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MINIMUM IMPULSE TRANSFERS TO ROTATE THE LINE OF APSIDES

Connie Phong and Theodore H. Sweetser*

While an optimal scenario for the general two-impulse transfer between coplanar orbits is not known, there are optimal scenarios for various special cases. We consider in-plane rotations of the line of apsides. Numerical comparisons with a trajectory optimization program support the claim that the optimal ΔV required by two impulses is about half that required by a single impulse, regardless of semi-major axes. We observe that this estimate becomes more conservative with larger angles of rotation and eccentricities, and thus also present a more accurate two-impulse rotation ΔV estimator.

INTRODUCTION

While an optimal scenario for the general two-impulse transfer between coplanar orbits is not known, there are optimal scenarios for various special cases. In this paper we examine the case of an in-plane rotation of an elliptical orbit.

A single impulse can be used to rotate the line of apsides (the major axis) of an orbit of eccentricity e and semi-major axis a through an angle $\Delta\omega$ in the plane of the orbit, where the size and shape of the orbit remain constant, since the initial and final orbits intersect. Specifically we can apply an impulse of magnitude

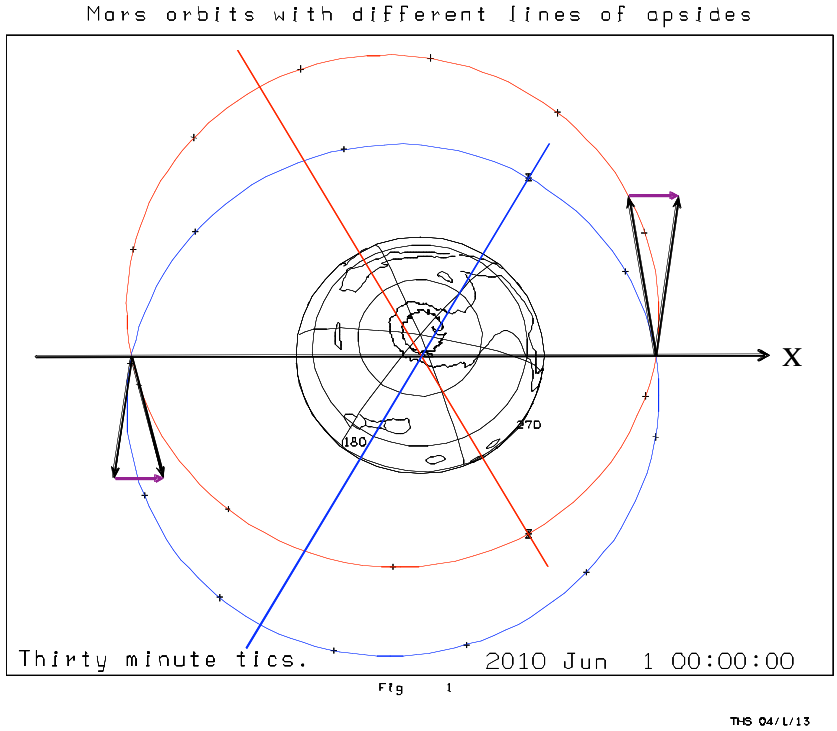
$$\Delta V_{single} = 2e\sin\left(\frac{\Delta\omega}{2}\right)\sqrt{\frac{\mu}{a(1-e^2)}} \tag{1}$$

either at the intersection of the orbits near apoapse or at the intersection near periapse as shown in Figure 1. Specifically Figure 1 shows a 120º rotation of apsides where a single impulse at position 1 (near periapse) rotates the blue orbit to the red orbit, while an impulse at position 2 (near apoapse) rotates the red orbit to the blue orbit.

Optimization of such a single impulse transfer, however, is not possible since the transfer is completely constrained by the initial and final orbits. Two-impulse transfers, on the other hand, are possible between any two terminal orbits, and while optimal scenarios are not known for the general two-impulse case there are various approximate solutions to many special cases. Accordingly, a rule of thumb used for first level mission

* Senior Engineer, Jet Propulsion Laboratory, California Institute of Technology

analysis suggests that a two-impulse transfer to rotate lines of apsides uses about half the total ΔV of a single impulse transfer for the same rotation.

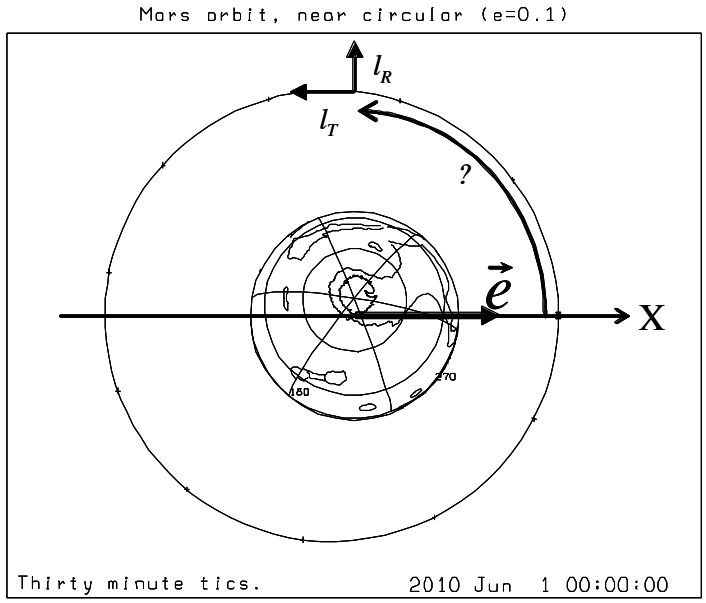


The source for this rule of thumb is unknown and the original justification for it remains unclear. It is possible that the basis of the rule of thumb comes from Edelbaum's consideration of the problem of minimum impulse transfers for nearly circular orbits (Edelbaum, 1967). Following Edelbaum, we linearize the variation of parameter equations about a circular reference orbit to obtain the equations of motion and focus primarily on

$$\frac{de_y}{du} = \sqrt{\frac{a}{\mu}}\left(2l_T \sin\theta - l_R \cos\theta\right) \tag{2}$$

where e_y is the y component of the eccentricity vector e, u is the time integral of thrust acceleration, θ is the angle of the maneuver point from the X-axis, and l_T and l_R are the circumferential and radial components of the direction vector of the maneuver respectively as illustrated in Figure 2. From Eq. (2) it is clear that for nearly circular orbits a circumferential maneuver at $\theta = 90°$ is twice as effective as a radial maneuver at periapse. This simple observation seems to have led to our rule of thumb that a two-impulse transfer requires maneuvers whose total magnitude (ΔV_{RoT}) is half the magnitude of a single-impulse transfer to rotate the line of apsides:

$$\Delta V_{RoT} = \frac{\Delta V_{single}}{2} = e\sin\left(\frac{\Delta\omega}{2}\right)\sqrt{\frac{\mu}{a(1-e^2)}} \tag{3}$$



Applicability of the Rule of Thumb to Elliptical Orbits

Thus Edelbaum directly supports the applicability of the rule of thumb to near-circular orbits with circumferential maneuvers. It has been assumed that the rule holds for more general elliptical orbits, but the accuracy and limitations of the approximation have not previously been addressed.

It was Lawden (1962) who first considered the problem of optimal slewing of the orbital axis for elliptical orbits. In agreement with our rule of thumb he states and assumes, without further discussion, that a two-impulse transfer is more economical than a single impulse transfer. He then presents an algorithmic solution to the problem of optimizing the two-impulse transfer that requires the satisfaction of six simultaneous equations to yield the magnitude, true anomaly, and direction of the symmetric impulses. Unfortunately this process is tedious and iterative and gives little insight as to the relationship between eccentricity and ΔV requirement. It therefore remains of limited practical usefulness for first level mission design analysis.

More recently Baker (1995) has derived an explicit approximate solution to Lawden's problem in which the impulses are assumed to be strictly circumferential but maintain the symmetry and equality of magnitude of Lawden's optimized impulses. Baker's solution provides the locations of the impulse points as well. However, his solution has the disadvantage of being more complex and less accessible than the rule of thumb.

$$\Delta V_{Baker} = \frac{e\sin(\Delta\omega/2)F}{\sqrt{(1-e^2)}\sin(f_{opt}-\Delta\omega/2)}\sqrt{\frac{\mu}{a}} \tag{4}$$

$$F = 1 + \frac{1}{2}\frac{e\sin f_{opt}}{\tan(f_{opt}-\Delta w/2)} - \frac{\frac{1}{4}e\sin(\Delta\omega/2)}{\sin(f_{opt}-\Delta\omega/2)} \tag{5}$$

$$f_{opt} \approx 90 + \frac{\Delta\omega}{2} + \frac{\frac{1}{2}e\cos(\Delta\omega/2)}{1+\frac{1}{2}e\sin(\Delta\omega/2)} \tag{6}$$

Our rule of thumb, while not optimal and heuristically only applicable to near-circular orbits and circumferential maneuvers, is at least a non-iterative solution and relatively uncomplicated. We seek to understand how well the rule of thumb applies to more eccentric orbits and how it compares to Baker's solution.

ANALYSIS

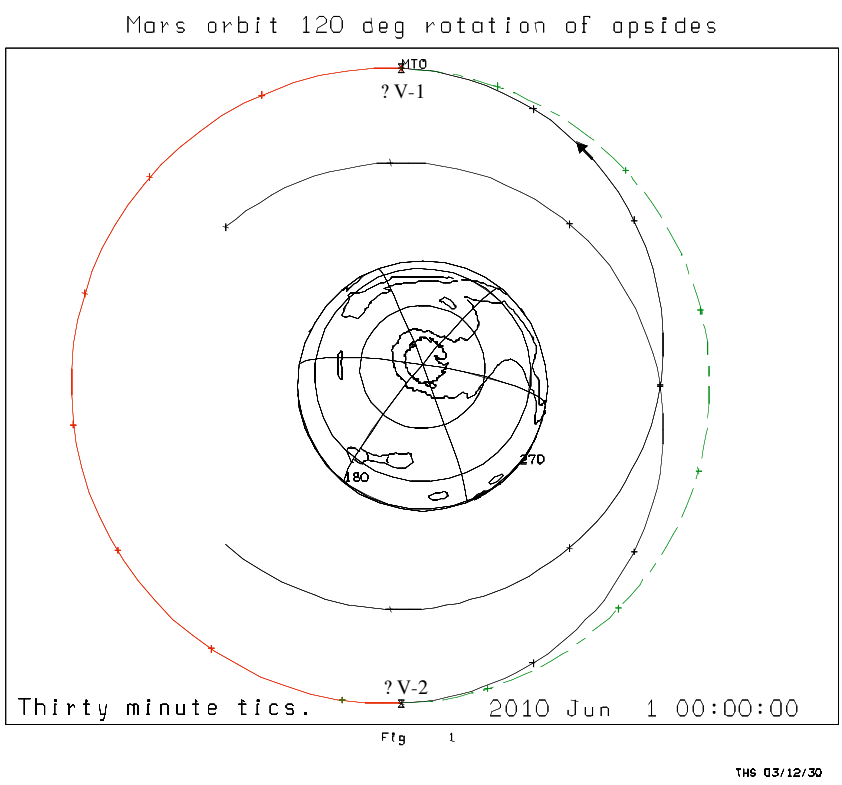
The accuracy of the rule of thumb as a function of semi-major axis and eccentricity was examined via numeric comparisons with results from the trajectory optimization program CONSAT (Version 3.14). CONSAT, developed at JPL by Carl Sauer, optimizes transfers between different orbits around a central body by using patched conic analysis and finite parameter optimization and uses primer vector theory to add maneuvers if such additional maneuvers can reduce the total ΔV. A slightly inclined (i = 10°, Ω = 0°), near-circular orbit (a = 5000, e = 0.15) of Mars (r = 3397 km) was used as a baseline. We used CONSAT to vary the positions of the maneuvers (defined by their true anomalies) on the initial and final orbits to minimize the total ΔV for rotations around the orbit normal by amounts ranging from $\Delta\omega$ = 10° to $\Delta\omega$ = 340°; these rotations were then repeated for a = 7000 km and for four other eccentricities. Table 1 presents a summary of the cases considered and the results of the comparison.

Figure 3 shows an actual transfer scenario from CONSAT outputs for a 120º rotation of apsides. Notice that the positions of ΔV-*1* and ΔV-*2* are symmetric about the major axis of the transfer ellipse. In all cases CONSAT also found the optimum to use exactly two symmetric and nearly circumferential burns as predicted by the rule of thumb.

Δ? (degrees)	a = 7400 e = 0.15	a = 5000 e = 0.15	a = 7400 e = 0.2	a = 5000 e = 0.2	a = 7400 e = 0.4	a = 5000 e = 0.4	a = 7400 e = 0.6	a = 5000 e = 0.6	a = 7400 e = 0.8	a = 5000 e = 0.8
10	0.993	0.993	0.998	0.989	0.961	0.961	0.908	0.908	0.794	0.794
20	0.990	0.990	0.984	0.984	0.952	0.952	0.893	0.893	0.771	0.771
40	0.983	0.983	0.976	0.976	0.935	0.935	0.865	0.865	0.729	0.729
60	0.978	0.978	0.968	0.968	0.919	0.919	0.840	0.840	0.696	0.696
80	0.972	0.972	0.961	0.961	0.905	0.905	0.820	0.820	0.670	0.670
100	0.968	0.968	0.955	0.955	0.894	0.894	0.803	0.803	0.650	0.650
120	0.964	0.964	0.951	0.951	0.885	0.885	0.791	0.791	0.635	0.635
140	0.962	0.962	0.947	0.947	0.878	0.878	0.782	0.782	0.626	0.626
160	0.960	0.960	0.945	0.945	0.874	0.874	0.777	0.777	0.620	0.620
180	0.959	0.959	0.944	0.944	0.873	0.873	0.775	0.775	0.618	0.618
200	0.960	0.960	0.945	0.945	0.874	0.874	0.777	0.777	0.620	0.620
220	0.962	0.962	0.947	0.947	0.878	0.878	0.782	0.782	0.626	0.626
240	0.964	0.964	0.951	0.951	0.885	0.885	0.791	0.791	0.635	0.635
260	0.968	0.968	0.955	0.955	0.894	0.894	0.803	0.803	0.650	0.650
280	0.972	0.972	0.961	0.961	0.905	0.905	0.820	0.820	0.670	0.670
300	0.978	0.978	0.968	0.968	0.919	0.919	0.840	0.840	0.696	0.696
320	0.983	0.983	0.976	0.976	0.935	0.935	0.865	0.865	0.729	0.729
340	0.990	0.990	0.984	0.984	0.952	0.952	0.893	0.893	0.771	0.771

Table 1. Ratio of the optimal transfer ΔV to the estimated ΔV_{RoT} given by the rule of thumb.

As expected from the heuristic argument derived from Edelbaum the rule of thumb is excellent for nearly circular orbits with only a maximum 4% error on the solution. The rule of thumb, however, quickly becomes less precise with increasing eccentricity. Yet at least it consistently remains a conservative approximation to the optimal ΔV. Table 1 also shows identical ratios for each value of a at every value of e, indicating that rule of thumb is independent of semi-major axis. It is also clear that across all eccentricities the error increases with the magnitude of rotation (if we think of rotations greater than 180º as negative rotations) and is greatest at the maximum possible rotation, $\Delta\omega = 180°$.



Bounding the Error

Thus the percentage error for the rule of thumb is dependent on the eccentricity and the degree of rotation of the line of apsides. We seek to bound the error by addressing these factors and developing corrections based on the empirical data collected from the CONSAT runs.

Although we do not know an analytical formula for the general two-impulse case, there is an analytical formula for an optimal 180° rotation. Lawden (1962) also found that for intersecting initial and final orbits that have aligned axes, the optimal transfer orbit will be tangential to both the initial and final orbits at apses and will pass through the farthest apse. In the case of a 180° rotation the optimal transfer orbit will thus begin with a circularization at the apoapse of the initial ellipse before a restoration of the periapse altitude 180° later at the point which becomes the apoapse of the final ellipse. The total ΔV required is given by

$$\Delta V_{180} = 2\left(1-\sqrt{1-e}\right)\sqrt{\frac{\mu}{a(1+e)}} \tag{7}$$

With a fair amount of algebraic manipulation we find that for $\Delta\omega = 180°$ the ratio of Eq. (7) to the rule of thumb given by Eq. (3) is

$$\frac{\Delta V_{180}}{\Delta V_{RoT}} = \frac{\sqrt{1-e}+\sqrt{1-e}}{1+\sqrt{1-e}} \tag{8}$$

The ratio given by Eq. (8) agrees with the values listed in Table 1 for a 180° rotation up to five decimal places. A quadratic fit using this eccentricity fraction and with endpoints (the points of no rotation: $\Delta\omega = 0°$ and $\Delta\omega = 360°$) set to 1.000 would give a good fit through the data points at low eccentricities. But at higher eccentricities the fit is only tight around $\Delta\omega = 180°$.

In order to gain a better fit through the data across all eccentricities we needed to bring down the values at the endpoints. We first extrapolated ratios for $\Delta\omega = 0°$ from their corresponding values at $\Delta\omega = 10°$ and $\Delta\omega = 20°$. It was then observed that

$$0.63e \leq (1 - \text{ratio}_{\Delta\omega=0})/(1 - \text{ratio}_{\Delta\omega=180}) \leq 0.56e \tag{9}$$

So we choose to simply bring down the endpoints by half the eccentricity, which is easy to remember and maintains a certain amount of conservatism. A quadratic through these endpoints results in curves that better fit our data, as shown in Figures 4 and 5 where the data points are the data from Table 1 and the gray curves are the quadratic fits through the centers and adjusted endpoints. Therefore we can improve the accuracy of the rule of thumb by including this additional eccentricity factor and adjustment so that

$$\Delta V_{imp} = \left\{ \left(\frac{\Delta\omega - 180}{180} \right)^2 (1 - 0.5e) + \frac{\sqrt{1-e} + \sqrt{1-e}}{1 + \sqrt{1-e}} \left[1 - \left(\frac{\Delta\omega - 180}{180} \right)^2 (1 - 0.5e) \right] \right\} \cdot \Delta V_{RoT} \qquad (10)$$

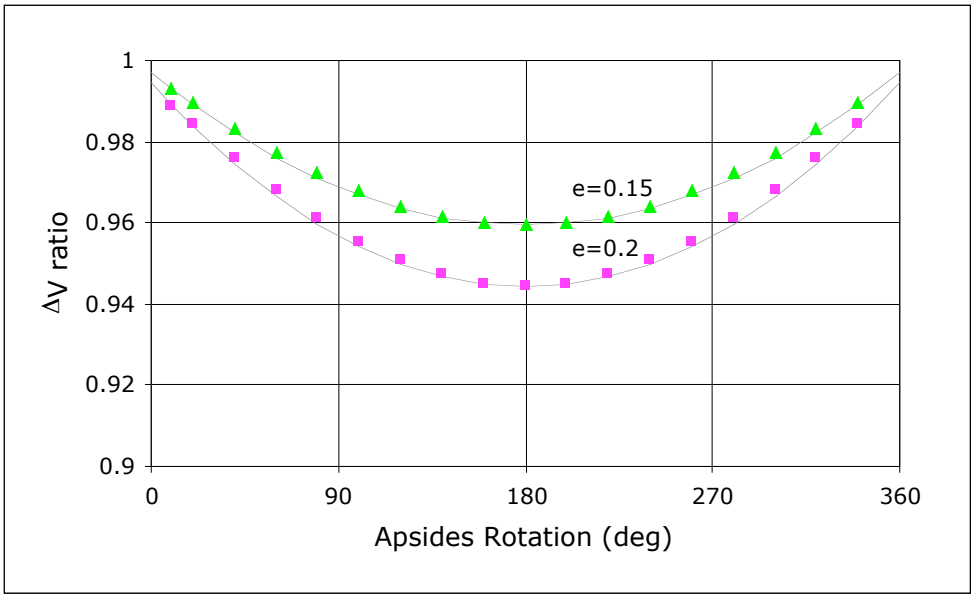


Figure 4. Optimal ΔV vs. the improved rule of thumb ΔV at lower eccentricities.

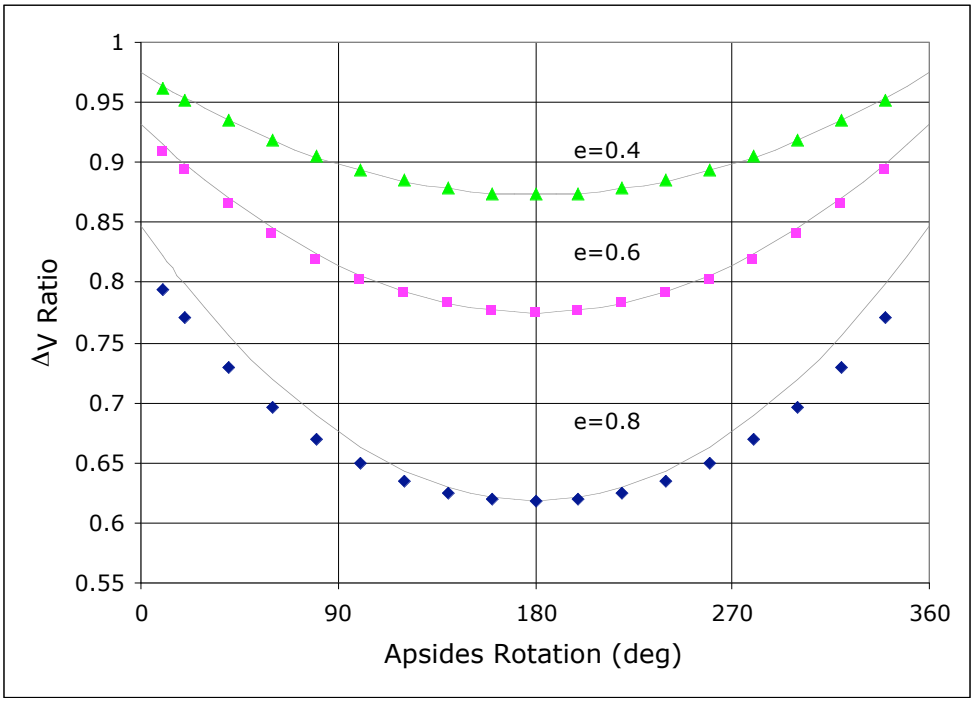


Figure 5. Optimal ΔV vs. the improved rule of thumb ΔV at higher eccentricities.

The Improved Rule of Thumb vs. Baker's Approximate Solution

Baker derived his approximate solution starting from the energy equation. It is thus interesting to compare it against our purely empirically improved rule of thumb. Figure 6 plots our CONSAT results against all ΔV approximations discussed thus far at a few different eccentricities. Notice that it includes a "Simplified Baker" approximation as well as a "Baker" approximation. Unlike Eq. (10) Baker's ΔV approximation, Eq. (4), includes a true anomaly term, and his optimal true anomaly is essentially $90° + \Delta\omega/2$ + a small correction term. In Figure 6, the Simplified Baker curve disregards the correction term whereas the Baker curve includes it.

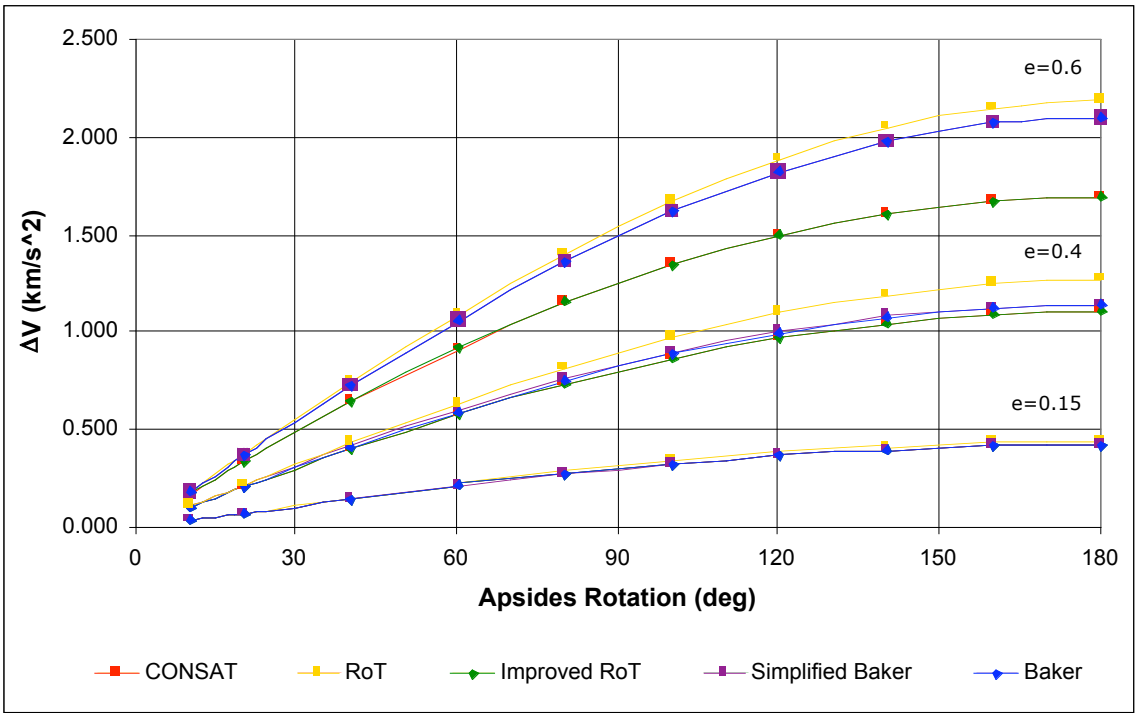


Figure 6: Comparison of various ΔV approximations to CONSAT results.

Not surprisingly for all eccentricities our original rule of thumb is the most conservative approximation and the error only increases with eccentricity. At lower eccentricities the difference between Baker's approximation and our simplification of it are also negligible. It is only at $e = 0.6$ when a clear differentiation of the approximation curves appear. This difference is also somewhat expected since Baker assumes low eccentricities for his derivation. Notice that our improved rule of thumb remains consistently better than both forms of Baker's approximation since we have excluded an explicit true anomaly term. Overall our confidence in Eq. (10), which has a purely empirical basis, is supported by ΔV approximations that are very similar to Baker's carefully derived analytical solutions.

The rule of thumb, our improvement to the rule, and Baker's solution are all for circumferential maneuvers (at least heuristically for the rules of thumb). Baker also computed Lawden's solutions for a few low eccentricity cases and compared them to his approximations as well as to Karrenberg's approximate solution which assumes tangential burns. He found that his circumferential burns more accurately approximated Lawden's solutions in all cases.

CONCLUSION

We have documented here an old and rather arcane bit of astrodynamics folklore, which gives an easy rule of thumb for the ΔV needed to rotate an orbit's line of apsides in the plane of the orbit: the optimal transfer uses circumferential maneuvers at the semi-latus rectum of an orbit halfway between the beginning and ending orbits and each circumferential maneuver is one-fourth as large as the velocity difference where the beginning and ending orbits cross. By comparing this result to actual optimal transfers we have found an improved estimation formula that also compares well with a more complicated analytic approximation that has specific impulse directions. These more complicated approximations can be used when more precision is needed in the design of space missions.

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