetric positive definite matrices X such that for all t_1, t_2 and z_1, z_2,

$$\delta q^T X \delta\psi \geq 0 \qquad \text{for all } X \in \mathcal{X}, \tag{35}$$

where δq and $\delta\psi$ are as defined in (34). It is clear from (35) that, without loss of generality, we can assume that the set $\mathcal{X}$ of matrices is invariant under multiplication by a positive scalar. Note that nondecreasing nonlinearities satisfy (35) with $\mathcal{X} = \{\lambda I : \lambda > 0\}$. It readily follows from (35) that a set $\mathcal{M}$ of multiplier matrices for the nonlinearities under consideration is given by

$$\mathcal{M} = \left\{ \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} \ : \ X \in \mathcal{X} \right\}.$$

To satisfy Condition 1 choose any scalar $\sigma > 0$ such that $D_{qp} - \sigma I$ is nonsingular. Then, we can readily show that

$$2\begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{\sigma}} I & \sqrt{\sigma} I \\ I & -\sigma I \end{pmatrix}^T \begin{pmatrix} X & 0 \\ 0 & -\frac{1}{\sigma} X \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\sigma}} I & \sqrt{\sigma} I \\ I & -\sigma I \end{pmatrix}.$$

Consequently, if we let

$$T = \begin{pmatrix} \frac{1}{\sqrt{\sigma}} I & \sqrt{\sigma} I \\ I & -\sigma I \end{pmatrix}, \qquad \mathcal{N} = \left\{ (X, \ \frac{1}{\sigma} X) \ : \ X \in \mathcal{X} \right\},$$

then $\Gamma_{22} = D_{qp} - \sigma I$ is nonsingular and Condition 1 holds.

C. Nonlinearities with Polytopic Parameterizations

These nonlinearities are assumed to satisfy

$$\delta\psi = \Omega(s)\delta q \tag{36}$$

where $\delta\psi$ and δq are defined in (34), $s := (t_1, t_2, z_1, z_2)$ and

$$\Omega(s) \in Co\{\Omega_1, \ldots, \Omega_\nu\} \quad \text{for all } s,$$

that is, for any s, $\Omega(s) = \sum_{k=1}^{\nu} \lambda_k \Omega_k$, where $\lambda_k \geq 0$ and $\sum_{k=1}^{\nu} \lambda_k = 1$.

Since $\delta\psi = \Omega(s)\delta q$, a symmetric matrix M satisfies (3) if

$$\begin{pmatrix} I \\ \Omega(s) \end{pmatrix}^T M \begin{pmatrix} I \\ \Omega(s) \end{pmatrix} \geq 0 \quad \text{for all} \quad s.$$

Let

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{pmatrix}$$

where partitioning is in accordance with $(\delta q, \delta\psi)$. Then the above inequalities can be expressed as

$$M_{11} + M_{12}\Omega(s) + \Omega^T(s) M_{12}^T + \Omega^T(s) M_{22} \Omega(s) \geq 0 \ \text{ for all } s.$$

Consider now those matrices M which satisfy $M_{22} \leq 0$. When $D_{qp} = 0$, we need only consider this case. With $M_{22} \leq 0$, the above inequalities can be shown to be equivalent to:

$$M_{11} + M_{12}\Omega_k + \Omega_k^T M_{12}^T + \Omega_k^T M_{22} \Omega_k \geq 0 \ \text{for } k = 1, \ldots, \nu. \tag{37}$$

Thus, the set $\mathcal{M}$ of symmetric matrices M which satisfy

$$\begin{pmatrix} I \\ \Omega_k \end{pmatrix}^T M \begin{pmatrix} I \\ \Omega_k \end{pmatrix} \geq 0 \ \text{ for } k = 1, \ldots, \nu \ \text{ and } M_{22} \leq 0, \tag{38}$$

is a set of multipliers matrices.

The above set of multiplier matrices does not necessarily satisfy Condition 1 with a single transformation T. To obtain a set satisfying Condition 1, choose any nonsingular matrix T and consider multiplier matrices of the form given in (18) where X and Y are symmetric matrices with $X > 0$ and $Y \geq 0$. A matrix M of this structure satisfies inequalities (37) if and only if X and Y satisfy

$$\begin{aligned} \begin{pmatrix} I \\ \Omega_k \end{pmatrix}^T T^T \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} T \begin{pmatrix} I \\ \Omega_k \end{pmatrix} &\geq 0 \ \text{ for } k = 1, \ldots, \nu \\ T_{12}^T X T_{12} - T_{22}^T Y T_{22} &\leq 0. \end{aligned} \tag{39}$$

Then, provided $T_{22} + T_{21} D_{qp}$ is invertible, Condition 1 is satisfied with

$$\mathcal{N} = \{(X, Y) : X^T = X > 0 \text{ and } Y^T = Y \geq 0 \text{ satisfy (39)}\}.$$

Once T is chosen, (39) is a set of linear matrix inequalities in X and Y. However, the choice of T to yield a large subset of multipliers in some sense is not clear. Therefore, T is treated as a design parameter at this point. For example, the simple choice of $T = I$ satisfies Condition 1 with $\mathcal{N}$ defined by

$$\mathcal{N} = \{(X, Y) : X^T = X > 0 \text{ and } Y^T = Y \geq 0 \text{ satisfy (40)}\}$$

where

$$X - \Omega_k^T Y \Omega_k \geq 0 \quad \text{for } k = 1, \ldots, \nu. \tag{40}$$

VI. An Example: Underwater Vehicle

In this section we consider a simple model of an underwater vehicle dynamics. This example is taken from [11] where a similar objective of designing observers is considered in a different framework. A simplified model of the vehicle is given by

$$\begin{aligned} \ddot{\phi}_1 &= -3\dot{\phi}_1|\dot{\phi}_1| + u \\ \ddot{\phi}_2 &= \dot{\phi}_1|\dot{\phi}_1| - 10\dot{\phi}_2|\dot{\phi}_2|, \end{aligned}$$

where ϕ_1 is propeller angle, ϕ_2 is vehicle position and u is the torque input to the propeller. It is assumed that we can only measure ϕ_1 and ϕ_2. The angular velocity $\dot{\phi}_1$ of propeller and the speed $\dot{\phi}_2$ of the vehicle will be estimated using an observer. In this model, $\dot{\phi}_1|\dot{\phi}_1|$ represents the propeller thrust and $10\dot{\phi}_2|\dot{\phi}_2|$ represents the hydraulic drag on the vehicle.

Introducing the state $x = (\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2)$, and the output $y = (\phi_1, \phi_2)$, and letting $p = (\dot{\phi}_1|\dot{\phi}_1|, \dot{\phi}_2|\dot{\phi}_2|)$, we can write this system in state space form (1) with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad B_p = \begin{pmatrix} 0 & 0 \\ -3 & 0 \\ 0 & 0 \\ 1 & -10 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad D_p = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

With $z = (x_2, x_4)$, the nonlinear term is described by (2) where

$$\psi(t,z) = \begin{pmatrix} z_1|z_1| \\ z_2|z_2| \end{pmatrix}, C_q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, D_q = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that the nonlinear function given by $f(\nu) = \nu|\nu|$ is a nondecreasing function. Considering $D_{qp} = 0$, the nonlinear term here is an incrementally positive real nonlinearity satisfying (35) with $\mathcal{X}$ being the set of matrices X of the form

$$X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \text{with} \qquad \lambda_1, \lambda_2 > 0.$$

Therefore, we can design an observer using the results in Corollary 2. Solving the linear matrix inequalities using the LMI toolbox in MATLAB [8], the observer gains obtained for $\alpha = 4$ are

$$L = \begin{pmatrix} -9.4678 & -0.0134 \\ -21.6510 & 0.3072 \\ -0.0039 & -19.0395 \\ -0.2699 & -211.0569 \end{pmatrix}, \; L_n = \begin{pmatrix} -4.4758 & 0.0189 \\ -0.3196 & -13.0741 \end{pmatrix}.$$

A two second simulation was carried out with initial state $x(0) = (0, 0, 0, 5)$, initial state estimate, $\hat{x}(0) = (0, 4, 0, -10)$, and control input

$$u(t) = \begin{cases} 5 & \text{for } 0 \le t < 1 \\ -10 & \text{for } 1 \le t < 2 \end{cases}.$$

In these simulations, dotted lines represent the state estimate which converged to the vehicle state in less than 0.5 seconds.

VII. Conclusions

We consider the problem of state estimation for nonlinear time-varying systems whose nonlinearities satisfy an incremental quadratic inequality. We demonstrate that many common nonlinear/time-varying terms satisfy such an inequality. We present observers which guarantee that the resulting state estimation error exponentially converges to zero. Observer design involves solving linear matrix inequalities (LMIs) for the observer gain matrices. These LMIs can be efficiently treated using commercially available software. Results are illustrated by application to a simple model of an underwater vehicle.

The results of this paper will be useful in obtaining observer based output feedback controllers for systems with nonlinear/time-varying terms satisfying an incremental quadratic inequality.

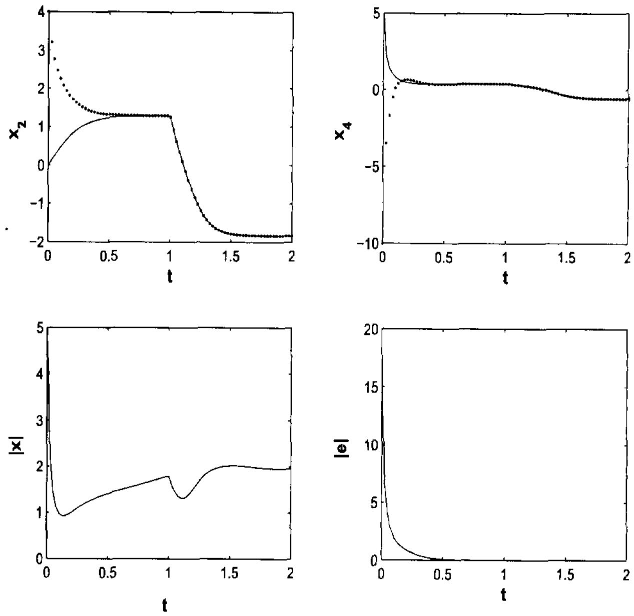

Fig. 1. Estimating the state of an underwater vehicle

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