

# Local-Oscillator Limited Frequency Stability for Passive Atomic Frequency Standards Using Square-Wave Frequency Modulation

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## Abstract

Atomic frequency standards using square-wave frequency modulation effectively interrogate the atomic line by switching back and forth between two frequencies with equal atomic absorption values. For a symmetric absorption line, the slope of the responses will also be equal. In the quasi-static limit this would seem to be an ideal interrogation process: the sign reversal of frequency slope can be removed by detection electronics to give an essentially unvarying sensitivity to local-oscillator frequency variations. Such an interrogation would seem to eliminate L.O. aliasing and so relieve stringent requirements on L.O. phase noise. Nevertheless, sign changes in the interrogation and detection processes mean that the sensitivity is actually zero at some point in the cycle. We derive consequences of this fact by an analysis in terms of the sensitivity function  $g(t)$ . For white phase noise we derive an optimal form for  $g(t)$ , and show that the aliased noise diverges as  $g(t)$  approaches a constant. For flicker phase noise we find a limiting form that could, in principle, eliminate the aliasing effect; in practice, however, the improvement is limited by a slow dependence on available bandwidth. Finally, we derive optimized forms for any phase noise spectrum.

## 1 Introduction

Limitations to passive frequency standards performance by aliased local-oscillator fluctuations have previously been analyzed for both pulsed and continuous interrogation methods. Until recently, frequency standards using continuous frequency modulation interrogation, such as cesium beam

tubes and rubidium gas cells, have been analyzed in terms of the Fourier sine coefficients of the modulation and demodulation waveforms. On the other hand, the analysis of pulse-mode standards, such as the linear ion trap and cesium fountain, is carried out in terms of a sensitivity function  $g(t)$  that is derived in the time domain by a quantum mechanical analysis of the atomic interrogation [1][2][3][4]. Then the aliasing effect is obtained as a function of the Fourier cosine coefficients of  $g(t)$ . Here, we present an application of the sensitivity-function methodology to a limiting case of square-wave frequency modulation for passive frequency standards interrogated by continuous FM and treated in the quasi-static limit.

The effectiveness of square-wave modulation or demodulation has been shown in recent work to be surprisingly smaller than might be expected. Venot et al. [5] (see [6] also) find only minimal ( $\approx 1\%$ ) performance improvement by including successively higher numbers of optimized harmonics in the detection waveform for the case of white L.O. phase noise. The surprise is that, mathematically, a constant sensitivity can be approached by increasing harmonic content, and a constant (unvarying) sensitivity would not cause any aliasing at all.

On the other hand, we should not be surprised to find that the actual limiting value of the aliased noise for square-wave frequency modulation and detection is not zero. For systems with finite bandwidth, both the time-varying sensitivity of the atomic discriminator and the transfer function of the modulator must in fact cross zero while reversing sign. Thus, it makes sense to consider a general continuous form of the sensitivity  $g(t)$  that goes from 0 to a positive value and back to 0 during half a cycle of the square wave, and to study the limiting behavior of the aliased noise as the graph of  $g(t)$  approaches a rectangle. For this form, which always includes zeros, it is not obvious that the aliasing effect is small.

In the following sections, we show how previous analyses of continuous FM interrogation can be put into the sensitivity-function setting, and we specify the mathematical framework used in the remainder of the paper. We investigate the behavior of the aliasing effect for white and flicker phase noise processes in terms of a dimensionless variance  $v_g$  that measures how much  $g(t)$  differs from a constant. Working from explicit time dependencies for  $g(t)$ , as well as from its Fourier coefficients, we calculate the zero-frequency spectral density of aliased frequency variations for the locked local oscillator. For atomic frequency standards, this provides a limit to achievable white frequency noise in the locked L.O. For white L.O. phase noise we prove that this aliased noise spectral density diverges as  $1/v_g$  as  $v_g \rightarrow 0$ . We also derive the optimal shape of  $g(t)$ , a sequence of parabolic arches that reduces the aliasing effect from our nominal case by a factor of  $6/\pi^2$ . The sine coefficients of

the corresponding demodulation waveform agree with those obtained previously [5]. Nevertheless, the aliasing effect for the optimal case is only 1.5% smaller than for square-wave modulation with sine-wave demodulation ( $\pi^2/16$ ), explaining why improvements to that widely used strategy have proven so hard to come by. For flicker phase noise, we present a form for  $g(t)$  for which the aliasing effect approaches zero with decreasing  $\nu_g$ . Finally, we show how to obtain optimized sensitivity functions for arbitrary phase noise spectra.

## 2 The sensitivity function

### 2.1 Previous work

The sensitivity function originated as a tool for studying the L.O. noise aliasing effect in passive atomic frequency standards with pulsed interrogation [1]. In these standards, the detected frequency error is available only at multiples of a cycle period  $T_c$ . The error signal at time  $nT_c$  has the form

$$\int_{(n-1)T_c}^{nT_c} g(t) y(t) dt, \quad (1)$$

where  $y(t)$  is the instantaneous normalized L.O. frequency error. The sensitivity function  $g(t)$ , which has been derived for different interrogation methods by quantum-mechanical computations, is a periodic function with Fourier series

$$g(t) = \sum_{n=-\infty}^{\infty} g_n e^{i2n\pi\nu_c t}, \quad (2)$$

where  $\nu_c = 1/T_c$ . When the error signal (1) is used in a frequency control loop, the L.O. frequency noise near the cycle frequency  $\nu_c$  and its harmonics is aliased into the loop passband by the Fourier components of the periodic function  $g(t)$ , causing a spurious random low-frequency correction voltage to be delivered to the L.O.. This effect induces a noise component in the controlled L.O. frequency inside the loop passband; the spectral density of this induced noise at Fourier frequency zero is given by

$$S_y(0) = \frac{2}{g_0^2} \sum_{n=1}^{\infty} |g_n|^2 S_y^{\text{LO}}(n\nu_c), \quad (3)$$

where  $S_y^{\text{LO}}(\nu)$  is the spectrum of the normalized frequency of the uncontrolled L.O. This noise appears as an Allan deviation component

$$\sigma_y(\tau) = \sqrt{\frac{S_y(0)}{2\tau}} \quad (4)$$

in the output of the controlled L.O. for  $\tau$  greater than the response time of the loop. Control loop analyses leading to (3) from different points of view are available in the literature [1][2][7][8][9].

The present paper deals with a situation in which the L.O. frequency error  $y(t)$  is detected continuously. Let  $M(t)$  and  $D(t)$  be the synchronous modulation and demodulation waveforms, assumed to be periodic with period  $T_m$ , odd about 0, and even about  $T_m/4$ . In the quasi-static approximation, the error signal, linearized with respect to small fluctuations  $y(t)$ , is  $D(t)M(t)y(t)$ . Under these conditions, the function  $g(t) = D(t)M(t)$  is periodic with period  $T_c = T_m/2$  and even about 0 (and about  $T_c/2$ ); naturally, we call  $g(t)$  the sensitivity function for this detection scheme. Figure 1 shows an example in which  $M(t)$  is a square wave,  $D(t)$  a sine wave, and  $g(t)$  a rectified sine wave.

Passive frequency standards with this kind of error signal have been analyzed in the literature for different cases of  $M(t)$  and  $D(t)$ . In all cases (unless  $M(t)$  and  $D(t)$  are both perfect square waves) there is an aliasing effect analogous to the one for pulsed interrogation, and (3) holds. These analyses, after expanding  $M(t)$  or  $D(t)$  as a sine series

$$2 \sum_{n=0}^{\infty} C_{2n+1} \sin((2n+1)\pi\nu_c t), \quad (5)$$

calculate the aliasing effect in terms of the sine coefficients  $C_{2n+1}$  without mentioning  $g(t)$  explicitly; nevertheless, formulas for its Fourier coefficients  $g_n$  can be seen in the results, which are all of form (3). We summarize three of these analyses, giving the  $g_n$  formula for each.

Audoin et al. [10] consider a modulation waveform  $M(t)$  that depends on the applied modulation and the atomic response curve, along with sine-wave demodulation  $D(t) = 2 \sin(\pi\nu_c t)$ . If  $M(t)$  is expanded as (5), then  $g(t) = 2M(t) \sin(\pi\nu_c t)$  has Fourier coefficients

$$g_0 = 2C_1, \quad g_n = C_{2n+1} - C_{2n-1} \text{ for } n \neq 0. \quad (6)$$

Venot et al. [5] consider square-wave modulation and a general demodulation waveform  $D(t)$  given by (5). In this case, we can assume that  $g(t) = D(t)$  for  $0 \leq t \leq T_c$ ; thus, on this interval,  $g(t)$  has the two orthogonal expansions

$$g(t) = g_0 + 2 \sum_{n=1}^{\infty} g_n \cos(2n\pi\nu_c t) = 2 \sum_{n=0}^{\infty} C_{2n+1} \sin((2n+1)\pi\nu_c t). \quad (7)$$

Taking the inner product of both expansions with  $\cos(2n\pi\nu_c t)$ , we find that

$$g_n = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{C_{2k+1} (2k+1)}{(2k+1)^2 - 4n^2} \text{ for all } n. \quad (8)$$

By an analysis in the frequency domain, De Marchi et al. [11] obtain an aliasing result equivalent to that of Venot et al. They assume square-wave modulation and demodulation, but with a lowpass filter  $h_F$  applied between them in order to represent bandwidth limitations of the system. If  $q(t)$  denotes a unit square wave of frequency  $\nu_m$ , their error signal is  $q(t) \int h_F(t-u) q(u) y(u) du$ , which is not of form  $g(t)y(t)$ . Nevertheless, a study of their derivation shows that, as far as the aliasing effect is concerned, this error signal is equivalent to  $q(t) [\int h_F(t-u) q(u) du] y(t)$ , which does have the desired form. We can regard  $M(t)$  as a filtered square wave and  $D(t)$  as a square wave; again, (8) holds, provided that  $(\pi/2) C_{2n+1}(2n+1)$  is interpreted as the filter response  $H_F((2n+1)\nu_m)$ .

## 2.2 Present approach

Although we shall keep the factors  $M(t)$  and  $D(t)$  in mind, it is reasonable to investigate how the aliasing effect depends on their product  $g(t)$ , regarded as an independent object of study. For convenience we set  $T_c = 1$ , and we shall drop the assumption that  $g(t)$  is even. There is precedent in the literature for doing so: Lemonde et al. [4] and Makdissi et al. [12] derive asymmetric sensitivity functions for periodic and continuous interrogation disciplines; Barillet et al. [13] allow cosine terms in the expansion of  $D(t)$ . Also, we wish to leave open the possibility that a dynamic analysis might lead to a periodic, asymmetric  $M(t)$ . Consequently, we must allow the coefficients  $g_n$  to be complex, satisfying  $g_{-n} = \overline{g_n}$ . Nevertheless, all of the examples and optimization solutions to follow are even functions. Two sensitivity functions that are proportional to each other are equivalent for our purposes.

Let us place the following mathematical conditions on  $g(t)$ ,  $0 \leq t \leq 1$ :

1. Its Fourier coefficients

$$g_n = \int_0^1 e^{-i2n\pi t} g(t) dt \quad (9)$$

satisfy

$$\sum_{n=-\infty}^{\infty} |g_n| < \infty; \quad (10)$$

2.  $g(0) = 0$ ;
3.  $g_0 = \int_0^1 g(t) dt > 0$ .

Condition 1 says that the Fourier series (2) converges absolutely and uniformly; this forces  $g(t)$  to extend to a continuous function on the real line with period 1. Condition 2 says that the modulation or demodulation waveform has to pass continuously through zero, even though the

other might be an ideal square wave. Because of Condition 1,  $g(1) = 0$  also. Condition 3 is needed in order that  $g(t)y(t)$  be a reasonable error signal on average; the open-loop gain contains  $g_0$  as a factor.

A function  $g(t)$  that fulfils these conditions will be called an admissible sensitivity. For such a function, (2) and Condition 2 gives

$$g_0 = - \sum_{n \neq 0} g_n. \quad (11)$$

In order to measure how much  $g(t)$  differs from a constant sensitivity, we define its normalized variance  $v_g$  by

$$v_g = \frac{1}{g_0^2} \int_0^1 [g(t) - g_0]^2 dt = \frac{1}{g_0^2} \int_0^1 g^2(t) dt - 1. \quad (12)$$

Then

$$v_g = \frac{2}{g_0^2} \sum_{n=1}^{\infty} |g_n|^2. \quad (13)$$

A previous study [14] used a different measure that applies to a class of sensitivity functions that are even about  $\frac{1}{2}$  and increasing on  $[0, \frac{1}{2}]$ .

To make a dimensionless version of the aliasing formula (3), let us normalize it by the case of sine-wave modulation and demodulation (sine-sine detection). For this case,  $g(t) = \sin^2(\pi t) = \frac{1}{2}(1 - \cos(2\pi t))$  (see Fig. 2), so that  $g_0 = \frac{1}{2}$ ,  $g_1 = g_{-1} = -\frac{1}{4}$ , and the other  $g_n$  are zero. The resulting aliasing formula is written

$$S_y(0; \sin^2) = \frac{1}{2} S_y^{\text{LO}}(\nu_c). \quad (14)$$

Dividing (3) by (14), we define a normalized aliasing factor by

$$A = \frac{4}{g_0^2} \sum_{n=1}^{\infty} |g_n|^2 \frac{S_y^{\text{LO}}(n\nu_c)}{S_y^{\text{LO}}(\nu_c)}. \quad (15)$$

It will be shown later (Section 5) that we can always make  $A < 1$  by a good choice of  $g(t)$ ; in other words, sine-sine detection can always be improved upon, in principle.

Our purpose is to study the behavior of  $A$  for different types of L.O. noise. For a power-law noise spectrum,  $S_y^{\text{LO}}(\nu) = h_\alpha \nu^\alpha$ , we write  $A = A_\alpha$ , where

$$A_\alpha = \frac{4}{g_0^2} \sum_{n=1}^{\infty} |g_n|^2 n^\alpha. \quad (16)$$

For  $\alpha = 0$  (white frequency), we have  $A_0 = 2v_g$ ; the aliasing factor always tends to zero as  $g$  approaches a constant sensitivity. Let us look at the more interesting cases  $\alpha = 2$  (white phase) and  $\alpha = 1$  (flicker phase).

### 3 White phase

#### 3.1 Optimal sensitivity function

For white phase noise, we show that the aliasing factor

$$A_2 = \frac{4}{g_0^2} \sum_{n=1}^{\infty} |g_n|^2 n^2 \quad (17)$$

has a positive minimum over all admissible  $g$ , and we derive the optimal  $g(t)$ . If  $A_2 < \infty$  for an admissible  $g(t)$ , then  $g(t)$  is absolutely continuous and has an  $L^2$  derivative  $g'(t)$ , whose  $n$ th Fourier coefficient is  $i2\pi n g_n$ . Since the  $L^2$  norm of  $g'(t)$  is given by

$$\|g'\|^2 = \int_0^1 |g'(t)|^2 dt = 8\pi^2 \sum_{n=1}^{\infty} |n g_n|^2,$$

we have

$$A_2 = \frac{\|g'\|^2}{2\pi^2 g_0^2}. \quad (18)$$

Knowing that  $0 = g(0) = g(1) = \int_0^1 g'(t) dt$ , we can express  $g_0$  in terms of  $g'(t)$  as follows:

$$\begin{aligned} g_0 &= \int_0^1 g(u) du = \int_0^1 \int_0^u g'(t) dt du \\ &= \int_0^1 g'(t) (1-t) dt = \int_0^1 g'(t) \left(\frac{1}{2} - t\right) dt. \end{aligned}$$

Thus, letting  $f(t) = \frac{1}{2} - t$ , we can rewrite (18) as

$$\frac{1}{A_2} = 2\pi^2 \left\langle \frac{g'}{\|g'\|}, f \right\rangle^2$$

in terms of the  $L^2$  inner product  $\langle f_1, f_2 \rangle = \int_0^1 f_1 \overline{f_2}$ . The quantity  $A_2^{-1}$  can be maximized by making the unit vector  $g'/\|g'\|$  point in the direction of  $f$ ; we are allowed to do that because  $f(t)$  is itself the derivative of the admissible sensitivity  $\frac{1}{2}t(1-t)$ , which, therefore, has to be the optimal choice.

Thus, the optimal sensitivity function for white FM is a parabolic arch, and

$$\min A_2 = \frac{1}{2\pi^2 \|f\|^2} = \frac{6}{\pi^2} \approx 0.60793,$$

which can be attained with square-wave modulation and a demodulation waveform  $D(t)$  consisting of a series of alternating positive and negative parabolic arches. This result confirms that of Venot et al. [5], whose optimal demodulation coefficients  $C_{2n+1} = (2n+1)^{-3}$  are indeed the sine coefficients of a parabolic-arch wave.

Since a parabolic arch looks much like a sinusoidal arch (see Fig. 2), it is interesting to compare the minimal aliasing factor to the aliasing factor for square-sine detection. A straightforward evaluation of (18) for  $g(t) = \sin(\pi t)$  gives  $A_2 = \pi^2/16 \approx 0.61685$ , which is greater than  $\min A_2$  by a factor of  $\pi^4/96 \approx 1.0147$ . For white phase noise, square-sine detection is close to being optimal.

The optimal sensitivity can be derived by yet another method, which will be used again repeatedly. After substituting (11) into (17) and setting  $x_n = ng_n$ ,  $a_n = -1/n$  ( $n \neq 0$ ), we have

$$\frac{1}{A_2} = \frac{\left(\sum_{n \neq 0} x_n a_n\right)^2}{2 \sum_{n \neq 0} |x_n|^2} = \frac{1}{2} \left\langle \frac{x}{\|x\|}, a \right\rangle^2,$$

where we are now using the inner product  $\langle u, v \rangle = \sum_{n \neq 0} u_n \bar{v}_n$  and its norm  $\|u\| = \sqrt{\langle u, u \rangle}$  for two-sided complex sequences with the term for  $n = 0$  omitted. For the optimal solution, set  $x_n = a_n$ , i.e.,  $g_n = -n^{-2}$  ( $n \neq 0$ ), which is proportional to the  $n$ th Fourier coefficient of  $t(1-t)$ ; then  $(\min A_2)^{-1} = \frac{1}{2} \|a\|^2 = \sum_{n=1}^{\infty} n^{-2} = \pi^2/6$  as before.

### 3.2 Comparison with experiments

The aliasing factor  $\pi^2/16$  for square-sine detection says that there is only a 2.1 dB improvement in the aliasing effect when we go from sine-sine detection to square-sine detection. This result disagrees with the measurements of Deng et al. [15] on a rubidium cell. They measured  $C_{2n} := ng_n/g_0$ , averaged over ranges of  $n$ , by injecting bandpass-filtered phase noise into the L.O. and observing the aliasing effect. Their results correspond to a decrease in the aliasing effect of approximately a factor of 10 in amplitude. To the authors' knowledge, this large discrepancy has not been explained.

There is better agreement with recent measurements of Barillet et al. [13] on a cesium tube. They observed the aliasing effect by injecting controlled amounts of phase noise in the vicinity of  $\nu_c, 2\nu_c, \dots, 5\nu_c$ . For square-sine detection with white phase noise, they measured  $\sigma_y(8\text{s}) = 7.3 \times 10^{-12}$ ; to calibrate our calculations, we choose the value of phase spectral density that makes our calculated value of  $\sigma_y(8\text{s})$  agree with the measured value. After attenuating the noise density at  $\nu_c$  by 14 dB, they measured  $3.9 \times 10^{-12}$ ; our calibrated calculations give  $3.7 \times 10^{-12}$ . Using the same noise conditions and square-wave modulation, they introduce a demodulation waveform

$$D(t) = 2 [\sin(\pi\nu_c t) + C_3 \sin(3\pi\nu_c t)]$$

with the object of minimizing the aliasing effect with respect to  $C_3$ . (Square-sine detection corresponds to  $C_3 = 0$ .) Although the measured values of  $\sigma_y(8\text{s})$  and our calculated values show a



rough qualitative agreement for  $0 \geq C_3 \geq -0.4$ , the calculated values vary between 0.94 and 0.55 times the measured values.

### 3.3 Approach to a rectangle

Suppose that the graph of the admissible sensitivity function on  $[0, 1]$ , which passes through the points  $(0, 0)$  and  $(1, 0)$ , approaches a rectangle in the sense that  $v_g$  tends to 0. Then, for white phase noise, the aliasing factor  $A_2$  tends to infinity at least as fast as  $1/v_g$ . More precisely,

$$A_2 v_g \geq \frac{1}{2\pi^2} \quad (19)$$

for all admissible  $g(t)$ ; moreover, the constant in (19) is best possible.

To prove (19), let  $f(t) = 1 - g(t)/g_0$ . Then

$$f(0) = f(1) = 1, \quad \int_0^1 f(t) dt = 0. \quad (20)$$

We may assume that  $A_2 < \infty$ . From (12) and (18), we have

$$v_g = \int_0^1 f(t)^2 dt, \quad A_2 = \frac{1}{2\pi^2} \int_0^1 f'(t)^2 dt;$$

thus, (19) is equivalent to

$$\int_0^1 f^2 \int_0^1 f'^2 \geq 1. \quad (21)$$

To prove (21), observe that the continuous function  $f$  assumes its mean value 0 at some point  $a$  in  $(0, 1)$ . Since  $f$  is absolutely continuous, so is  $f^2$ , implying that  $f^2$  is the integral of its derivative  $2ff'$ . Then

$$\begin{aligned} 1 &= \frac{1}{2} \left[ f(0)^2 - f(a)^2 + f(1)^2 - f(a)^2 \right] \\ &= - \int_0^a ff' + \int_a^1 ff' \\ &\leq \int_0^a |ff'| + \int_a^1 |ff'| = \int_0^1 |ff'| \\ &\leq \left( \int_0^1 f^2 \int_0^1 f'^2 \right)^{1/2} \end{aligned}$$

by Schwarz's Inequality. This proves (21).

We give an example to show that (21) is not valid in general with a constant greater than 1 on the right side. To make Schwarz's Inequality efficient, we try to make  $|f'(t)|$  proportional to

$|f(t)|$ , while retaining the conditions (20). Let  $f(t) = be^{-ct} - d$  for  $0 \leq t \leq \frac{1}{2}$ ,  $f(t) = f(1-t)$  for  $\frac{1}{2} \leq t \leq 1$ . After solving for  $b$  and  $d$  as functions of  $c$  such that  $f(0) = 1$ ,  $\int_0^{1/2} f = 0$ , we find that

$$\int_0^1 f^2 \int_0^1 f'^2 = 1 + \frac{4}{c} + O\left(\frac{1}{c^2}\right) \quad \text{as } c \rightarrow \infty.$$

This quantity can be made as close to 1 as we like by making  $c$  large. The sensitivity function  $g(t)$  that corresponds to this example approaches a rectangle as  $c \rightarrow \infty$ .

## 4 Flicker phase

Previous work [14] suggested that as  $g(t)$  approaches a rectangle in some reasonable way, the aliasing factor

$$A_1 = \frac{4}{g_0^2} \sum_{n=1}^{\infty} |g_n|^2 n \quad (22)$$

approaches a positive constant. This is true for the method of horizontal shrinking used in [14] for symmetric  $g(t)$ , in which we split the graph of  $g(t)$  down the middle, shrink both halves horizontally towards the vertical lines  $t = 0$  and  $t = 1$  while retaining their shapes, and bridge the gap by the constant  $g(1/2)$ .

For another example, taken from [11], let

$$M(t) = \frac{4}{\pi} \sum_{n=0}^{N-1} \frac{\sin((2n+1)\pi t)}{2n+1},$$

which is a partial sum of the sine series of a square wave  $q(t)$  with period 2 and amplitude 1. (In terms of the lowpass filter referred to earlier,  $H_F((2n+1)\nu_m) = 1$  if  $0 \leq n \leq N-1$ , and 0 otherwise.) This function will be called the  $N$ -term Gibbs square wave (shown in Fig. 2 for  $N = 10$ ). Letting  $D(t) = q(t)$  makes  $g(t)$  the even extension of  $M(t)$ . Plots of  $g_n$  as a function of  $n$  and  $N$  are given in [11]. For large  $N$ , the summations in (8) and (22) can be approximated by integrals that can be evaluated in terms of known functions. Also, because  $g(t) = M(t)$  on  $[0, 1]$  and (5) is an orthogonal expansion,  $v_g$  can be evaluated directly from (12). (In fact,  $v_g = 1/g_0 - 1$  with this particular scaling.) We find that  $v_g \sim 2/(\pi^2 N)$  as  $N \rightarrow \infty$ , and

$$\lim_{N \rightarrow \infty} A_1 = \frac{28\zeta(3)}{\pi^4} \approx 0.3455, \quad (23)$$

where  $\zeta(p) = \sum_{n=1}^{\infty} n^{-p}$ .

These findings lead to the conjecture that  $A_1$ , like  $A_2$ , has a positive lower bound over all admissible  $g(t)$ . To show that this conjecture is false, we restrict the harmonic content of the

Fourier series (2) of  $g(t)$ . Assume that  $g_n = 0$  for  $|n| > N$ . Set  $x_n = g_n\sqrt{|n|}$ ,  $a_n = -1/\sqrt{|n|}$ ; from (22) and (11) we obtain

$$\frac{1}{A_1} = \frac{(\sum x_n a_n)^2}{2 \sum x_n^2} = \frac{1}{2} \left\langle \frac{x}{\|x\|}, a \right\rangle^2,$$

where the summations are over  $1 \leq |n| \leq N$ . As before, to maximize  $A_1^{-1}$  we set  $x = a$ , i.e.,  $g_n = -1/|n|$ , and obtain  $g_0$  from (11). Write  $L_N = \sum_{n=1}^N n^{-1}$ . Then  $L_N = \ln N + \gamma + O(1/N)$  as  $N \rightarrow \infty$ , where  $\gamma$  is Euler's constant, and

$$A_1 = \frac{1}{L_N} \sim \frac{1}{\ln N},$$

$$v_g = \frac{1}{2L_N^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sim \frac{\pi^2}{12(\ln N)^2}.$$

The actual sensitivity function,

$$g_N(t) = 2 \sum_{n=1}^N \frac{1 - \cos(2n\pi t)}{n} = 4 \sum_{n=1}^N \frac{\sin^2(n\pi t)}{n}, \quad (24)$$

will be called the  $N$ -term logarithmic arch because  $\sum_{n=1}^{\infty} n^{-1} \cos(2n\pi t) = -\ln |\sin(\pi t)|$  for  $0 < t < 1$  [16]; it is shown in Fig. 2 for  $N = 10$ . Although  $A_1$  decreases only logarithmically with  $N$ , it still goes below the limiting value (23) for the Gibbs square wave when  $N \geq 10$ .

## 5 Arbitrary phase noise

The previous argument generalizes easily to any L.O. phase noise spectral density. Let

$$s_n = \frac{S_y^{\text{LO}}(|n| \nu_c)}{S_y^{\text{LO}}(\nu_c)}$$

for  $|n| \geq 1$ ; for simplicity, assume all  $s_n > 0$ . To minimize (15) over admissible  $g(t)$  whose Fourier coefficients  $g_n$  vanish for  $|n| > N$ , we need only set  $g_n = -1/s_n$  for  $1 \leq |n| \leq N$ , and determine  $g_0$  from (11). For a given  $N$ , the optimal sensitivity and minimal aliasing factor are given by

$$g_N(t) = 2 \sum_{n=1}^N \frac{1 - \cos(2n\pi t)}{s_n} = 4 \sum_{n=1}^N \frac{\sin^2(n\pi t)}{s_n}, \quad (25)$$

$$(\min A)_N = \left( \sum_{n=1}^N \frac{1}{s_n} \right)^{-1}. \quad (26)$$

Since  $s_1 = 1$  we see that  $(\min A)_N < 1$ ; in other words, we can always improve on sine-sine detection in principle. On the other hand, other studies [17][18][5][6] have suggested that the

aliasing effect could be reduced by suppressing the L.O. phase noise severely at  $\nu_c = 2\nu_m$  before the atomic resonance sees it. In this situation, the sensitivity  $g(t) = \sin^2(\pi t)$  is nearly optimal because  $s_1^{-1} = 1$  dominates the other  $s_n^{-1}$ . This confirms another result of Venot et al. [5]: assuming square-wave modulation and white phase noise with noise suppression at  $\nu_c$ , they compute optimal sine coefficients of the demodulation waveform. For three-harmonic optimization and complete noise suppression at  $\nu_c$ , their values of  $C_3/C_1$  and  $C_5/C_1$  differ by only 1% from  $-\frac{1}{5}$  and  $-\frac{1}{35}$ , which are the exact values from the expansion of  $\sin^2(\pi t)$  as a sine series (5).

In a narrow mathematical sense, of course, there is nothing special about the term for  $n = 1$  in (3); for any  $m > 1$ , the optimal detection also improves on detection using  $g(t) = \sin^2(m\pi t)$ , which selects only the noise at frequency  $m\nu_c$ .

We can distinguish two behaviors for large  $N$ . If  $\sum s_n^{-1} = \infty$  (e.g.,  $s_n \sim n^\alpha$  with  $\alpha \leq 1$ ), then  $A$  can be made as small as we like by making  $N$  large, subject to the limitations of the quasi-static approximation; we have already treated the borderline case  $\alpha = 1$ . If  $\sum s_n^{-1} < \infty$  (e.g.,  $s_n \sim n^\alpha$  with  $\alpha > 1$ ), then  $g_N(t)$  converges uniformly as  $N \rightarrow \infty$  to an admissible sensitivity that is optimal for this noise spectrum, and  $(\min A)_N$  converges to the positive minimum  $(\sum_{n=1}^{\infty} s_n^{-1})^{-1}$  of the aliasing factor. For  $\alpha = 2$  we found that the optimal  $g(t)$  is a parabolic arch. For the sake of curiosity, we plot the optimal  $g(t)$  for  $\alpha = 3$  in Fig. 2 (opt 3); this function has zero slope at  $t = 0$  and 1.

After obtaining a sensitivity function  $g(t)$  that is judged to be desirable, we can try to realize it as  $M(t)D(t)$ . If one of the factors is a square wave, then the other factor is  $g(t)$  on  $[0, 1]$ . If one of the factors is a sine wave, we can either express the other factor as  $g(t)/\sin(\pi t)$  in the time domain, or obtain its sine expansion (5); in fact, it follows from (6) and (11) that

$$C_{2n+1} = - \sum_{k=n+1}^{\infty} g_k. \quad (27)$$

Here, if  $g(t)$  has a finite cosine series, then the sine series (5) is also finite. The results can be unattractive, however; see Fig. 3, which is the result of trying to obtain a logarithmic-arch sensitivity (Fig. 2) in this way.

## 6 Conclusions

Using the quasi-static theory, we have shown how to exploit the sensitivity function as a tool for studying the aliasing effect that arises when an atomic resonator is probed by a frequency-

modulated signal in a control loop, and we have seen that simplicity and insight can be gained by direct use of the sensitivity function in analyses. In particular, we have quantitatively examined the attractive and almost obvious idea that as a constant interrogation sensitivity is approached, the aliased L.O. noise must go away. This idea turns out not to be true because the sensitivity must actually be zero at some point. In fact, a sensitivity that is almost constant has little time to get down to zero, and thereby tends to excite high-frequency noise components. This problem is especially severe for white phase noise: the aliased noise has an absolute minimum and grows as the reciprocal of the normalized sensitivity variance as the sensitivity approaches a constant. Even for flicker phase noise, the minimal aliased noise tends to zero only like  $1/\log N$ , where  $N$  is the maximum degree of the cosine components of the sensitivity function; thus, in a practical sense, the aliased noise cannot be made arbitrarily small. Nevertheless, for a known phase noise spectrum, the sensitivity function can in principle be tailored to achieve the least possible aliased noise for a prescribed  $N$ . Whether this is accomplished by tailoring the modulation waveform, the demodulation waveform, or both is a matter of practical design.

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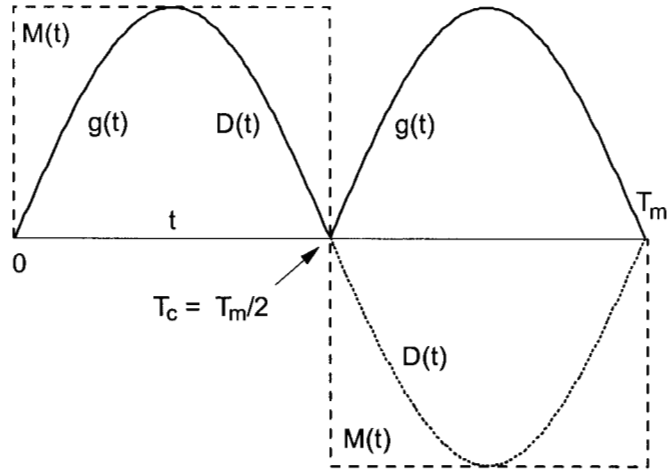


Figure 1: Square wave modulation  $M(t)$  and sine wave demodulation  $D(t)$ , both with period  $T_m$ , combine to give a rectified sine-wave sensitivity function  $g(t)$  with period  $T_c = T_m/2$ .

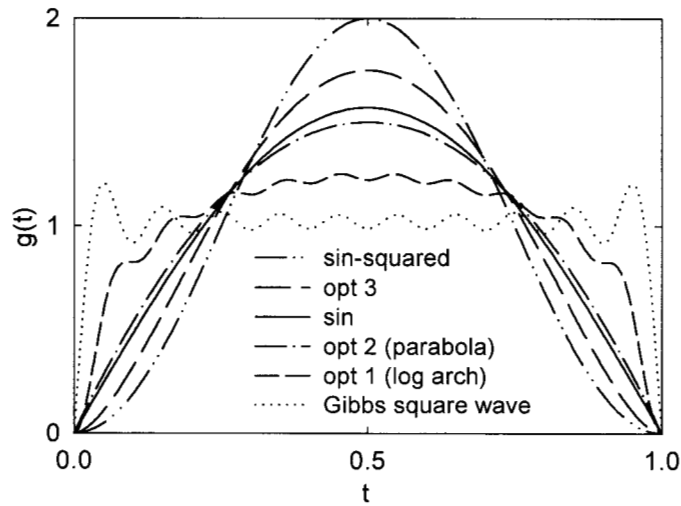


Figure 2: A collection of sensitivity functions, normalized to have mean value  $g_0 = 1$ . The functions labeled opt  $\alpha$  are optimized for  $\nu^{\alpha-2}$  phase noise. The logarithmic arch (opt 1) is a constant plus 10 cosine terms; the Gibbs square wave is a sum of 10 sine terms.



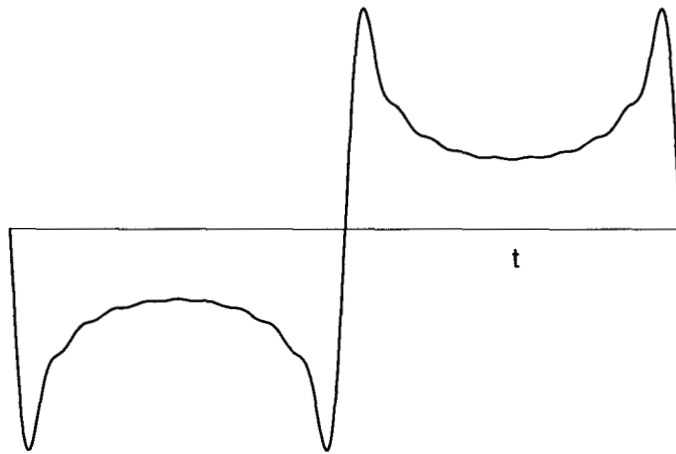


Figure 3: This is the mod./demod. waveform that you have to use to turn a sine-wave demod./mod. into a 10-term logarithmic-arch sensitivity.