Searching for the second harmonic

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Abstract

To mathematically mode] the oscillations of a spherical bubble, either the time-dependent volume V(t) or the radius R(t) can be used as the dependent variable. A recent exchange between the advocates of the volume approach [Zabolotskaya, E.A., J. Acoust. Sot. Am. *94,* 2448 *(1993)]* and the radius approach [Wu, J. and G, Du, J. Acoust. Sot. Am. *94,* 2446 *(1993)]* concerns an apparent discrepancy in the predicted nonlinear acoustic cmi ssions from a bubble subject to a t ime-periodic external pressure field. This Letter attempts to resolve the discrepancy via an examination of the arguments offered by both sides. In addition, some general comments on nonlinear harmonic and subharmonic emissions from acoustical y-forced bubble oscillations are presented.

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I. The debate.

It is rather well documented experimentally that both single bubbles and bubble fields oscillate nonlinearly.[1] One consequence of those nonlinear oscillations is that the radiated sound from bubbles will also contain harmonics and, in some cases, subharmonic. The Fourier spectrum of the radiated sound can, properl y interpreted, yield useful information about the bubble mechanics, and about other parameters of both the external driving field and material conditions of the bubble. Many researchers have used the presence of one or more of the spectral components arising in acoustical experiments as proof of a (or many) bubble's presence and/or activity.

It is therefore important that any model purporting to mathematically describe the dynamics of bubbles reproduce the spectral features of the experimental observable. Any model that, for the same parametrical conditions, failed to yield (at least qualitatively) the same spectrum would be considered simply deficient with respect to the prediction of observable radiative effects,

1 wish to turn to a recent series of exchanges between two groups who have considered models of periodically-forced bubble oscillations.[3] Their discussions relate to the differing predictions of the "radial displacement" (RD) and the "volume displacement" (VD) approaches. The VD camp obtains a different expression for the radiated acoustic pressure from the bubble at the second harmonic frequency than the RD camp. The purpose of this letter is to resolve the apparent discrepancy between the two approaches, while not entering into the meleé proper.

Some background illustrating how the different groups arrive at their results is necessary. *Both* models use as a framework the hydrodynamic theory of bubble oscillations developed by Rayleigh, Plesset, Noltingk, Neppiras and Poritsky[4], commonly referred to as the RP equation. This model follows from the Navier-Stokes fluid equations if one assumes incompressible, irrotational flow with normal-stress-continuity boundary conditions including surface tension and bulk viscous stress. The present discussion is limited to the inviscid model, although this will have no relevance to the conclusions drawn. The two groups part ways over the proper choice of the dependent variable, ZS1 and DW1 giving the respective justifications for their choice. As may be surmised, the RD approach uses the bubble radius R(t) as the dependent variable, while the VD approach uses the bubble volume V(t).

In DW1, the authors compare the two approaches. They note (as ZS 1 had pointed out) that at a particular acoustic driving frequency $\omega^* = ((1+\eta)\omega_0)$, with η (1 for isothermal, γ for adiabatic oscillations) the polytropic exponent, and ω_0 the bubble's fundamental linear resonance frequency) the amplitude of the second-harmonic component of the radiated acoustic field from the bubble goes to zero for the VD model, but not for the RD model. DW1 argue that the VD approach is *inadequate* primarily by showing that an equivalent expression to the RD result can be derived by approaching the VD derivation differently.

As an aside in IZ2, the authors argue that DW1 incorrectly infer a radiated pressure from a near field term which should, according to IZ2, decay as $1/r^4$, and thus not contribute to the sound field. They maintain that the zero-amplitude result is correct for the far-field radiation.

In the most recent exchange of letters (WD2 and Z3) both sets of authors reiterate their positions, the arguments centering around the issue. of compressibility (which is, as regards the zero-amplitude result, beside the point). In an apparent attempt to reconcile the two approaches and the fact that experiments show a non-zero second-harmonic component, Z3 complicates the situation by arguing that the $P_2 = 0$ result arises from the internal gas nonlinearity cancelling the RP equation nonlinearity at ω^*. Further, Z3 says that the near-field hydrodynamic nonlinear term can yield the missing 2nd harmonic (implying that experiments were measuring near-field hydrodynamic fluctuations, and not radiated acoustic pressure!)

The main aim of this Letter is to point out that this ongoing debate concerns essentially a *straw man,* born of injudicious application of perturbation methods within the modelling process. Both the VD and RD approaches, the reader will recall, share the same physics, subject to the limiting assumptions involved in arriving at the RP equation. If only substitution between $R(t)$ and $V(t)$ is performed, then the results of integrating either equation of motion will yield *exactly equivalent results.* Note that even though the substitution involves a cubic nonlinearity, it is both smooth and invertible over the allowed (open and strictly real) domain of R and V , which is $(0,\infty)$.

Why does the process break down? The reason, simply stated, is that the arguments leading to an expression for the bubble's second harmonic pressure amplitude P_2 utilize a perturbation approach: expansion followed by truncation. The results obtained following the truncation *do not preserve the original transformation*, and hence should not even be expected to yield the same behavior. Specifically, an expansion of the form $R = R_0 + R'$, when substituted into the RP equation and followed by truncation of resulting terms of 0(>2) in R differs from performing the same process with an expansion of the form $V = V_0$ 4 V', truncating at O(>2) in V. The resulting expressions are not correct to the *same* order in the *same* variables. in fact, in this inst ante, the exactness of the geometry has been compromi sed !

11. The physics briefly considered.

So far I have limited myself to clarifying the confusion over the source of the apparent discrepancy between the VD and RD approaches. But which approach yields the better approximation for the radiated acoustic field? The first line of attack would be a detailed comparison with a numerical solution to the full RP equat ion, Much work has been done in this area.[5] in particular, Eatock has calculated the second-hamonic velocity component (both radius and volume velocity) for the full solution as a function of changing driving frequency. For driving frequencies spanning m* (from ω_0 up to $3\omega_0$), there is no indication of a z,ero-amplitude result, although the relative magnitude of the second-hamonic does vary, governed by the resonance behavior of the bubble. It must be noted, however, that his driving pressure consisted of short pulse trains, and thus steady-state oscillations were not attained.

Ultimately, the question can only be decided by comparison with experiment. There are man y experiments involving bubbles exposed to acoust ic waves.6 'J "he on] y experiments to date which can be compared exactly to the expressions developed for the bubble motion are the single bubble experiments of Holt and Crum.[1] Results there show a changing but non-zero second-harmonic component. However, those experiments were for driving frequencies up to $0.8\omega_0$, and thus do not include ω^*, which is greater than ω_0. Single bubble experiments involving purely volume oscillations driven *above* their linear resonance frequency have not been performed.

III. The big picture.

A larger question which is not treated by either set of authors is why one might mathematically or physically expect the 2nd harmonic to ever disappear as the driving frequency is changed (or, equivalently, as the bubble size is changed in a fixed driving frequency experiment). To address this question, we must first understand why a second harmonic appears at all in the solution to a time-dynamic evolution process forced at a single frequency.

The generic answer to this question is nonlinearity. Specifically, a quadratic nonlinearity in the model equations describing the physics will yield a solution (analytically, the intrepid reader can verify) varying at 2ω. If we rewrite the invi scid version of the RP equation to remove the interpretation of radius and wall velocity by simply substituting x for R, and y for dR/dt, we obtain two nonlinear coupled first-order ODE's:

$$dx/dt = y$$

$$dy/dt = (-3/2)y^2/x + (P_0 R_0^{-3\eta})\, x^{3\eta-1} - P_0(1 + \varepsilon\cos(\omega t)).$$

We thus we have a quadratic nonlinearity in y, and either a quadratic or an approximately cubic nonlinearity in x, depending on the value of η. It is clear that we expect a 2ω component in the solution.

IV. The debate reengaged.

Considering all the available evidence, there is no *physical or mathematical reason* why, for any driving pressure and frequency (assuming a fixed equilibrium bubble radius) a particular harmonic would disappear. Except, of course, the asymptotic limits of very small driving pressure and very high order harmonics. The case is different for subharmonics, since (considering only purely volume oscillations) they only arise via a saddle-node bifurcation, and thus can have exactly zero amplitudes for finite, sharply demarcated regions of the driving parameter space. While Z3's suggestion that "one nonlinearity cancels the other, leading to the zero amplitude result" at ω^* is intriguing, it is rather to be viewed as an *artifact* of the VD truncation than a feature of bubble dynamics,

However, being a thoroughgoing empiricist and experimentalist, I would consider the question open until the experiment has been performed, and 1 offer a brief description of a way one could go about it.

Optically levitate[7] a single, suitably small bubble in water. 'levitate", because the bubble must be isolated to provide the correct boundary conditions for the RP equation; "single" for the same reason. "Suitably" means small enough so that the pressure threshold for onset of stable shape oscillations is finite, but large enough to provide a large optical cross-section at visible wavelengths.

Determine the bubble's static radius optically, or via rise-time techniques. Drive the bubble into volume oscillations with a standing or traveling acoustic wave at about a tenth of an atmosphere. Using a lock-in amplifier, monitor the amplitude and phase of the second-harmonic component of the scattered light from the bubble due to an incident laser beam of sufficient power (and different wavelength from the levitating laser) satisfying the conditions for Mie scattering. Keeping the static bubble radius constant, and the acoustic pressure amplitude constant, vary the driving frequency ω_d from ω_0 to $3\omega_0$. Alternatively, vary the static radius from RI (resonant at ω_d) to R_3 (resonant at $\omega_d/3$), keeping the driving pressure and frequency constant. This last suggestion, radius variation via controlled rectified diffusion during short bursts of hi gh-amplitude ultrasound, is perhaps the easier of the two parameter variations proposed.

References

1. Crum, L.A., J. Acoust. Soc. Am. **73**, 116 (1983); Lauterborn, W. and E. Cramer, Phys. RCV. Lett. 47, 1445 (1984); Holzfuss, J. and W. Lauterborn, Phys. Lett. 115A, 369 (1986); Holt R.G. and L.A. Crum, J. Acoust. Sot. Am, 91, 1924 (1992).

2. Welsby, V.G. and M.H. Safar, Acustica 22, 177 (1969); Miller, 1).1.., Ultrasonics **19**, 217 1981); Newhouse, V.L. and P.M. Shankar, J. Acoust. Sot. Am. 7S, 1473 (1984).

3. **DW1:** Du, G. and J. Wu, J. Acoust. Sot. Am, 87, 1965 (1990); **WD2:** Wu, J. and G. Du, J. Acoust. Sot. At-n. 94, 2446 (1993); **ZS1:** Zabolotskaya, E.A. and Soluyan, S.1., Sov. Phys. Acoust. **18**, 396 (1973); **IZ2:** Il'insky, Yu.A. and E.A. Zabolotskaya, J. Acoust. Sot. Am. 92, 2837 (1992); **Z3:** Zabolotskaya, E. A., J. Acoust. Sot. Am. 94, 2448 (1993).

4. Rayleigh, J. W. S., Philos. Msg. 34, 94 (1917); Plesset, M, S., J. Appl. Mech. **16**, 277 (1 949); Noltingk, B.E. and EA. Neppiras, Proc. Phys. Sot. B **63B**, 674 (1 950); Noltingk, B.E. and E.A. Neppiras, Proc. Phys. Sot. B 6411, 1032 (1 951); Poritsky, H., Proc. 1st U. S. National Congress in Applied Mechanics (A. S. M.E.) 813, (1952).

5. Lauterborn, W., J. Acoust. Sot. Am. S9, 283 (1976); Eatock, B. C., PhD thesis, University of Toronto, 1981; Prosperetti, A., L.A. Crum and K .W. Commander, J, Acoust. Sot, Am. **83**, 502 (1988); Parlitz, U., V. Englisch, C. Scheffczyk and W. Lauterborn, J. Acoust. Sot. Am. **88**, 1061 (1990).

6. See, for example, the reviews Neppiras, E. A., Phys. Rep. **61**, 1S9 (] 980); Prosperetti, A., in *Frontiers in Physical Acoustics, 145* (Corso, Italy, 1986).

7. Unger, B.T. and P.].. Marston, J. Acoust. Sot. Am. **83**, 970 (1988).