Reply to the "Comment" by J. Bloxham and W. Kuang on a paper entitled "The topographic torque on the bounding surface of a rotating gravitating fluid and the excitation by core motions of decadal fluctuations in the Earth's rotation" (R. Hide, *Geophysical Research Letters 22,961-964,* 1995).

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The findings of my recent paper (Hide [1 995, cited as 9511] include *inter alia* a complete refutation of an oft-repeated naive claim by Professors Bloxham and Kuang (BK) that there is a serious flaw in the theoretical basis of my method for investigating the topography of the Earth's core-mantle boundary (CM B) using Earth-rotation and other geophysical data, and that geophysical conclusions based on applications of the method arc therefore unacceptable. Stripped of rhetoric and semantic complications (including the inconsistent use of the term "geostrophic," which has no agreed definition out of context amongst geophysical fluid dynamicists (see e.g. *Greenspan* [1968], *Pedlosky [1979], Gill [1982])* but which is precisely defined and used in the context of 95H), the "Comment" (CBK) by BK on 951 I contains little more than an unsupported refusal by its authors to accept the clear anti unambiguous evidence given in 951 I against their unsubstantiated and demonstrably-err oneous claim. So far as I am aware, physical and mathematical arguments underlying their claim have yet to be divulged by BK in a scientific paper, the article "Bloxham and Kuang 1993" cited in CBK being the first of several unrefereed abstracts they have published on the same theme in the programs of various AGU and other major scientific meetings. As shown in 95}[, their claim is unacceptable on the grounds that it is incompatible with the laws of fluid dynamics and the theoretical basis of my method- which in essence is the same as onc put forward and applied independently by Professor f-l, Le Mouël and his colleagues, scc 9511—is upheld. It is hoped that this reply to CBK will be found helpful by those involved in the international SEDI (Study of the Earth's Deep Interior) program of research on the structure and dynamics of the Earth's core and lower mantle, where good ideas arc nccdcd for exploiting a wide range of geophysical data.

Consider a fluid bounded by a closed, rigid, impermeable and irrcgular]y-shaped surface S which rotates with angular velocity $\Omega(t)\hat{z}$ relative to an inertial frame. Suppose that the density of the fluid at a general point 1' (with coordinates $r = x\hat{x} + y\hat{y} + z\hat{z}$ in a frame fixed in S with its origin O at the center of mass of the system) is p(r, t), where t denotes time, and (x, y, z) are Cartesian coordinates, $\hat{z}$ being a onit vector in the axial direction and $(\hat{x}, \hat{y})$ unit vectors in the equatorial plane, and that $p(r, t)$ and $u(r, t)$ arc, respectively, the pressure and the Eulerian flow velocity at P. The *instantaneous* "topographic torque" $\Gamma_S(t)$ due to the action of normal pressure forces on S is given by

$$\Gamma_S(t) = \iint\limits_S p(\boldsymbol{r},t)\boldsymbol{r} \times d\boldsymbol{S} \tag{1}$$

(see H3.1, i.e. equation (3.1) of 9511), where the integral is taken over the whole of S, the vector element of which is $d\boldsymbol{S}$ directed generally away from O. The vector $\Gamma_S(t)$ fluctuates about a time-average equal and opposite to that of other fluctuating torques (electromagnetic, viscous, gravitational) exerted by the fluid on S (for the time-average of the net torque must be zero).

Given the shape of S and determinations of $p(\boldsymbol{r},t)$ on S, the vector $\Gamma_S(t)$ could be calculated directly using (1). However, as in the case of the Earth's core and in other situations where $p(\boldsymbol{r},t)$ is *not* known from direct measurements but other information *is* available, such as $\boldsymbol{u}(\boldsymbol{r},t)$ and p(r, t) in the vicinity of S, it is still possible to investigate $\Gamma_S(t)$, by using the equations of fluid dynamics to relate the local pressure gradient ∇p to "observable" quantities. It is readily shown that

$$\Gamma_S(t) = \iiint\limits_{S-C} \boldsymbol{r} \times \nabla p(\boldsymbol{r},t) d\tau \tag{2}$$

(see H3.3) where C is any spherical surface concentric with the origin, upon which r is therefore constant, that lies everywhere within the fluid and $d\tau$ is an element of the volume of fluid lying between S and C, over the whole of which the integral is taken. The magnetohydrodynamic equation of motion is conveniently expressed as follows:

$$\nabla p = \nabla' p^{(V)} + \nabla' p^{(G)} + \nabla' p^{(A)} \tag{3}$$

(see H 1.1 and H3.5). The three vectors $\nabla' p^{(V)}$, $\nabla' p^{(G)}$ and $\nabla' p^{(A)}$ define unambiguously the respective contributions to the instantaneous pressure gradient $\nabla p(\boldsymbol{r},t)$ at P that can be associated with (a) the buoyancy force (per unit volume), (b) the Coriolis force $2\boldsymbol{\Omega}(t) \times \boldsymbol{U}(\boldsymbol{r},t)$ (where $\boldsymbol{U}(\boldsymbol{r},t) \equiv \rho(\boldsymbol{r},t)\boldsymbol{u}(\boldsymbol{r},t)$) and (c) all the remaining ("ageostrophic") terms in the full momentum equation, notably those representing relative acceleration, viscous friction, Lorentz forces and etc. In general, none of the three vectors on the right hand side of (3) is irrotational (even though their sum has this property); hence the notation involving the use of the prime (see (9) below). The quantity

$$\Gamma_S^{(G)}(t;C) \equiv -2\Omega \iiint\limits_{S-C} \boldsymbol{r} \times (\hat{z} \times \boldsymbol{U}) d\tau \tag{4}$$

(see H3. 11) expresses unambiguously the contribution made to $\Gamma_S(t)$ by Coriolis forces acting on the fluid occupying the volume "S- C".

There are corresponding unambiguously-defined expressions (see H3.10 and H3.9) for the respective contributions associated with $\nabla' p^{(V)}$ and $\nabla' p^{(A)}$. The first of these contributions depends on the non-radial component of the acceleration due to gravity plus centripetal effects, There can be circumstances in which the spherical surfaces C can be chosen such that this and the "ageostrophic" contribution are both negligible in comparison with $\Gamma_S^{(G)}(t;C)$, in which case (4) provides a good first approximation to $\Gamma_S(t)$. This is the basis of the abovementioned method for investigating topographic torques at the CMB. It is important however to emphasize here that (4) is a *diagnostic* relationship, from which $\Gamma_S^{(G)}(t;C)$ can be determined when

$U(r,t)$ is already known from measurement or theory. In the determination of U from first principles, by obtaining simultaneous solutions of the governing equations (now used *prognostically* rather than diagnostically), ageostrophic terms *always* play a crucial role and can never be completely neglected !

Equation (4) leads directly to the following expression for the vector $\Gamma_S^{(G)}(t;C)$:

$$\Gamma_S^{(G)}(t;C) = 2\Omega(t)\iiint_{S-r-} z\left[U_x(t)\hat{x} + U_y(t)\hat{y} + U_z(t)\hat{z}\right]d\tau \qquad (5)$$

(scc }14. 1). This equation (contrary to tbc views expressed in CBK) is mathematically correct, physically meaningful and geophysically useful! When, as in the case of the Earth's CMB, the surface S is almost spherical, (5) applied to that region "S- C" of the outer core lying below the very thin viscous boundary layer on S and above the spherical surface C that just touches the lowest point in S, (5) reduces to an expression for the "geostrophic" (or "Coriolis" or "gyroscopic") contribution to the vector $\Gamma_S(t)$ derived previously by another method *(hide [1989,* 1986]), sec 9511.11 is also noteworthy, as Professor Le Mouël has kindly pointed out to me recently, that the z-component of (5) leads to an expression for the axial component of $\Gamma_S(t)$ which is essentially equivalent to onc derived by *Jault and Le Mouël [1989], in* a detailed study of the transfer of axial angular momentum between different parts of the Earth's liquid core and between the core and mantle.

It follows from (5) and the impermeability condition on S that (inasmuch as very weak suction due to the thin viscous E3knlan-1 lartmann boundary layer on S can be neglected)

$$\Gamma_S^{(G)}(t;C)\cdot\hat{z} = -\Omega\iint_C z^2 U\cdot dS \qquad (6)$$

(see 114.4; cf. }14.2 and }14.3), remembering that dS, by definition, is directed generally *away* from the origin of coordinates O. *Of particular importance in connection with the controversy started by BK is the implication of this equation that*—in virtue of the implication of the equation of mass conservation $\nabla\cdot U = 0$ (see }13.4) that

$$\iint_C U\cdot dS = 0 \qquad (7)$$

—the axial component of $\Gamma_S^{(G)}(t;C)$ is *not in general identically equal to zero, i.e.*

$$\Gamma_S^{(G)}(t;C)\cdot\hat{z} \neq 0. \qquad (8)$$

As shown in 95H, (8) suffices to refute the unsupported claim that BK have repeatedly made, in CBK and elsewhere, concerning the validity of my method. 1 was unable to discover the basis of their claim until November 1994, when I received a referee's report by the senior author of CBK recommending the rejection of my first attempt to publish a response to BK's pronouncements. The basis of that proposed rejection was a certain expression (in the terminology of the present paper) for $\Gamma_S^{(G)}(t;C)\cdot\hat{z}$. Had BK carried out tbc mathematical analysis leading to that expression correctly they would have found results equivalent to (6) and (8). In the event, owing to an unfortunate mathematical slip they obtained an erroneous expression, equivalent to (6) with r^2 in the place of z^2, which, crucially, does *not* satisfy (8)! Apparently believing this inaccurate result that $\Gamma_S^{(G)}(t;S)\cdot\hat{z}\equiv 0$, and also in their consistent but fallacious physical interpretation of the result, BK chew incorrect conclusions about the theoretical basis of my work and that of others on topographic core-mantle coupling and gave them wide publicity (see (2.2) of 95 H), presumably for the intended benefit of the SEDI community! It is possible that BK have been influenced in their confused thinking by a well-known but often misunderstood result in the theory of rotating fluids (which follows directly from (6)), namely that in the very-special (and hypothetical, see 95H) case when ageostrophic effects are negligible *everywhere*, the axial component of the topographic torque exerted by the fluid on its boundaries is equal to zero (see *Greenspan* [1968], *Pedlosky* [1979], *Roberts* [1988]; cf. ***Hide*** [1989]),

Having brought these mistakes to the attention of BK in December 1994 (when I made a further attempt to resolve the controversy Professor Bloxham had started nearly two years earlier), I was astonished to discover in May 1995, on being invited by the Editor of GRL to reply to CBK, that they continue to hold the view that the axial component of $\Gamma_S^{(G)}(t;C)$ must be identically equal to zero, notwithstanding evidence to the contrary presented clearly in 95H (see equations (6) to (8) above). In their new attempt to justify this belief, they make in CBK the. remarkable and inaccurate (see below) assertion that my expression H3.11 (see (4) above) must be meaningless! In support of this assertion, BK give an "argument" which invokes an "equation" labeled "Hide (1.1)", apparently without realizing that there is a crucial difference between that "equation" and equation (1 .1) of 95H. in 95H, the hydrodynamical equation of motion is expressed for convenience in the form given by (3) above (see also H1.1, 3.5), the corresponding vorticity equation being

$$\nabla\times\left[\nabla' p^{(V)} - \nabla' p^{(G)} + \nabla' p^{(A)}\right] = \mathbf{0}, \qquad (9)$$

since $\nabla\times\nabla p\equiv 0$. The authors of CBK apparently failed to notice that the term $\nabla' p^{(G)}$ is *not* the same as $\nabla p^{(G)}$, so *the question of "single valuedness" does not arise.* The vector $\nabla' p^{(G)}$ ***is*** uniquely defined by H1.1 (see (3) above) and there is no ambiguity whatsoever in the expression H3.11 for the vector $\Gamma_S^{(G)}(t;C)$ with which $\nabla' p^{(G)}$ is directly associated (see (4) to (8) above).

Before concluding, it is necessary to comment on the vague statements made in CBK concerning the magnitude of $\Gamma_S(t)\cdot\hat{z}$, which presumably refer to theoretical (including numerical) models of core motions. in realistic models, all the dependent variables $p(\mathbf{r},t)$, $\mathbf{u}(\mathbf{r},t)$, $\rho(\mathbf{r},t)$ etc. would be determined simultaneously from the full equations of magnetohydrodynamics, solved under appropriate boundary conditions. Acceptable solutions obtained in this way would satisfy H3.11 and H4.1 to H4.5 (see (4) to (6) above) automatically; indeed, these equations could prove useful in diagnostic studies needed to validate such models, in none of these solutions would ageostrophic effects be negligible ***everywhere,*** for such solutions are impossible on general mathematical and physical grounds (see 95H and discussion following (4) above). How $\Gamma_S(t)$ depends on the various dimensionless parameters required to characterize the model is of course a matter of great interest in the study of core motions. But **the statements made** in CBK on this matter are largely irrelevant to the simple point at issue here, which they serve to obscure rather than illuminate,

, The time-wasting controversy provoked by Professor Bloxham started with widely-pob]icizcd pronouncements offered as a (presumably) serious contribution to the geophysical discussions taking place within the SEDI community, at meetings I was unable to attend. The future interest of that community as it develops strategies for research on the Earth's deep interior will be served if BK can now ac.knowledge that their pronouncements were wrong and misguided, They should revise their opinions in the light of 95} 1, where the simple but important scientific issue at the heart of the controversy is clearly identified and folly resolved,

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References

Gill, A. E., *Atmosphere-ocean dynamics*, New York: Academic Press, 1982.

Greenspan, t I. P., *The theory of rotating fluids*, Cambridge University Press, 1968,

Hide, R., Presidential address: The Earth's differential rotation, *Quart. J. Roy. Astron. Sot.* **,276',** 3-14, 1986.

Hide, R., fluctuations in the Earth's rotation and the topography of the core-mantle interface, *Phil. Trans. Roy. Sot, London A* 328, 351-363, 1989.

Hide, R., The topographic torque on a bounding surface of a rotating gravitating fluid and the excitation by core motions of decadal fluctuations in the Earth's rotation, *Geophys. Res. Lett.*, 22,961-964, 1995; cited as "9511".

Jault, D. and Le Mouël, J-L., The topographic torque associated with tangentially geostrophic motion at the core surface and inferences on the flow inside the core, *Geophys. Astrophys. Fluid Dyn.*, 46',273-296, 1989.

Pedlosky, J., *Geophysical fluid dynamics*, New York: Springer Verlag, 1979.

Roberts, P. 11., On topographic core-mantle coupling, *Geophys. Astrophys. Fluid Dyn.*, 44, 181-187, 1988.

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