PREDICTION OF CHAOS USING STABILIZATION PRINCIPLE

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ABSTRACT

In this paper several comments concerning chaos from the viewpoint of theory of stability are made. Special attention is paid to dependence of orbital instability (leading to chaos) upon frames of reference, metrics of configuration space, and class of functions selected for mathematical models of physical phenomena. New representation of chaos is discussed. The theory is illustrated by examples.

1. INTRODUCTION

In recent years an increasing amount of interest has been addressed to the fact that, in many different domains of science (physics, chemistry, biology, engineering), systems with a similar strange behavior are frequently encountered. These systems display irregular (aperiodic) time evolution, and are called chaotic.

One of the most obvious limitation of existing methodologies for analysis of chaotic behavior is the following: based upon governing differential equations, in general, chaos cannot be analyzed, or even predicted without actual numerical runs. The "fatality" of this limitation was recently established by da Costa, N. and Doria, F.[2] who, based upon Godel's incompleteness theorem, presented a rigorous proof to the algorithmic impossibility of deciding whether a given system of equations has chaotic domains or not in the class of "elementary" functions. But the validity of numerical runs in the presence of persisting instability is always in question. Indeed, quoting L. D. Landay (1959), "Yet not every solution of the equation of motion, even if it is exact, can actually occur in Nature. The flows that occur in Nature must not only obey the equations of fluid dynamics, but also be stable.", one concludes that stability is an attribute of a mathematical model rather than of a physical phenomenon.

In this note we will make several comments concerning chaos from the viewpoint of general theory of stability.

2. BACKGROUND

Most of the dynamical processes are so complex that their universal theory which would capture all the details during all the time periods is unthinkable. That is why the art of mathematical modeling is to extract only the fundamental aspects of the process and to neglect its insignificant features, without losing the core of information. But "insignificant features" is not a simple concept. In many cases even vanishingly small forces can cause large changes in the dynamical system parameters, and such situations are intuitively associated with the concept of the instability. Obviously the destabilizing forces cannot be considered as " insignificant features," and therefore, they cannot be ignored. But since they may be humanly indistinguishable in the very beginning, there is no way to incorporate them into the model. This simply means that the model is not adequate for quantitative description of the corresponding dynamical process: it must be changed or modified. At the same time, it should be stressed that stability is not an invariant of motion. Firstly, it may depend upon the frame of reference. For instance, the same inviscid flow can be stable in Eulerian representation and unstable in Lagrangian one[1], or in a frame of reference moving with the streamlines,[12]. Secondly, the stability may depend upon the metric of configuration space: the same solution can be stable in one metric. and unstable in another depending upon the definition of the " distance" between two solutions (in the same way in which the same sequence can converge in one space and diverge in another). For example, as shown by Synge, [6] the motion of a particle:

$$q_1 = \frac{1}{2}t^2 + t, \; q_2 = 0 \tag{1}$$

derived from the potential

$$\Pi = -\frac{1}{2}q_1^2 + q_2$$

is stable in kinemato-statical sense when the distance between the basic and disturbed trajectories is established by the length of the geodesic perpendicular to the disturbed trajectory. But the same motion is unstable in kinematical sense when the distance is measured by the length of the line connecting points for which time t has the same value. In both cases the metric of configuration space is defined via the kinetic energy $W = \frac{1}{2}a_{ij}\dot{q}^i\dot{q}^j$:

$$g_{ij} = a_{ij} \tag{2}$$

Thirdly, the stability may depend upon the class of functions in which the motion is described. For instance, as shown by Zak, M .[7], the motion of a filament suspended in a gravity field, is stable in the class of functions defined in open interval (which does not include the free end), but it is unstable in the closed interval (snap of a whip). In this connection one should notice that the governing equations of classical dynamics, and in particular, of continuous systems, in addition to Newton's laws, are based upon a pure mathematical assumption that all the functions describing the system motions, must be differentiable "as many times as necessary". But since this assumption is not always consistent with the physical nature of motions, such an inconsistency leads to instability (in the class of smooth functions) of the governing equations[7].

Hence, the occurrence of chaos or turbulence in description of mechanical motions means only that these motions cannot be properly described by smooth functions if the scale of observations is limited. These arguments can be linked to Godel's incompleteness theorem[3], and the Richardson's[5] proof that the theory of elementary functions in classical analysis is undecidable.

Thus, since instability is not an invariant of motions, the following question can be posed: is it possible to **find** such a new (enlarged) class of functions, or a new metric of configuration space, or a new frame of reference *in* order to eliminate instability? Actually such a possibility would lead to different representative parameters describing the same motion in such a way that small uncertainties in external forces cause small changes of these parameters. For example, in turbulent and chaotic motions, mean velocities, Reynolds stresses, and power spectra, represent "stable" parameters, although classical governing equations neither are explicitly expressed via these parameters, nor uniquely define them.

The first step toward the enlarging of the class of functions for modeling turbulence was made by O. Reynolds (1895) [4] who decomposed the velocity field into the mean and pulsating components, and actually introduced a multivalued velocity field. However, this decomposition brought new unknowns without additional governing equations, and that created a "closure" problem. In 1986 Zak[9-1 1] has shown that the Reynolds equations can be obtained by referring the Navier-Stokes equations to a rapidly oscillating frame of reference, while the Reynolds stresses represent the contribution of inertia forces. From this viewpoint the "closure" has the same status as " proof" of Euclid's parallel postulate, since the motion of the frame of reference can be chosen arbitrarily. In other words, the " closure" of Reynolds equations represents a case of undecidability in classical mechanics. However, based upon the interpretation of the Reynolds stresses ss inertia forces, it is reasonable to choose the motion of the frame of reference such that the inertia forces eliminate the original instability. In other words, the enlarged class of functions should be selected such that the solution to the original problem in that class of functions will not possess an exponential sensitivity to changes in initial conditions, This stabilization principle has been formulated and applied to chaotic and turbulent motions by Zak [7-1 1]. As shown there, the motions which are chaotic (or turbulent) in the original frame of reference can be represented as a sum of the "mean" motion and rapid fluctuations, while both components are uniquely defined. It is worth emphasizing that the amplitude of velocity fluctuation is proportional to the degree of the original instability, and therefore, the rapid fluctuations can be associated with the measure of the uncertainty in the description of the motion. It should be noticed that both "mean" and "fluctuation" components representing the originally chaotic motion are stable, i.e., they are not sensitive to changes of initial conditions, and are fully reproducible.

3. CHAOS IN FAST OSCILLATING FRAME OF REFERENCE

Formally, chaos is caused by instability of trajectories (orbital instability). If the velocity of a particle is decomposed as $\vec{v} = v\vec{\tau}$, ($\vec{\tau}$ is the unit vector along the trajectory), then orbital instabilities are identified with instabilities of $\vec{\tau}$. In other words, the orbital instability leads only to redistribution of the energy between different coordinates, and it can be associated with an ignorable variable which does not contribute into kinetic energy. Therefore, an unlimited growth of this variable does not violate the boundedness of energy. That is why the orbital instability may not lead to classical attractors and chaos can emerge. In dissipative systems the persisting instability can be " balanced" by dissipative forces in a sense that exponentially diversing trajectories are locked up within a contracting phase-space volume, and this leads to chaotic attractors. In both conservative and dissipative systems, exponential divergence of trajectories within a constant or a contracting volume causes their mixing, so that the motion cannot be traced unless the initial conditions are known to infinite accuracy. It means that in configuration space, two different trajectories which may be initially indistinguishable (because of finite scale of observation), diverge exponentially, so that a "real" trajectory can fill up all the spacing between these exponentially diverging trajectories. In other words, in the domain of exponential instability, each trajectory "multiplies" and therefore, the predicted trajectory become multivalued, so the velocities can be considered as random variables:

$$\dot{q}^i = \dot{q}^i(t, \varepsilon), \quad 0 \leq \varepsilon \leq 1 \tag{3}$$

where $\dot{q}$ and ε for a fixed t are a function and a point on a probability space, respectively. Let us refer the original equations of motions to a non-inertial frame of reference which rapidly oscillate with respect to the original inertial frame of reference. Then the absolute velocity q can be decomposed into the relative velocity $\dot{q}_1$ and the transport velocity $\dot{q}_2 = 2\dot{q}_{2(o)}$:

$$\dot{q} = \dot{q}_1 + 2\dot{q}_{2(o)} \cos\omega t, \quad \omega \to \infty \tag{4}$$

while $\dot{q}_1$ and $\dot{q}_2^o$ are "slow" functions of time in the sense that

$$\omega \gg \frac{1}{T} \tag{5}$$

where τ is the time scale upon which the changes q_1 and $\dot{q}_{2(o)}$ can be ignored

Then for the mean $\bar{q}$:

$$q \cong q_1 \text{ since } \int_0^{t \gg \tau} \dot{q}_{2(o)} \cos\omega t\, dt \simeq \frac{1}{\omega}\dot{q}_{2(o)} \sin\omega t \to 0 \text{ if } w \to \infty \tag{6}$$

In other words, a fast oscillating velocity practically does not change the displacements.

Taking into account that

$$\frac{w}{2\pi}\int_o^{2\pi/\omega} \dot{q}_1 dt \simeq \dot{q}_1, \quad \int_0^{2\pi/\omega} \dot{q}_{2(o)} \sin\omega t\, dt = 0, \quad \int_0^{2\pi/\omega} \ddot{q}_{2(o)} \cos\omega t\, dt = 0 \tag{7}$$

$$\text{and} \quad \int_0^{2\pi/\omega} \dot{q}^2_{2(o)} \cos^2\omega t\, dt = \frac{1}{2}\dot{q}^2_{2(o)}$$

one can transform a system:

$$\dot{x}^i = a^i_j x^j + b^i_{jm} x^i x^m, \; i = 1, 2, \cdots n \tag{8}$$

into the following form:

$$\dot{\bar{x}}_i = a^i_j \bar{x}^j + b^i_{jm} \bar{x}^j \bar{x}^m, + b^i_{jm} \overline{x^j x^m}, i = 1, 2, \cdots n \tag{9}$$

where $\bar{x}^i$ and $\overline{x^i x^j}$ are means and double-correlations of x^i as random variables, respectively.

Actually the transition from (8) to (9) is identical to the Reynolds transformation: indeed, being applied to the Navier-Stokes equations, it leads to the Reynolds equations, and therefore, the last terms in (9) (which is a contribution of inertial forces due to fast oscillations of the frame of reference) can be identified with the Reynolds stresses. From a mathematical viewpoint, this transformation is interpretable as an enlarging the class of smooth functions to multivalued ones. Indeed, as follows from (5), for any arbitrarily small interval At, there always exists such a large frequency $\omega > \Delta t/2\pi$ that within this interval the velocity q runs through all its values, and actually the velocity field becomes multivalues.

The most significant advantage of the Reynolds-type equations (9) is that they are explicitly expressed via the physically reproducible parameters $\bar{x}^i \overline{x^i x^j}$ which describe, for instance, a mean velocity profile in turbulent motions, or a power spectrum of chaotic attractors. However, as a price for that, these equations require a closure since the number of unknowns in there is larger than the number of equations. Actually the closure problem has existed for almost hundred years since the Reynolds equations were derived, In the next sections, based upon the stabilization principle introduced by Zak, M.[8- 10] this problem will be discussed.

4. CLOSURE OF REYNOLDS-TYPE EQUATIONS USING STABILIZATION PRINCIPLE

The main purpose of the transition from the form (8) to the form (9) is to change the representative parameters describing the motion in such a way that they become physically reproducible, i.e., mathematically stable. Hence, the next logical step is to utilize the extra-variable-s, i.e., the Reynolds stresses, for elimination of the original instability. In other words, one can seek such an additional relationships:

$$\varphi(\overline{x^i x^j}, \bar{x}^i, \bar{x}^j, \cdots) = 0 \tag{lo}$$

which makes the system (9), (10) stable. Obviously, in this posedness of the problem, the solution to the system (9), (10) is not unique: the system can be overstablized to any degree, while each of these stable solutions will have physical meaning. But for the best solution one has to minimize the uncertainties represented by the Reynolds stresses, and therefore, the system should be brought to the boundary of instability. Since the orbital instability causing chaos is characterized by positive Lyapunov exponents λ_i^+, one should select the Reynolds stresses in (9) such that

$$\lambda_i^+ = 0 \tag{11}$$

while keeping the rest of the Lyapurrov exponents without changes:

$$\lambda_i^o = \tilde{\lambda}_i^o, \lambda_i^- = \tilde{\lambda}_i^- \tag{12}$$

where $\lambda_i^o, \lambda_i^-, \tilde{\lambda}_i^-$ and $\tilde{\lambda}_i^-$ are non-positive Lyapunov exponents of the system (9), (10) and equation (8), respectively.

Clearly, those components of the Reynolds stresses which do not affect the Lyapunov exponents, must be omitted. In general, the solution to equations (9- 12) will eventually approach a set of periodic attractors which "replaces" the chaotic. attractor of equation (8). However one should consider these sets not as an approximation to the original chaotic attractor, but, rather as a different way of mathematical representation) of the same physical phenomenon. This representation is provided by a new frame of reference whose oscillations are coupled with the dynamical variables such that the inertia forms (i.e. the Reynolds stresses) generated by transport motion, eliminate the original instability. In other words, the new frame of reference provides the best " view" of the motion.

The decomposition (9) applied to equation (8), generates not only pair correlations $\overline{x^i x^j}$, but also correlations of higher order, such as triple correlations $\overline{x^i x^j x^k}$, quadruple correlations $\overline{x^i x^j x^k x^m}$, etc. Indeed, multiplying equation (8) by x^k and averaging and combining the results, one **obtains** the **governing equations** for the pair correlations $\overline{x^i x^k}$;

$$\overline{x^i x^k} = a^i_j \overline{x^j x^k} + a^k \overline{x^j x^i} + b^i_{jm}(\overline{x^k x^j x^m} + \overline{x^k x^j}\bar{x}^m + \overline{x^k x^m}\bar{x}^j) + b_{jm}(\overline{x^i x^j x^m} + \overline{x^i x^j}\bar{x}^m + \overline{x^i x^m}\bar{x}^j), \tag{13}$$

which contain nine additional triple correlations $\overline{x^i x^j x^k}$.

Now the application of the stabilization principle will lead to the system (9- 13) which will define $\bar{x}^i, \overline{x^i x^j}$ and those components of triple corrections $\overline{x^i x^j x^m}$ which affect the Lyapunov exponents in equations (11) and (12). Hence, the solutions to the systems (9- 12) and (9- 13) can be regarded as the first and the second approximation, respectively, to the problem. Theoretically speaking, by considering next order approximations, a complete probabilistic structure of the solution to equation (8) can be reproduced.

Applications of the stabilization principle is significantly simplified for those systems whose boundaries of instability can be formulated analytically. For some cases of con- servative chaos and simple turbulent flows *new* representations of solutions were given by Zak, M.[7- 11].

In the next section we will demonstrate application of the stabilization principle to a dissipative chaotic system known as the Lorentz attractor by numerical elimination of positive local Lyapunov exponents.

5. PREDICTION OF MEAN FLOW

Applying the transformation (4) to the Rossler attractor:

$$\dot{x}_1 = -x_2 - x_3, \dot{x}_2 = x_1 + \frac{1}{5}x_2, \dot{x}_3 = \frac{1}{5} - \mu x_3 + x_1 x_3 \tag{14}$$

represents it in the form (9):

$$\dot{\bar{x}} = -\bar{x}_2 - \bar{x}_3; \dot{\bar{x}}_2 = \bar{x}_1 + \frac{1}{5}\bar{x}_2; \dot{\bar{x}}_3 = \frac{1}{5} - \mu\bar{x}_3 + \bar{x}_1\bar{x}_3 + \overline{x_1 x_3} \tag{15}$$

where $\bar{x}_1, \bar{x}_2, \bar{x}_3$ are the mean values of x_1, $X_{2)}$ and x_3, while $x_1 x_3$ and $x_1\overline{x_2}$ are double correlations representing the Reynolds “stresses”.

As extra-variables, these double correlations must be found from the condition that they suppress the positive Lyapunov exponent down to zero. In this case both the mean and the double-correlations components of the motion will be represented by periodic attractors, i.e. in a fully deterministic way.

Numerical implementation of this strategy performed for p = 5.7 leads to the following results. The mean flow expressed via the variables $\bar{x}_1, \bar{x}_2, \bar{x}_3$ is represented by a multi-periodic (but not chaotic) attractor plotted in Fig. 1. The initial error in $\bar{x}_i$ (which is of order of 10^{-6}) does not grow exponentially: it oscillates between 20.10^{-6} and (–40.10-6).

It should be stressed that the solution to Eq. (15) plotted in Fig. 1 is stable (in the new class of functions which includes "multivalued” fluctuations): small changes in initial conditions will lead to small changes in the solution.

One should recall that although equations (15) is different from the original Rossler equations (14), they describe the same physical phenomenon in a specially selected fast oscillating frame of reference.

The second example illustrates prediction of the mean flow in chaotic behavior of a charged particle in a uniform magnetic field:

$$\dot{v}_x = -\frac{x}{r^3} - v_y,\ \dot{v}_y = \frac{y}{r^3} + v_x,\ \dot{v}_z = -\frac{z}{r^3},\ (r^2 = x^2 + y^2 + z^2),\ \dot{x} = v_x\ ,\ \dot{y} = v_y\ ,\ \dot{z} = v_z \tag{16}$$

The system is chaotic if, for instance,

$$x = 1.5;\ y = 0,\ z = 4.0;\ v_x = v_y = v_Z = 0.01 \ \text{ at } \ t = 0 \tag{17}$$

The Fig. 2 demonstrates chaotic region in the space x, y, z. Application of the stabilization principle leads to the multi-periodic (but not chaotic) mean-flow plotted in Fig. 3.

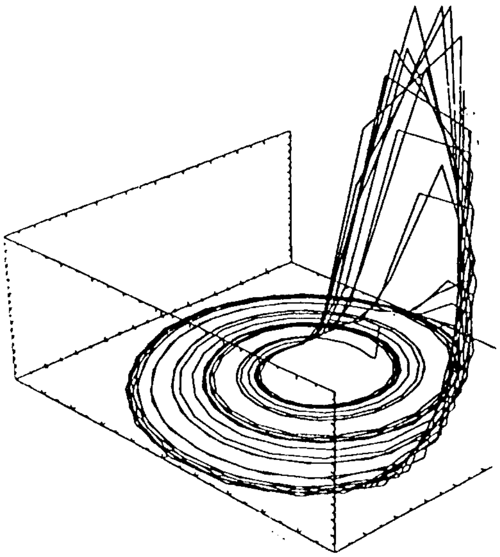

Figure 1

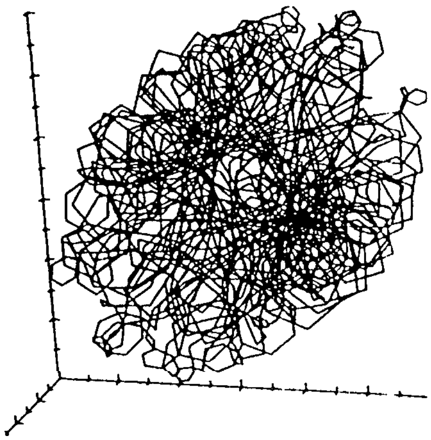

Figure 2

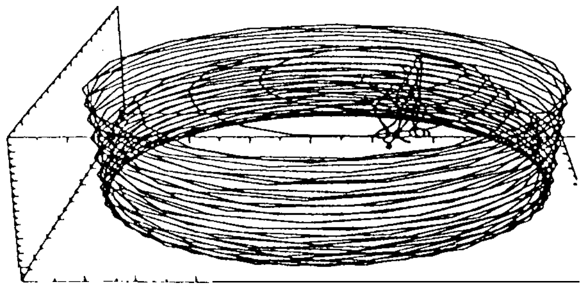

Figure 3

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