

# Optimum Receiver Structure for PPM Signals with Avalanche Photodiode Statistics

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## Abstract

The maximum likelihood decision statistic for detection of pulse-position modulated signals with an avalanche photodiode is derived, using the more accurate Webb density rather than Poisson or Gaussian approximations for the distribution of avalanche photodiode output electrons. It is shown that for Webb-distributed output electrons, the maximum likelihood rule is to choose the PPM word corresponding to the slot with the maximum electron count.

## 1 Introduction

The detection of information-bearing laser pulses by means of avalanche photodiode (APD) devices is a topic of interest in the design of optical communications systems. Although several key papers have been published that derive the output statistics of APD's [1, 2], provide accurate approximations to these statistics [3], and discuss applications to problems in optical detection (e. g., [4]), the authors have not found a derivation of the optimal receiver structure for pulse-position modulation (PPM) detection with valid APD statistics. Most studies assume an overly simplistic Poisson or Gaussian approximation, which may not yield good approximations to the true link performance over some relevant parameter ranges [4]. Using these approximations, one can easily show that the maximum likelihood detection rule is to pick the PPM word that corresponds to the slot with the maximum electron count. Here we provide a derivation of the structure of the optimum (maximum likelihood and maximum a posteriori) receiver for PPM pulses using the Webb density to model avalanche electrons generated by an APD in response to absorbed signal and background photons. The exact distribution of these electrons was derived in [1] and [2], and is highly difficult to work with. The Webb density [3] provides a very good approximation in most regions of interest, models both Poisson distributed photon absorption and random avalanche gain as

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described by McIntyre and Conradi [1, 2], but does not take into account additive thermal noise generated by the follow-on electronics.

## 2 The Log-Likelihood Function

In the optical communication system that we are considering, the transmitter sends one of  $Q$  PPM symbols in the form of a laser pulse located in one of  $Q$  time slots, where each slot is  $T_s$  seconds long. The output of the APD detector is integrated synchronously within each slot to form a vector of  $Q$  observables,  $\mathbf{m}$ , on which to base the PPM decision. Only one of the slots contains the signal, and it is assumed that all slots are equally likely to contain the signal. Since the slots are disjoint in time, the components of the observable vector are independent. The probability density of each component depends upon the presence of background and signal photon intensities. Let the background photon intensity be denoted by  $\lambda_b T_s$  and the signal photon intensity be denoted by  $\lambda_s T_s$ . Let  $m$  be the number of output APD electrons in a particular time slot, and let  $x = m - \lambda_b T_s G$ , where  $G$  is the gain of the APD device. We shall now deal with the vector of observables  $\mathbf{x}$ . According to the Webb approximation, the conditional density of  $x$  given the presence of background photons only is

$$p(x|\lambda_b T_s) = \frac{1}{\sqrt{2\pi\sigma_b^2(1+x/\beta_b)^3}} \exp\left(-\frac{x^2}{2\sigma_b^2(1+x/\beta_b)}\right), \quad (1)$$

for  $x \geq -\lambda_b T_s G$ , and the conditional density of  $x$  given the presence of signal and background photons is

$$p(x|(\lambda_b + \lambda_s)T_s) = \frac{1}{\sqrt{2\pi\sigma_{sb}^2(1+(x-\lambda_s T_s)/\beta_{sb})^3}} \exp\left(-\frac{(x-\lambda_s T_s)^2}{2\sigma_{sb}^2(1+(x-\lambda_s T_s)/\beta_{sb})}\right), \quad (2)$$

for  $x \geq -\lambda_b T_s G$ , where  $\sigma_b = \sqrt{\lambda_b T_s G^2 F}$ ,  $\sigma_{sb} = \sqrt{(\lambda_b + \lambda_s) T_s G^2 F}$ ,  $\beta_b = \frac{\lambda_b T_s G F}{F-1}$ , and  $\beta_{sb} = \frac{(\lambda_b + \lambda_s) T_s G F}{F-1}$ . Here,  $F = kG + (2 - 1/G)(1 - k)$ , where  $k$  is the ionization ratio of the APD device.

For maximum likelihood detection, the log-likelihood function corresponding to each hypothesis is computed, and the hypothesis corresponding to the largest value of the log-likelihood function is selected. Note that under equally likely hypotheses, the maximum likelihood decision is also the maximum a posteriori decision. Suppose hypothesis  $q$  is true, that is, a PPM word with a signal pulse in the  $q$ -th slot is received,

and denote this event as  $H_q$ . The log-likelihood function given  $H_q$  can be expressed as

$$\ln \Lambda(\mathbf{x}|H_q) = \ln p(x_q | (\lambda_b + \lambda_s)T_s) + \sum_{\substack{i=1 \\ i \neq q}}^Q \ln p(x_i | \lambda_b T_s). \quad (3)$$

Note that  $x_i = m_i - \lambda_b T_s G$ , where  $m_i$  is the output electron count in slot  $i$ . Substituting equations (1) and (2) into (3), and adding and subtracting the quantity  $\frac{x_q^2}{2\sigma_b^2(1+x_q/\beta_b)} + \frac{3}{2} \ln(1+x_q/\beta_b) + \ln \sigma_b$  in order to complete the sum on the right hand side of (3), we obtain the following:

$$\ln \Lambda(\mathbf{x}|H_q) = \Psi(x_q) - \ln(\sigma_{sb}/\sigma_b) + \sum_{i=1}^Q \ln p(x_i | \lambda_b T_s), \quad (4)$$

where

$$\Psi(x_q) = \frac{x_q^2}{2\sigma_b^2(1+x_q/\beta_b)} - \frac{(x_q - \lambda_s T_s G)^2}{2\sigma_{sb}^2(1+(x_q - \lambda_s T_s G)/\beta_{sb})} + \frac{3}{2} \ln \left( \frac{1+x_q/\beta_b}{1+(x_q - \lambda_s T_s G)/\beta_{sb}} \right). \quad (5)$$

Note that the last two terms on the right-hand side of equation (4) do not depend on  $q$  and hence cannot contribute any information to the final decision. These terms can therefore be ignored, and the decision based entirely upon the sufficient part of the log-likelihood function,  $\Psi(x_q)$ , as defined in (5).

### 3 Monotonicity of $\Psi(x_q)$

In order to decide which PPM word was received, the decision function  $\Psi(x_q)$  is calculated for each time slot  $q$ , and the PPM word corresponding to the largest  $\Psi(x_q)$ ,  $1 \leq q \leq Q$ , is selected. However,  $\Psi(x)$  is a somewhat complicated function of its argument that must be computed in real time to avoid creating a bottleneck in a high data rate system – a requirement that may exceed the real-time capabilities of the digital signal processing assembly. Hence, a simpler decision algorithm would be of value. If it can be shown that the decision function defined in (5) is a monotonically increasing function of its argument, then the PPM decisions could be based entirely upon the statistics  $x_i$ , and hence on the APD electron counts  $m_i$ , avoiding the need for more complicated time consuming calculations.

Since  $\Psi(x)$  is continuous over its domain, we can show that it is monotonically increasing by showing that its derivative is positive over its domain. After much simplification, the derivative can be expressed

as

$$\begin{aligned} \frac{\partial}{\partial x} \Psi(x) &= \frac{3\alpha^3 G((G + \alpha)\lambda_b T_s + \alpha x) - G^2(G + \alpha)^2(\lambda_b T_s)^2 \alpha}{(G(G + \alpha)\lambda_b T_s + \alpha x)^2} \\ &+ \frac{G^2(G + \alpha)^2(\lambda_b T_s + \lambda_s T_s)^2 \alpha - 3\alpha^3(G(G + \alpha)\lambda_b T_s + G^2\lambda_s T_s + \alpha x)}{(G(G + \alpha)\lambda_b T_s + G^2\lambda_s T_s + \alpha x)^2} \end{aligned} \quad (6)$$

where  $\alpha = (G - 1)(1 + (G - 1)k) > 0$ . By cross multiplying, (6) may be written as

$$\begin{aligned} \frac{\gamma^2(\gamma + G^2\lambda_s T_s)^2}{\alpha} \frac{\partial}{\partial x} \Psi(x) &= 3\alpha^2 \gamma(\gamma + G^2\lambda_s T_s)^2 - G^2(G + \alpha)^2(\lambda_b T_s)^2(\gamma + G^2\lambda_s T_s)^2 \\ &+ G^2(G + \alpha)^2(\lambda_b T_s + \lambda_s T_s)^2 \gamma^2 - 3\alpha^2(\gamma + G^2\lambda_s T_s)\gamma^2 \end{aligned} \quad (7)$$

where  $\gamma = G(G + \alpha)\lambda_b T_s + \alpha x > 0$ . Since  $\alpha > 0$ , we need only show that the right-hand side of (7) is positive. This quantity may be written as

$$\begin{aligned} RHS &= 3\alpha^2 \gamma G^2 \lambda_s T_s (\gamma + G^2 \lambda_s T_s) + 2G^3 (G + \alpha)^2 (\lambda_b T_s)^2 \lambda_s T_s \alpha \gamma + 2G^2 (G + \alpha)^2 \lambda_b T_s \lambda_s T_s \alpha^2 x^2 \\ &+ G^4 (G + \alpha)^2 (\lambda_b T_s)^2 (\lambda_s T_s)^2 (2G\alpha + \alpha^2) + G^2 (G + \alpha)^2 (\lambda_s T_s)^2 \alpha x (2G(G + \alpha)\lambda_b T_s + \alpha x). \end{aligned} \quad (8)$$

Substituting  $x = m - G\lambda_b T_s$  into (8) yields

$$\begin{aligned} RHS &= 3\alpha^2 G^2 \lambda_s T_s (G^2 \lambda_b T_s + \alpha m)(G^2 (\lambda_b T_s + \lambda_s T_s) + \alpha m) \\ &+ 2\alpha G^3 (G + \alpha)^2 (G^2 \lambda_b T_s + \alpha m)(\lambda_b T_s)^2 \lambda_s T_s + 2\alpha^2 G^2 (G + \alpha)^2 \lambda_b T_s \lambda_s T_s (m - G\lambda_b T_s)^2 \\ &+ \alpha G^2 (G + \alpha)^2 (2G^2 \lambda_b T_s + \alpha m)(\lambda_s T_s)^2. \end{aligned} \quad (9)$$

For  $m \geq 0$  (or  $x \geq -G\lambda_b T_s$ ), the right-hand side of (9) is positive. Therefore,  $\Psi(x)$  is monotonically increasing, and it suffices to base the PPM word decision upon its argument  $x$ , or equivalently, upon  $m$ , the number of electrons generated by the APD. Thus, the maximum likelihood receiver selects the PPM word corresponding to the time slot containing the greatest APD electron count.

## References

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