SIMULTANEOUS PLACEMENT OF ACTUATORS AND SENSORS

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ABSTRACT

A placement of actuator and sensors for flexible structures is studied using the H_2, H_∞, and Hankel modal norms. The norm of a flexible structure with multiple actuators and sensors is a root-mean-square sum of norms of structures equipped with single sensor and single actuator. This property allows for the evaluation of the importance of a combination of sensors and actuator, in terms of their participation in the overall structural dynamics. This simple and powerful tool is applicable for structures only, since in a general case, a system norm cannot be decomposed into a combination of single-input-single-output norms, and consequently complex searching techniques need to be used.
The algorithm consists of the determination either H_2, or H_∞, or Hankel norms for the *i*th mode, *j*th actuator, and *k*th sensor. Using the norms, the sensor and actuator placement matrices are generated for each considered mode. The matrices are used to evaluate each sensor and actuator combination, and to determine the simultaneous actuator and sensor locations such that the norm of the each mode is maximized. The approach is illustrated with the truss example that includes the physical interpretation of the selection.

NOMENCLATURE

M	mass matrix
K	stiffness matrix
D	damping matrix
q	displacement
$\dot{q}$	velocity
B	matrix of actuator location
C_q	matrix of displacement sensor location
C_v	matrix of velocity sensor location
n_d	number of degrees of freedom
f	structural input forces
y	structural output
Φ	mode shape matrix
M_m	modal mass matrix
K_m	modal stiffness matrix
D_m	modal damping matrix
q_m	modal displacement
$\dot{q}_m$	modal velocity
B_m	modal matrix of actuator location
C_{qm}	modal matrix of location of displacement sensors
C_{vm}	modal matrix of location of velocity sensors
n	number of modes
ω_i	natural frequency of *i*th mode
ζ_i	damping ratio of *i*th mode
$H(\omega)$	FRF
$\Vert H\Vert_2$	H_2 norm of FRF
$\Vert H\Vert_\infty$	H_∞ norm of FRF
$\Vert H\Vert_h$	Hankel norm of FRF
H_{ijk}	FRF for *i*th mode, *j*th actuator, and *k*th sensor
σ_{ijk}	actuator and sensor placement index for *i*th mode, *j*th actuator, and *k*th sensor

1 INTRODUCTION

Simultaneous selection of sensor and actuator locations presented in this paper is an extension of the actuator and sensor placement algorithm from Refs.[1,2]. The latter algorithm describes either actuator placement for given sensor locations, or sensor placement for given actuator locations. The simultaneous placement is an issue of certain importance, since fixing the locations of sensors while placing actuators (or vice versa) limits the improvement of system performance.

The presented placement algorithm is developed for flexible structures using either H_2, or H_∞, or Hankel modal norm. The norms of a flexible structure with multiple actuators and sensors are expanded into a root-mean-square (rms) sum of norms of structures equipped with a single sensor and single actuator. This property is used to evaluate the importance of an arbitrary combination of sensors and actuators. This comparatively simple tool is applicable to structures only; in a general case system norms cannot be decomposed into a combination of single-input-single-output norms.

The algorithm consists of determination either H_2, or H_∞, or Hankel norms for a single mode, single actuator, and single sensor. Based on these norms the sensor and actuator placement matrices are generated for each considered mode to evaluate sensor and actuator combinations, and to determine the simultaneous actuator and sensor locations that maximize each modal norm. The approach is illustrated with a beam example.

2 MODAL MODEL

A structural model is described by its mass, stiffness, and damping matrices, as well as by the sensors and actuators locations. These parameters are imbedded in the structural second-order differential equation:

$$M\ddot{q} + D\dot{q} + Kq = Bf, \tag{1a}$$

$$y = C_q q + C_v \dot{q} \tag{1b}$$

In this equation q is the structural displacement vector of dimension n_d, f is the input vector of dimension r, y is the output vector of dimension s, and M, D, K are the mass, damping, and stiffness matrices, respectively, of dimensions $n_d \times n_d$. The input matrix B of dimensions $n_d \times r$ characterizes the actuator locations; the output displacement and rate matrices C_q and C_v of dimensions $s \times n_d$ characterize the displacement and rate sensor locations. The mass matrix is positive definite, and the stiffness and damping matrices are positive semidefinite; n_d is the number of degrees of freedom, r is the number of actuators, and s is the number of sensors.

Using modal transformation the above equation can be written in the modal coordinates. For a small and proportional damping, let ω_i be the ith natural frequency and ϕ_i be the ith natural mode, or mode shape. Define the matrix of natural frequencies $\Omega = diag(\omega_1, \omega_2, ..., \omega_n)$, and the mode shape matrix $\Phi = [\phi_1 \quad \phi_2 \quad \dots \quad \phi_n]$, of dimensions $n_d \times n$ that consists of n natural modes. A new variable, the modal displacement vector, q_m, is introduced, such that $q = \Phi q_m$. This transformation produces the modal mass matrix $M_m = \Phi^T M \Phi$, and the modal stiffness matrix $K_m = \Phi^T K \Phi$, both diagonal. For a proportional damping the modal damping matrix $D_m = \Phi^T D \Phi$ is diagonal as well.

Left-multiplying Eq.(1) by Φ^T, and subsequently by M_m^{-1} one obtains the following modal model

$$\ddot{q}_m + 2Z\Omega\dot{q}_m + \Omega^2 q_m = B_m u, \tag{2a}$$

$$y = C_{mq} q_m + C_{mv} \dot{q}_m, \tag{2b}$$

where $Z = 0.5 M_m^{-1} D_m \Omega^{-1}$ is a diagonal matrix of the modal damping, and B_m is the modal input matrix, $B_m = M_m^{-1}\Phi^T B$, while $C_{mq} = C_q \Phi$ and $C_{mv} = C_v \Phi$ are the modal displacement and rate matrices, respectively.

The modal equations (2a,b) can be re-written as a set of n independent equations for each modal displacement

$$\ddot{q}_{mi} + 2\zeta_i \omega_i \dot{q}_{mi} + \omega_i^2 q_{mi} = b_{mi} u, \tag{3a}$$

$$y_i = c_{mqi} q_{mi} + c_{mvi} \dot{q}_{mi}, \qquad i = 1, ..., n \tag{3b}$$

where damping ratio of the ith mode, ζ_i, is the ith diagonal entry of Z. In the above equations y_i is the system output due to the ith mode dynamics, while b_{mi} is the ith row of B_m and c_{mqi} and c_{mvi} are the ith columns of C_{mq}, and C_{mv}, respectively. Define c_{mi} as an equivalent output matrix of the ith mode

$$c_{mi} = \frac{c_{mqi}}{\omega_i} + c_{mvi} \tag{4}$$

then $\|b_{mi}\|_2$ and $\|c_{mi}\|_2$ are the input and output gains of the ith mode, see [1], where $\|x\|_2$ denotes the Euclidean norm of x, i.e., $\|x\|_2 = \sqrt{tr(x^T x)}$.

3 MODAL NORMS

In the following the H_2, H_∞, and Hankel norms are used to measure the system, its modes, and the importance of the actuator and sensor locations. Let $H(\omega)$ be a frequency response function of a structure. Its H_2 norm is defined as follows

$$\|H\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} tr(H^*(\omega)H(\omega))d\omega \tag{5a}$$

while the H_∞ norm is defined as

$$\|H\|_\infty = \max_\omega \sigma_{\max}(H(\omega)), \tag{5b}$$

where $\sigma_{\max}(H(\omega))$ is the largest singular value of $H(\omega)$. The Hankel norm of a structure is its largest Hankel singular value, i.e.,

$$\|H\|_h = \sqrt{\lambda_{\max}(W_c W_o)}. \tag{5c}$$

where $\lambda_{\max}(.)$ denotes the largest eigenvalue, and W_c, W_o are the system controllability and observability grammians, respectively.

In modal coordinates the equations of a flexible structure are uncoupled, see (3). In this case the norms of the *i*th mode and *j*th actuator and *k*th sensor can be approximately determined from the following equations, c.f. [1]

$$\|H_{ijk}\|_2 \cong \frac{\|b_{mij}\|_2 \|c_{mik}\|_2}{2\sqrt{\zeta_i \omega_i}}, \tag{6a}$$

$$\|H_{ijk}\|_\infty \cong \frac{\|b_{mij}\|_2 \|c_{mik}\|_2}{2\zeta_i \omega_i}, \tag{6b}$$

$$\|H_{ijk}\|_h \cong \frac{\|b_{mij}\|_2 \|c_{mik}\|_2}{4\zeta_i \omega_i} \tag{6c}$$

(The approximate equality sign "$\cong$" is used in the following sense: $x \cong y$ if $\|x-y\| << \|x\|$). Note that the H_∞ norm is approximately twice the Hankel norm, that is, $\|H_{ijk}(\omega)\|_\infty \cong 2\|H_{ijk}(\omega)\|_h$.

The H_2, H_∞ and Hankel norms of the *i*th mode are determined as the root-mean-square sum over all actuators and sensors, see [2]:

$$\|H_{mi}\|_2^2 \cong \sum_{j=1}^{r}\sum_{k=1}^{s} \|H_{ijk}\|_2^2, \quad i = 1, \ldots, n. \tag{7a}$$

$$\|H_{mi}\|_\infty^2 \cong \sum_{j=1}^{r}\sum_{k=1}^{s} \|H_{ijk}\|_\infty^2, \quad i = 1, \ldots, n. \tag{7b}$$

$$\|H_{mi}\|_h^2 \cong \sum_{j=1}^{r}\sum_{k=1}^{s} \|H_{ijk}\|_h^2, \quad i = 1, \ldots, n. \tag{7c}$$

The H_2 norm of the total system is approximately the root-mean-square sum over all its modal norms, that is

$$\|H\|_2^2 \cong \sum_{i=1}^{n} \|H_{mi}\|_2^2, \tag{8}$$

and the H_∞ and Hankel norms of the total system are approximately the largest of its modal norms, that is

$$\|H\|_\infty \cong \max_i \|H_{mi}\|_\infty, \quad \text{and} \quad \|H\|_h \cong \max_i \|H_{mi}\|_h \tag{9}$$

for $i = 1, \ldots, n$.

4 ACTUATOR AND SENSOR PLACEMENT

In this section the symbol $\|.\|$ will denote either H_2, or H_∞, or Hankel norm. For the set R of the candidate actuator locations, one shell select a subset r of actuators, and concurrently for the set S of the candidate sensor locations, one shell select a subset s of sensors. The criterion is the maximization of the system norm.

Recall that the norm $\|H_{ijk}\|$ characterizes the *i*th mode equipped with *j*th actuator and *k*th sensor. For the *i*th mode define the actuator and sensor placement index as follows

$$\sigma_{ijk} = \frac{\|H_{ijk}\|}{\|H_{mi}\|} \tag{10}$$

The placement index σ_{ijk} is a measure of the participation of the *j*th actuator and *k*th sensor in the impulse response of the *i*th mode. Note also that the following property of the placement indices holds

$$\sigma_{ijk}\sigma_{ilm} \cong \sigma_{ijm}\sigma_{ilk} \tag{11}$$

This property can be proven by the substitution of the norms as in Eqs.(6a,b,c) into the definition (10) of the index.

Using this index the actuator and sensor placement matrix of the *i*th mode is generated,

$$\Sigma_i = \begin{bmatrix} \sigma_{i11} & \sigma_{i12} & \cdots & \sigma_{i1k} & \cdots & \sigma_{i1S} \\ \sigma_{i21} & \sigma_{i22} & \cdots & \sigma_{i2k} & \cdots & \sigma_{i2S} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{ij1} & \sigma_{ij2} & \cdots & \sigma_{ijk} & \cdots & \sigma_{ijS} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{iR1} & \sigma_{iR2} & \cdots & \sigma_{iRk} & \cdots & \sigma_{iRS} \end{bmatrix} \begin{array}{l} \\ \\ \\ \Leftarrow \quad j\text{th actuator} \\ \\ \\ \end{array}, \tag{12}$$

$$\Uparrow$$

kth sensor

$i=1,\ldots,n$

For the *i*th mode the *j*th actuator index is the rms sum over all selected sensors,

$$\sigma_{aij} = \sqrt{\sum_{k=1}^{s} \sigma_{ijk}^2} \tag{13a}$$

For the same mode the *k*th sensor index is the rms sum over all selected actuators

$$\sigma_{sik} = \sqrt{\sum_{j=1}^{r} \sigma_{ijk}^2} \tag{13b}$$

These indices, however, cannot be readily evaluated, since in order to evaluate the actuator index one needs to know the sensor locations (which have not been yet selected), and vice versa. This difficulty can be overcome by using the property (11). Note that by choosing the two largest indices for the ith mode, say σ_{ijk} and σ_{ilm} (and $\sigma_{ijk} > \sigma_{ilm}$) the corresponding indices σ_{ijm} and σ_{ilk} are also large. In order to show it note that $\sigma_{ilm} \le \sigma_{ijm} \le \sigma_{ijk}$ and $\sigma_{ilm} \le \sigma_{ilk} \le \sigma_{ijk}$ as a result of (11) and the fact that $\sigma_{ijm} \le \sigma_{ijk}$ and $\sigma_{ilk} \le \sigma_{ijk}$. In consequence, by selecting individual actuator and sensor locations with the largest indices one automatically maximize the indices (13a) and (13b) of the sets of actuators and sensors.

The determination of locations of large indices is illustrated with the following example. Let σ_{123}, σ_{138}, and σ_{164} be the largest indices selected for the first mode. They correspond to 2,3, and 6 actuator locations, and 3, 4, and 8 sensor locations. They are marked in dark color in Fig.1. According to (11) the indices σ_{124}, σ_{128}, σ_{133}, σ_{134}, σ_{163}, and σ_{168} are also large. They are marked with light color in Fig.1. Now we see that the rms summation for actuators is over all selected sensors (3, 4, and 8), and the rms summation for sensors is for over all selected actuators (2, 3, and 6), and that both summations maximize the actuator and sensor indices.

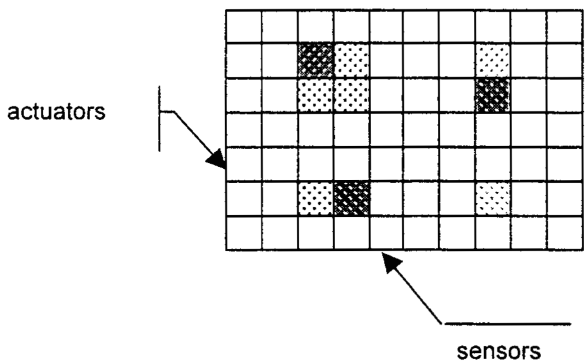

Figure 1: An example of the actuator and sensor placement matrix for the first mode.

5 EXAMPLE

An actuator and sensor placement procedure is illustrated with a clamped beam in Fig.2. The beam is of 150 cm length, cross-section of 1 cm², divided into 15 equal elements. The external (filled) nodes are clamped. The candidate actuator locations are the vertical forces at nodes 1 to 14, and the candidate sensor locations are the vertical rate sensors located at nodes 1 to 14. Using H_∞ norm, and considering the first four modes, we shall determine at most 4 actuator and 4 sensor locations (one for each mode).

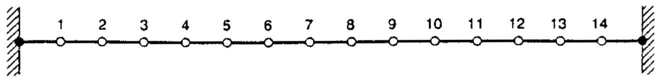

Figure 2: A clamped beam.

In this example, n=4, and R=S=14. Using Eqs.(6), (10), and (12) the placement matrices for the first four modes were determined and plotted in Figs.3-6. Before the placement procedure is applied the accuracy of Eq.(11) is checked. For this purpose the second mode is chosen, i.e., i=2, and the following actuator and sensor locations are selected: j=k=3, l=m=q, and q=1,...,14. For these parameters Eq.(11) is as follows

$$\sigma_{233}\sigma_{2qq} \cong \sigma_{23q}\sigma_{2q3}, \qquad q=1,\dots,14$$

The plots of the left- and right-hand side of the above equations are shown in Fig.7, showing good coincidence.

The maximal values of the actuator and sensor index in the placement matrix determine the preferred location of actuator and sensor for each mode. Note that for each mode four locations: two sensor locations and two actuator locations have the same maximal value. Moreover, they are symmetrical with respect to the beam center, see Table1. We selected four collocated sensors and actuators at the left-hand side of the beam center, one for each mode. Namely, for mode 1 – node 8, for mode 2 – node 4, for mode 3 – node 3, and for mode 4 – node 2.

	(Actuator, Sensor) location
Mode 1	(8,8), (7,7), (7,8), (8,7)
Mode 2	(4,4), (11,11), (4,11), (11,4)
Mode 3	(3,3), (12,12), (3,12), (12,3)
Mode 4	(2,2), (13,13), (2,13), (13,2)

TABLE 1. The best actuator and sensor locations for the first four modes

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[2] **Gawronski, W.** and **Lim, K.B.,** *Balanced Actuator and Sensor Placement for Flexible Structures*. International Journal of Control , vol. 65, p.131-. 1996

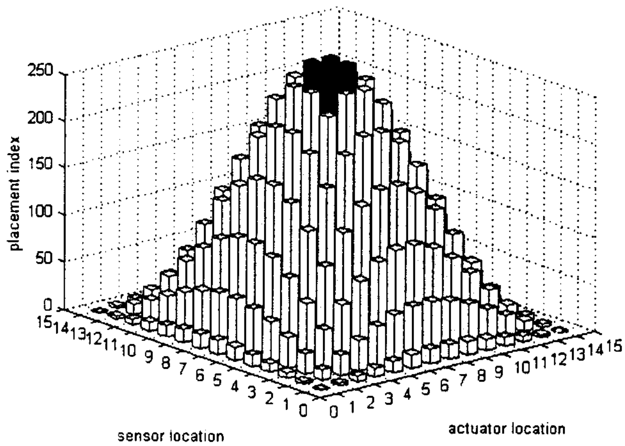

Figure 3: Actuator and sensor placement matrix for mode 1. The maximal placement indices, in dark color, correspond to the following (actuator, sensor) locations: (8,8), (7,7), (7,8), and (8,7)

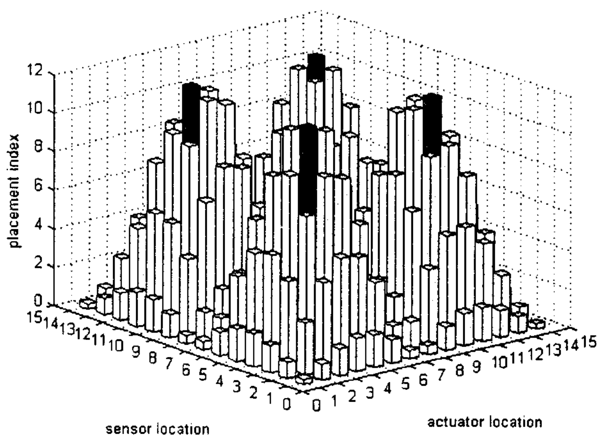

Figure 4: Actuator and sensor placement matrix for mode 2. The maximal placement indices, in dark color, correspond to the following (actuator, sensor) locations: (4,4), (4,11), (11,11), (11,4)

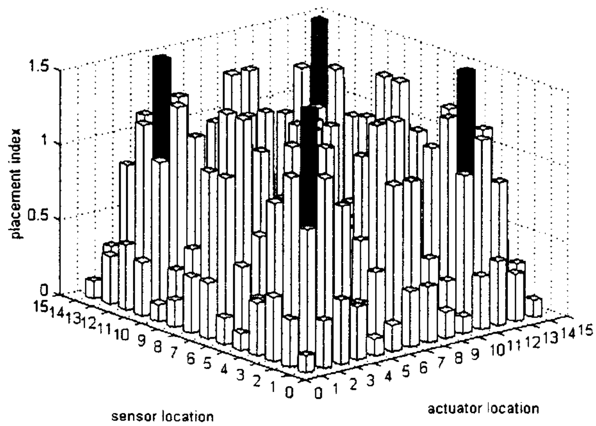

Figure 5. Actuator and sensor placement matrix for mode 3. The maximal placement indices, in dark color, correspond to the following (actuator, sensor) locations: (12,12), (3,12), (12,3), (3,3)

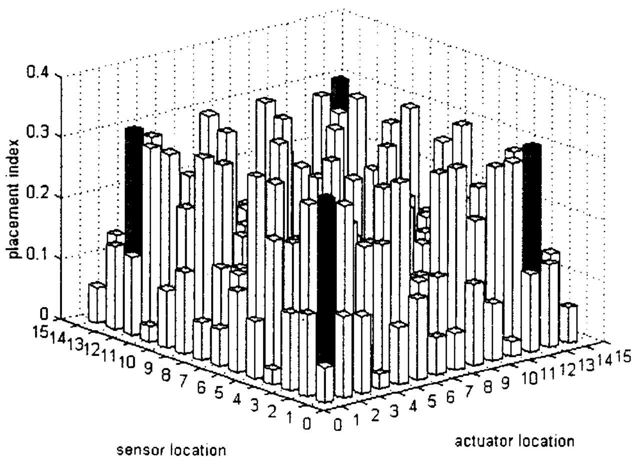

Figure 6: Actuator and sensor placement matrix for mode 4. The maximal placement indices, in dark color, correspond to the following (actuator, sensor) locations: (2,2), (2,13), (13,2), (13,13)

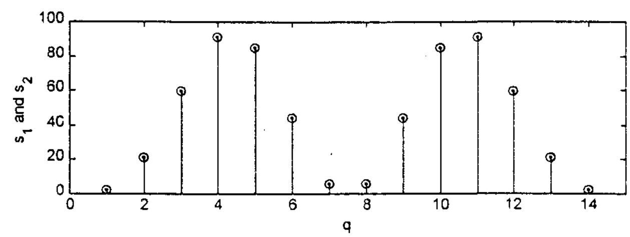

Figure 7. The verification of Eq.(11): ○ denotes $s_1 = \sigma_{233}\sigma_{2qq}$, **and • denotes** $s_2 = \sigma_{23q}\sigma_{2q3}$.