

Analytical Modeling of the White Light Fringe

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We developed technique for extracting the phase, visibility and amplitude information as needed for interferometric astrometry with the Space Interferometry Mission (SIM). Our analytical model accounts for a number of physical and instrumental effects, and is valid for a general case of bandpass filter. We were able to obtain general solution for polychromatic phasors and address properties of unbiased fringe estimators in the presence of noise. For demonstration purposes we studied the case of rectangular bandpass filter with two different methods of optical path difference (OPD) modulation – stepping and ramping OPD modulations. A number of areas of further studies relevant to instrument design and simulations are outlined and discussed.

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1. Introduction

SIM is designed as a space-based 10-m baseline Michelson optical interferometer operating in the visible waveband (see Ref. 1 for more details). This mission will open

up many areas of astrophysics, via astrometry with unprecedented accuracy. Over a narrow field of view SIM is expected to achieve a mission accuracy of $1 \mu\text{as}$. In this mode SIM will search for planetary companions to nearby stars by detecting the astrometric “wobble” relative to a nearby ($\leq 1^\circ$) reference star. In its wide-angle mode, SIM will be capable to provide a $4 \mu\text{as}$ precision absolute position measurements of stars, with parallaxes to comparable accuracy, at the end of a 5-year mission. The expected proper motion accuracy is around $3 \mu\text{as}/\text{yr}$, corresponding to a transverse velocity of 10 m/s at a distance of 1 kpc.

The SIM instrument does not directly measure the angular separation between stars, but the projection of each star direction vector onto the interferometer baseline by measuring the pathlength delay of starlight as it passes through the two arms of the interferometer. The delay measurement is made by a combination of internal metrology measurements to determine the distance the starlight travels through each arm, external metrology measurements that determine the length and local orientation of the baseline, and a measurement of the central white light fringe to determine the point of equal optical pathlength.

This paper discusses analytic model for the white light fringe data extraction. Our goal here is to establish functional dependency of the white light fringe parameters on the instrumental input parameters. The problem of interference of electromagnetic radiation is well studied and extensive number of publications on this subject are available (see Refs. 2-14 and references therein). Because of complexity of this problem in the general case of polychromatic light, most of the current research is done numerically. While numerical studies have proven to be extremely valuable in

analyzing the interference patterns and are very useful in addressing various instrumental effects, the analytical methods provide the much needed critical understanding of the white light interference phenomena. The primary motivation for the work was the idea to use the averaged and bias-corrected complex phasors to estimate the external pathlength difference for the incoming polychromatic light. A definition for the complex visibility phasors is stemming from the form of a complex visibility function, $\tilde{V} = ve^{i\phi}$, where v is the visibility and ϕ is its phase. Decomposing this expression onto real and imaginary parts as $\tilde{V} = X + iY$, one obtains the complex visibility phasors, $X = v \cos \phi$ and $Y = v \sin \phi$. SIM will be able to effectively determine both visibility and phase of the fringe, but for the astrometric purposes the phase must be determined to a much higher accuracy of a few tens of picometers. Phase determination in the presence of noise is a non-linear process which requires careful approach to averaging and correcting for biases in the data.¹⁵ It will be demonstrated below that analytic solution may be used as a tool to study the complex interferometric phenomena on a principally different qualitative level.^{2,10}

In this paper we derive analytic model that may be used to describe photoelectron detection process. We analytically describe the physical and instrumental processes that are important in estimating the fringe parameters (i.e. intensity of incoming radiation, it's visibility and the phase of the fringe). Effects that are not included in the model are due to polarization of both incoming light and the instrumental throughput, effect of the wavefront-tilt, low frequency vibrations, drifts, jitter, etc. We plan to address these issues in a subsequent publication.¹⁸

The paper is organized as follows: In Section 2 we introduce parameterization

for the polychromatic fringe pattern and define the quantities that are forming the astrometric signal on the CCD. Specifically, we derive solution for the white light fringe equation in the general case. In Section 3 we present general analytic solution for complex visibility phasors and will discuss a noise suppression approach. In Section 4 we develop technique for studying the rectangular bandpass filter. We also obtain functional dependency of our solution in the two cases of OPD modulation, namely the stepping and ramping modulations. In Section 5 we present conclusions and recommendations for future studies of accurate fringe reconstruction.

2. Parameterization of Polychromatic Fringe Pattern

The current algorithms and simulations for optical interferometry are all based on monochromatic light. This is a good approximation for some of existing testbed configurations that use as many as 80 spectral channels for dispersed light. Nominally the flight system will use four to eight channels for guide interferometers. Because of the large bandwidth of each channel (87.5 nm), the quasi-monochromatic assumptions are not valid, and modifications to the algorithms are necessary.^{16,17} In this Section we will introduce a method designed to address this issue.

Description of the interferometric pattern in the polychromatic case that involves a finite bandwidth of radiation - is a complicated task. Thus, the observational conditions in the case of polychromatic light are significantly altered compare to the simplicity of the monochromatic process. In general, all the quantities involved are complicated functions of the wavelength. A way to describe this process is to collect contributions of all infinitesimal constituents of polychromatic light at different wave-

lengths within the bandwidth of the incoming electromagnetic radiation.¹⁵ In other words, the total number of photo-electron counts, N , registered by a CCD detector per wavenumber and per unit time, may be given by the following expression:

$$dN(k, t) = \mathcal{F}(k)\mathcal{I}_0(k) \left(1 + V(k) \sin [\phi(k) + kx(t)]\right) dk dt, \quad (1)$$

where $\mathcal{F}(k)$ is a dimensionless factor representing the total instrumental throughput;^{16,17} $\mathcal{I}_0(k)$, $V(k)$ and $\phi(k)$ are the intensity, visibility and phase of the incoming light; $x(t)$ is modulated internal delay. We are using a nomenclature where a wavenumber k relates to the wavelength as follows $k = \frac{2\pi}{\lambda}$. We also accounted for the nominal $\frac{\pi}{2}$ phase shift due to the SIM beam splitter, which produces a sine fringe rather than a cosine one.

Note that the total instrumental throughput depends on a number of other factors, some of these are the collective area of the detector, quantum efficiency of CCD, and overall spectral response of the instrument. Our goal here is to derive observational equation that may be used to estimate the apparent fringe phase and visibility. To estimate the true source visibility and phase one would have to perform a set of additional calibration and estimation procedures that will be addressed elsewhere.¹⁸

A. *Integration Over the Spectral Bandwidth*

In this Section we will perform integrations of Eq. (1) over wavenumber space and time, that are necessary to derive analytical model. This model will be used further for the purposes of the fringe parameters estimation.

Let us first perform integration over the SIM wavenumber bandwidth $k \in [k_{\text{SIM}}^-, k_{\text{SIM}}^+]$, where $k_{\text{SIM}}^- = 450$ nm is the beginning of the SIM bandwidth, and $k_{\text{SIM}}^+ = 950$ nm is

the end of this bandwidth, thus $k \in [450, 950]$ nm. A formal integration of Eq.(1) over dk leads to the following result

$$N(t)\Delta k_{\text{SIM}} = \int_{k_{\text{SIM}}^-}^{k_{\text{SIM}}^+} N(k, t) dk, \quad \Delta k_{\text{SIM}} = k_{\text{SIM}}^+ - k_{\text{SIM}}^-. \quad (2)$$

In the case of channeled (or dispersed) spectrum output, the integration of this equation over the range of wavenumbers is straightforward. For this purpose, we designate index, ℓ , to denote a particular spectral channel. Suppose that there exists a total of L spectral channels, thus $\ell \in [1, L]$.

Our definition for the spectral channel ℓ implies the width of the channel $\Delta k_\ell = k_\ell^+ - k_\ell^-$ and existence of a “central” wavenumber k_ℓ within this channel. We also, assume continuous spectrum within the bandwidth, so that there is no gaps exist in the interval $k \in [k_{\text{SIM}}^-, k_{\text{SIM}}^+]$. A consequence of this is the equality $k_{\ell+1}^- = k_\ell^+$, which leads to the following discrete representation of the bandwidth

$$\Delta k_{\text{SIM}} = k_{\text{SIM}}^+ - k_{\text{SIM}}^- = \sum_{\ell=1}^L (k_\ell^+ - k_\ell^-) \equiv \sum_{\ell=1}^L \Delta k_\ell. \quad (3)$$

Result Eq. (3) allows us to rewrite Eq. (2) as follows:

$$N(t)\Delta k_{\text{SIM}} = \int_{k_{\text{SIM}}^-}^{k_{\text{SIM}}^+} N(k, t) dk = \sum_{\ell=1}^L \int_{k_\ell^-}^{k_\ell^+} N(k, t) dk = \sum_{\ell=1}^L N_\ell(t)\Delta k_\ell, \quad (4)$$

where $N_\ell(t)$ is the instantaneous number of photons within a particular spectral channel. As a result, Eq. (1) may be integrated as given below

$$N_\ell(t)\Delta k_\ell = \int_{k_\ell^-}^{k_\ell^+} N(k, t) dk = \int_{k_\ell^-}^{k_\ell^+} \mathcal{F}(k)\mathcal{I}_0(k) \left(1 + V(k) \sin [\phi(k) + kx(t)]\right) dk, \quad (5)$$

or, equivalently

$$N_\ell(t) = \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \mathcal{F}(k)\mathcal{I}_0(k) \left(1 + V(k) \sin [\phi(k) + kx(t)]\right) dk. \quad (6)$$

This equation is our first important result. It will help to focus our attention from the discussion of coherent processes within the whole wide bandwidth, onto addressing this processes on a smaller scale – within a particular narrow spectral channel, ℓ .

B. Definitions for the Fringe Parameters

At this point, it is convenient to introduce a set of useful notations. First of all, we define the average total intensity of incoming electromagnetic radiation, $\mathcal{I}_{0\ell}$, within the ℓ -th spectral channel as

$$\mathcal{I}_{0\ell} = \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \mathcal{F}(k) \mathcal{I}_0(k) dk. \quad (7)$$

It is natural to introduce normalized intensity of light $\hat{\mathcal{I}}_{0\ell}$ within the ℓ -th channel:

$$\hat{\mathcal{I}}_{0\ell}(k) = \frac{\mathcal{F}(k) \mathcal{I}_0(k)}{\mathcal{I}_{0\ell}} \quad \text{with} \quad \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \hat{\mathcal{I}}_{0\ell}(k) dk = 1. \quad (8)$$

These new notations allow to present Eq. (6) as given below

$$N_\ell(t) = \mathcal{I}_{0\ell} \left(1 + \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \hat{\mathcal{I}}_{0\ell}(k) V(k) \sin [\phi(k) + kx(t)] dk \right). \quad (9)$$

In the next Section we will define the fringe visibility, phase and mean wavenumber.

1. Fringe Visibility, Mean Wavenumber and Phase

To further simplify the obtained equation, we will introduce functional form the fringe visibility, the phase and the wavenumber notations. Thus, the fringe visibility, $V_{0\ell}$, within the ℓ -th channel is given as

$$V_{0\ell} = \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \hat{\mathcal{I}}_{0\ell}(k) V(k) dk. \quad (10)$$

Similarly to Eq. (8) we denote normalized visibility in the channel as

$$\hat{V}_{0\ell}(k) = \frac{\hat{\mathcal{I}}_{0\ell}(k)V(k)}{V_{0\ell}} \equiv \frac{\mathcal{F}(k)\mathcal{I}_0(k)V(k)}{\mathcal{I}_{0\ell}V_{0\ell}}, \quad \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \hat{V}_{0\ell}(k) dk = 1. \quad (11)$$

These definitions help us to re-write equation (9) in the following compact form

$$N_\ell(t) = \mathcal{I}_{0\ell} \left(1 + V_{0\ell} \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \hat{V}_{0\ell}(k) \sin[\phi(k) + kx(t)] dk \right). \quad (12)$$

To define mean wavenumber, k_ℓ , and mean phase, ϕ_ℓ , for the ℓ -th spectral channel we will use the following expression:

$$k_\ell = \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \hat{\mathcal{I}}_{0\ell}(k) k dk \equiv \frac{1}{\mathcal{I}_{0\ell} \Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \mathcal{F}(k)\mathcal{I}_0(k) k dk, \quad (13)$$

There are two ways to define the phase within the channel. Thus, it is tempting to define the mean phase as

$$\phi_\ell = \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \hat{\mathcal{I}}_{0\ell}(k) \phi(k) dk \equiv \frac{1}{\mathcal{I}_{0\ell} \Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \mathcal{F}(k)\mathcal{I}_0(k) \phi(k) dk. \quad (14)$$

This definition is acceptable for narrow spectral channel, however for a wide channel one needs a more convenient form, namely

$$\phi(k_\ell), \quad \text{which is} \quad \phi(k_\ell) \neq \phi_\ell, \quad (15)$$

and is simply the phase value at the specific wavenumber. In our further analysis we will be using this later definition. (The relationships between the two definitions for the phase Eqs. (14) and (15) will be addressed in Appendix A).

The three introduced quantities (i.e. the visibility, mean wavenumber k_ℓ and phase at the mean wavenumber $\phi(k_\ell)$) allow to proceed with integration of Eq. (12).

C. Complex Fringe Envelope Function

Definitions introduced in the previous Section allow us to separate functions k_ℓ and $\phi(k_\ell)$ from the functions with direct dependency on the wavenumber k . As a result, Eq. (12), for the total photon count, may be presented as below

$$\begin{aligned}
 N_\ell(t) = & \mathcal{I}_{0\ell} \left(1 + \right. \\
 & + V_{0\ell} \sin [\phi(k_\ell) + k_\ell x(t)] \cdot \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \hat{V}_{0\ell}(k) \cos [(k - k_\ell)x(t) + \phi(k) - \phi(k_\ell)] dk + \\
 & \left. + V_{0\ell} \cos [\phi(k_\ell) + k_\ell x(t)] \cdot \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \hat{V}_{0\ell}(k) \sin [(k - k_\ell)x(t) + \phi(k) - \phi(k_\ell)] dk \right). \quad (16)
 \end{aligned}$$

To further simplify the analysis, it is convenient to introduce the complex fringe envelope function, $\tilde{W}_\ell[\Delta k_\ell, \phi(k_\ell), x(t)]$, which is given as

$$\tilde{W}_\ell[\Delta k_\ell, \phi(k_\ell), x(t)] = \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \hat{V}_{0\ell}(k) e^{j((k - k_\ell)x(t) + \phi(k) - \phi(k_\ell))} dk \equiv \quad (17)$$

$$\equiv \frac{1}{\mathcal{I}_{0\ell} V_{0\ell} \Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \mathcal{F}(k) \mathcal{I}_0(k) V(k) e^{j((k - k_\ell)x(t) + \phi(k) - \phi(k_\ell))} dk. \quad (18)$$

As a complex function, \tilde{W}_ℓ (please refer to discussion of the fringe envelope function given in Appendix B) may be equivalently presented by its real, $\text{Re}\{\tilde{W}_\ell\}$, and imaginary, $\text{Im}\{\tilde{W}_\ell\}$, components:

$$\tilde{W}_\ell = \text{Re}\{\tilde{W}_\ell\} + j \text{Im}\{\tilde{W}_\ell\} \quad (19)$$

with

$$\begin{aligned}
 \text{Re}\{\tilde{W}_\ell\} &= \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \hat{V}_{0\ell}(k) \cos [(k - k_\ell)x(t) + \phi(k) - \phi(k_\ell)] dk, \\
 \text{Im}\{\tilde{W}_\ell\} &= \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \hat{V}_{0\ell}(k) \sin [(k - k_\ell)x(t) + \phi(k) - \phi(k_\ell)] dk. \quad (20)
 \end{aligned}$$

This definition of complex envelope function given by Eq.(17)-(20) allows us to present

expression (16) in a simpler form:

$$N_\ell(t) = \mathcal{I}_{0\ell} \left(1 + V_{0\ell} \sin [\phi(k_\ell) + k_\ell x(t)] \cdot \text{Re} \{ \tilde{W}_\ell [x(t)] \} + V_{0\ell} \cos [\phi(k_\ell) + k_\ell x(t)] \cdot \text{Im} \{ \tilde{W}_\ell [x(t)] \} \right). \quad (21)$$

The complex fringe envelope function, $\tilde{W}_\ell [x(t)]$, as any complex function, may also be represented by its amplitude and its phase, namely:

$$\tilde{W}_\ell [\Delta k_\ell, \phi(k_\ell), x(t)] = \mathcal{E}_\ell [\Delta k_\ell, \phi(k_\ell), x(t)] e^{i\Omega_\ell [\Delta k_\ell, \phi(k_\ell), x(t)]}, \quad (22)$$

where \mathcal{E}_ℓ and Ω_ℓ are the amplitude and phase correspondingly. For the complex envelope function Eq. (19) these two are given as follows:

$$\mathcal{E}_\ell(t) = \sqrt{\text{Re}^2 \{ \tilde{W}_\ell \} + \text{Im}^2 \{ \tilde{W}_\ell \}}, \quad \Omega_\ell(t) = \text{ArcTan} \left\{ \frac{\text{Im} \{ \tilde{W}_\ell \}}{\text{Re} \{ \tilde{W}_\ell \}} \right\}. \quad (23)$$

Finally, we re-write Eq. (21) in the following general form:

$$N_\ell(t) = \mathcal{I}_{0\ell} \left(1 + V_{0\ell} \mathcal{E}_\ell(t) \cdot \sin [\phi(k_\ell) + k_\ell x(t) + \Omega_\ell(t)] \right). \quad (24)$$

Note that the apparent visibility of the fringe now is the product of the true averaged visibility and the modulus of the phase corrected Fourier transform of the filter function, evaluated at the current delay or

$$\tilde{\Gamma}_x = \tilde{V}_{0\ell} \tilde{W}_\ell \equiv V_{0\ell} \mathcal{E}_\ell e^{j(\phi(k_\ell) + \Omega_\ell)}. \quad (25)$$

It is known that the transfer function \tilde{W}_ℓ describes the coherence envelope.² If $\hat{V}_{0\ell}(k) \sim \mathcal{F}(k)\mathcal{I}_0(k)V(k)$ is symmetric, then \tilde{W}_ℓ is real valued, $\Omega_\ell = 0$, and only at zero delay,¹¹ where the envelope is at peak, is the true visibility observed.

D. Temporal Integration

The last integration to be performed in Eq.(1) (or equivalently Eq.(24)), is the integration over time. The optical pathlength difference may be modulated either as a set of discrete

values corresponding to a number of steps in the OPD space (stepping PZT modulation) or by ramping PZT over the range of OPD values. The total integration time, Δt , is the sum of durations of eight temporal bins. (While our result is applicable for arbitrary number of temporal bins, the SIM design will utilize 8 temporal bins):

$$\Delta t = t^+ - t^- = \sum_{i=1}^{N=8} \Delta \tau_i, \quad \text{with} \quad \Delta \tau_i = t_i^+ - t_i^-. \quad (26)$$

Direct integration of Eq. (21) leads to expression for the total number of photons collected at each PZT stroke:

$$N_\ell \Delta t = \int_{t^-}^{t^+} N_\ell(t) dt = \sum_{i=1}^{N=8} \int_{t_i^-}^{t_i^+} N_\ell(t) dt = \sum_{i=1}^{N=8} N_{\ell i} \Delta \tau_i. \quad (27)$$

where $N_{\ell i} \Delta \tau_i$ is the total number of photons collected in a particular i -th temporal bin and for the ℓ -th spectral channel. Substituting $N_\ell(t)$ from Eq. (24) directly into Eq. (27), one obtains following expression for $N_{\ell i}$:

$$\begin{aligned} N_{\ell i} \Delta \tau_i &= \frac{1}{\Delta \tau_i} \int_{t_i^-}^{t_i^+} N_\ell(t) dt = \\ &= \frac{1}{\Delta \tau_i} \int_{t_i^-}^{t_i^+} dt \mathcal{I}_{0\ell} \left(1 + V_{0\ell} \sin \phi(k_\ell) \cdot \mathcal{E}_\ell(t) \cdot \cos [k_\ell x(t) + \Omega_\ell(t)] + \right. \\ &\quad \left. + V_{0\ell} \cos \phi(k_\ell) \cdot \mathcal{E}_\ell(t) \cdot \sin [k_\ell x(t) + \Omega_\ell(t)] \right). \end{aligned} \quad (28)$$

To complete this integration, we assume that quantities $\mathcal{I}_{0\ell}$, $V_{0\ell}$, $\phi(k_\ell)$, and k_ℓ do not change with time during the photon-counting intervals. The only quantity that is explicitly varies with time – is the optical pathlength difference $x(t)$.

The integration over time may be performed in a general form and corresponding expression for the photon count, $N_{\ell i}$, is given as follows:

$$N_{\ell i} = \mathcal{I}_{0\ell} \left(1 + V_{0\ell} \sin \phi(k_\ell) \cdot \text{Re}\{\tilde{\mathcal{P}}_{\ell i}\} + V_{0\ell} \cos \phi(k_\ell) \cdot \text{Im}\{\tilde{\mathcal{P}}_{\ell i}\} \right), \quad (29)$$

with quantities $\text{Re}\{\tilde{\mathcal{P}}_{\ell i}\}$ and $\text{Im}\{\tilde{\mathcal{P}}_{\ell i}\}$ given as below:

$$\begin{aligned}\text{Re}\{\tilde{\mathcal{P}}_{\ell i}\} &= \frac{1}{\Delta\tau_i} \int_{t_i^-}^{t_i^+} dt \mathcal{E}_\ell[\Delta k_\ell, \phi(k_\ell), x(t)] \cos [k_\ell x(t) + \Omega_\ell[\Delta k_\ell, \phi(k_\ell), x(t)]], \\ \text{Im}\{\tilde{\mathcal{P}}_{\ell i}\} &= \frac{1}{\Delta\tau_i} \int_{t_i^-}^{t_i^+} dt \mathcal{E}_\ell[\Delta k_\ell, \phi(k_\ell), x(t)] \sin [k_\ell x(t) + \Omega_\ell[\Delta k_\ell, \phi(k_\ell), x(t)]].\end{aligned}\quad (30)$$

For convenience of further analysis, we combined these two real-valued matrices into one complex matrix $\tilde{\mathcal{P}}_{\ell i}$

$$\tilde{\mathcal{P}}_{\ell i} = \text{Re}\{\tilde{\mathcal{P}}_{\ell i}\} + j \text{Im}\{\tilde{\mathcal{P}}_{\ell i}\}.\quad (31)$$

Furthermore, with definitions for $\text{Re}\{\tilde{\mathcal{P}}_{\ell i}\}$ and $\text{Im}\{\tilde{\mathcal{P}}_{\ell i}\}$, Eq. (30), this complex matrix may be presented as given below

$$\tilde{\mathcal{P}}_{\ell i} = \frac{1}{\Delta\tau_i} \int_{t_i^-}^{t_i^+} dt \mathcal{E}_\ell[\Delta k_\ell, \phi(k_\ell), x(t)] e^{j(k_\ell x(t) + \Omega_\ell[\Delta k_\ell, \phi(k_\ell), x(t)])},\quad (32)$$

which is conveniently transforms as follows

$$\tilde{\mathcal{P}}_{\ell i} = \frac{1}{\Delta\tau_i} \int_{t_i^-}^{t_i^+} dt e^{j k_\ell x(t)} \tilde{W}_\ell[\Delta k_\ell, \phi(k_\ell), x(t)],\quad (33)$$

where the complex envelope function \tilde{W}_ℓ given by Eqs. (17)-(18). Eq. (33) may further be transformed to establish its true dependency on time and wavenumber. To do this, we substitute the expression for the complex envelope function from Eq. (18). Thus, one obtains the following expression for the matrix $\tilde{\mathcal{P}}_{\ell i}$:

$$\begin{aligned}\tilde{\mathcal{P}}_{\ell i} &= \frac{1}{\mathcal{I}_{0\ell} V_{0\ell} \Delta k_\ell \Delta\tau_i} \times \\ &\times \int_{t_i^-}^{t_i^+} dt e^{j k_\ell x(t)} \int_{k_\ell^-}^{k_\ell^+} dk \mathcal{F}(k) \mathcal{I}_0(k) V(k) e^{j((k-k_\ell)x(t) + \phi(k) - \phi(k_\ell))}.\end{aligned}\quad (34)$$

Equivalently, one has following expression with explicit dependency

$$\tilde{\mathcal{P}}_{\ell i} = \frac{1}{\mathcal{I}_{0\ell} V_{0\ell} \Delta k_\ell \Delta\tau_i} \int_{t_i^-}^{t_i^+} \int_{k_\ell^-}^{k_\ell^+} dt dk \mathcal{F}(k) \mathcal{I}_0(k) V(k) e^{j(k x(t) + \phi(k) - \phi(k_\ell))}.\quad (35)$$

Finally, we have defined everything that is needed to study Eq. (29), for the polychromatic fringe, which may equivalently be presented in a matrix form as below

$$\begin{pmatrix} N_{\ell 1} \\ \dots \\ N_{\ell N} \end{pmatrix} = \begin{pmatrix} 1; & \text{Im}\{\tilde{\mathcal{P}}_{\ell 1}\}; & \text{Re}\{\tilde{\mathcal{P}}_{\ell 1}\} \\ \dots & \dots & \dots \\ 1; & \text{Im}\{\tilde{\mathcal{P}}_{\ell N}\}; & \text{Re}\{\tilde{\mathcal{P}}_{\ell N}\} \end{pmatrix} \begin{pmatrix} \mathcal{I}_{0\ell} \\ \mathcal{I}_{0\ell} V_{0\ell} \cos \phi(k_\ell) \\ \mathcal{I}_{0\ell} V_{0\ell} \sin \phi(k_\ell) \end{pmatrix}, \quad (36)$$

where the complex matrix $\tilde{\mathcal{P}}_{\ell i}$ is given by Eq. (35).

The obtained result Eq. (29), (35) (or, equivalently Eq. (36)) constitutes the general form of expression for the polychromatic fringe. We will use this result to finalize the development of the general form of the observational model for polychromatic case with arbitrary phase modulation.

Ideally, one would need to determine not only three quantities $\mathcal{I}_{0\ell} V_{0\ell} \cos \phi(k_\ell)$, $\mathcal{I}_{0\ell} V_{0\ell} \sin(k_\ell)$, and $\mathcal{I}_{0\ell}$, but the full functional dependency of the original quantities. However, the finite width of the observational band-width Δk_ℓ complicates the estimation process by bringing the non-linearity in the observational equation via the envelop function W . Note that if one neglects the size of the bandwidth Δk_ℓ with respect to the mean wavenumber k_ℓ or $\Delta k_\ell/k_\ell \rightarrow 0$ (the envelop function becomes unity $W \rightarrow 1$), one recovers the full simplicity of the monochromatic case.¹⁵

3. General Solution for Polychromatic Phasors With Noisy Data

Currently in use, there are two fringe estimators, one for visibility (the unbiased estimator is V^2), and one for the phase (the unbiased estimator is the complex phasor). The V^2 estimator is already worked out in much detail (i.e. Refs. 2-14, 16) if the complex phasor estimator is completed. So the development of the complex phasor

was the main purpose for the presented work. As it is known, the complex fringe visibility can be represented by a phasor; if the fringe is stable, we can add the phasors vectorially over multiple samples. This co-adding can provide an improved signal-to-noise ratio. To co-add the fringe phasors requires a phase reference, for instance the white light phase.^{3,4}

In this Section we will develop optimally-weighted solution that accounts for a number of noise sources and will be applicable for a general case of delay modulation.

A. *Parameterization of the Fringe Equation*

For the purposes of clarity we will omit spectral index ℓ . All the obtained results are valid for any channel and thus could be easily reconstructed, if needed.

In the case of noisy data, observations of phonon-counts N_i are actually done with errors and, in reality, we observe $N_i = \bar{N}_i + \epsilon_i$, where \bar{N}_i is the mean value of photon counts at the i -th temporal bin and ϵ_i is a random variable. We assume that ϵ_i are random variables that are primarily due to gaussian statistics. (This approach may be extended to incorporate other sources of noise.¹⁷ The corresponding results will be reported elsewhere.) that are distributed around zero and following relations are valid

$$N_i = \bar{N}_i + \epsilon_i, \quad E(\epsilon_i) = 0, \quad E(\epsilon_i^2) = \sigma_i^2. \quad (37)$$

In the general case one must account not only for the Gaussian statistics of read-out process, but also for the Poisson statistic that governs photo-emission (and photo-counting) process. Thus, a correct approach would be to assume that ϵ_i is a sum of two terms $\epsilon_i = \mu \epsilon_i^G + (1 - \mu) \epsilon_i^P$, where ϵ_i^G is Gaussian and ϵ_i^P is Poissonian

variables and μ is a number between 0 and 1. Expected complication arises from the fact that standard deviation computed for the photon-counting Poissonian bias is actually proportional to the signal $(\sigma_i^P)^2 \propto \bar{I}_i$ (see discussion in Ref. 14). This issue is out of scope of the present paper and we will address this issue at a later time.

We also assume that N_i are independent, therefore, we may form a diagonal covariance matrix for the quantities N_i (or equivalently for ϵ_i) with dispersions σ_i^2 on the diagonal:

$$C_y = \begin{pmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & \sigma_N^2 \end{pmatrix}, \quad G_y = C_y^{-1} = \begin{pmatrix} \sigma_1^{-2} & 0 & \dots & 0 \\ 0 & \sigma_2^{-2} & \dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & \sigma_N^{-2} \end{pmatrix}, \quad (38)$$

where G_y is the matrix of weights. Therefore, in the case when noise is present in the data, equation (29) has following form:

$$N_{\ell i} = \mathcal{I}_{0\ell} \left(1 + V_{0\ell} \sin \phi(k_\ell) \cdot \text{Re}\{\tilde{\mathcal{P}}_{\ell i}\} + V_{0\ell} \cos \phi(k_\ell) \cdot \text{Im}\{\tilde{\mathcal{P}}_{\ell i}\} \right) \quad (39)$$

Equation (39) may equivalently be presented in a matrix form as below

$$N_{\ell i} = \begin{pmatrix} 1 & \text{Im}\{\tilde{\mathcal{P}}_{\ell i}\} & \text{Re}\{\tilde{\mathcal{P}}_{\ell i}\} \end{pmatrix} \begin{pmatrix} \mathcal{I}_{0\ell} \\ \mathcal{I}_{0\ell} V_{0\ell} \cos \phi(k_\ell) \\ \mathcal{I}_{0\ell} V_{0\ell} \sin \phi(k_\ell) \end{pmatrix} \quad (40)$$

or, equivalently,

$$\bar{N}_i + \epsilon_i = A_{i\alpha} X^\alpha, \quad (41)$$

with indexes i and α running as $i \in \{1, \dots, N\}$ and $\alpha \in \{1, 2, 3\}$. Vector X^α is the vector to be determined and matrix $\mathbf{A}^T = A_{i\alpha}$ is the $3 \times N$ rotational matrix in the

phase space. A maximum likelihood solution to the system of equations (41) may be given by the following system of equations

$$X^\alpha = \sum_i^N \mathbf{A}_\diamond^\dagger{}^{i\alpha} \bar{N}_i, \quad \text{where} \quad \mathbf{A}_\diamond^\dagger = (\mathbf{A}^T G_y \mathbf{A})^{-1} \mathbf{A}^T G_y, \quad (42)$$

with $\mathbf{A}_\diamond^\dagger$ being an optimally-weighted pseudo-inverse matrix. Note that by choosing different gain matrix¹⁶ instead of optimally weighted least-squared matrix Eq. (38), one may obtain solution with different, specifically designed properties. Nevertheless, our solution has enough embedded generality as it allows for arbitrary properties of noise contribution, which will be further explored below.

B. Optimally-Weighted Pseudo-Inverse Matrix

In this Section we will find solution for the pseudo-inverse matrix $\mathbf{A}_\diamond^\dagger$ that was introduced by Eq.(42). To construct this matrix we will use the weights matrix G_y given by Eq.(38) and the rotation matrix \mathbf{A}_\diamond given by Eq.(33) as:

$$\mathbf{A}_i = \left(1; \text{Im}\{\tilde{\mathcal{P}}_i\}; \text{Re}\{\tilde{\mathcal{P}}_i\} \right) \equiv \left(1; s_i; c_i \right). \quad (43)$$

where we denoted $s_i = \text{Im}\{\tilde{\mathcal{P}}_i\}$, $c_i = \text{Re}\{\tilde{\mathcal{P}}_i\}$.

Let us construct matrix $(\mathbf{A}^T G_y \mathbf{A})$ first. Calculation of $(\mathbf{A}^T G_y \mathbf{A})$ is straightforward even for the most general case of arbitrary number of temporal bins ($N \geq 3$) and with arbitrary integration intervals ($\Delta\tau_i \neq \Delta\tau_j$ for $i \neq j$). Thus, after some algebra we find the following structure:

$$(\mathbf{A}^T G_y) = \begin{pmatrix} \frac{1}{\sigma_1^2}; \frac{1}{\sigma_2^2}; \dots \frac{1}{\sigma_N^2} \\ \frac{s_1}{\sigma_1^2}; \frac{s_2}{\sigma_2^2}; \dots \frac{s_N}{\sigma_N^2} \\ \frac{c_1}{\sigma_1^2}; \frac{c_2}{\sigma_2^2}; \dots \frac{c_N}{\sigma_N^2} \end{pmatrix}, \quad (\mathbf{A}^T G_y \mathbf{A}) = \begin{pmatrix} \sum_i^N \frac{1}{\sigma_i^2} & \sum_i^N \frac{s_i}{\sigma_i^2} & \sum_i^N \frac{c_i}{\sigma_i^2} \\ \sum_i^N \frac{s_i}{\sigma_i^2} & \sum_i^N \frac{s_i^2}{\sigma_i^2} & \sum_i^N \frac{s_i c_i}{\sigma_i^2} \\ \sum_i^N \frac{c_i}{\sigma_i^2} & \sum_i^N \frac{s_i c_i}{\sigma_i^2} & \sum_i^N \frac{c_i^2}{\sigma_i^2} \end{pmatrix}. \quad (44)$$

By inverting the obtained result one constructs the covariance matrix Λ of the following structure:

$$\Lambda = (\mathbf{A}^T G_y \mathbf{A})^{-1} = \frac{1}{\Delta_\diamond} \begin{pmatrix} \frac{1}{2} \sum_{ij} \frac{(s_i c_j - c_i s_j)^2}{\sigma_i^2 \sigma_j^2} & \frac{1}{2} \sum_{ij} \frac{(c_i - c_j)(s_i c_j - c_i s_j)}{\sigma_i^2 \sigma_j^2} & -\frac{1}{2} \sum_{ij} \frac{(s_i - s_j)(s_i c_j - c_i s_j)}{\sigma_i^2 \sigma_j^2} \\ \frac{1}{2} \sum_{ij} \frac{(c_i - c_j)(s_i c_j - c_i s_j)}{\sigma_i^2 \sigma_j^2} & \frac{1}{2} \sum_{ij} \frac{(c_i - c_j)^2}{\sigma_i^2 \sigma_j^2} & -\frac{1}{2} \sum_{ij} \frac{(s_i - s_j)(c_i - c_j)}{\sigma_i^2 \sigma_j^2} \\ -\frac{1}{2} \sum_{ij} \frac{(s_i - s_j)(s_i c_j - c_i s_j)}{\sigma_i^2 \sigma_j^2} & -\frac{1}{2} \sum_{ij} \frac{(s_i - s_j)(c_i - c_j)}{\sigma_i^2 \sigma_j^2} & \frac{1}{2} \sum_{ij} \frac{(s_i - s_j)^2}{\sigma_i^2 \sigma_j^2} \end{pmatrix},$$

where determinant of the matrix $(\mathbf{A}^T G_y \mathbf{A})$, $\Delta_\diamond = \det \|(\mathbf{A}^T G_y \mathbf{A})\|$, is given as

$$\Delta_\diamond = \frac{1}{2} \sum_{ijk} \frac{(s_i c_j - s_j c_i)}{\sigma_i^2 \sigma_j^2 \sigma_k^2} \left[(s_i c_j - s_j c_i) + (s_j c_k - s_k c_j) + (s_k c_i - s_i c_k) \right], \quad (45)$$

with a triple summation for all the indexes denoting the temporal bins and running from 1 to N , namely $\forall \{i, j, k\} \in [1, N]$.

These intermediate results allow us to write the solution for the $(N \times 3)$ optimally-weighted pseudo-inverse matrix $\mathbf{A}_\diamond^\dagger = \mathbf{A}_{\diamond k}^{\dagger \alpha}$ in the following compact form:

$$\mathbf{A}_\diamond^\dagger = (\mathbf{A}^T G_y \mathbf{A})^{-1} \mathbf{A}^T G_y = \frac{1}{\mathcal{D}^\diamond} \begin{pmatrix} \mathcal{A}_k^\diamond \\ \mathcal{B}_k^\diamond \\ \mathcal{C}_k^\diamond \end{pmatrix}, \quad (46)$$

where coefficients $\mathcal{A}_k^\diamond, \mathcal{B}_k^\diamond, \mathcal{C}_k^\diamond$ and \mathcal{D}^\diamond depend on duration of each temporal bin, mean

wavenumber and variances for the data taken in each bin, and are given by

$$\begin{aligned}
\mathcal{A}_k^\circ &= \sum_{ij}^N \frac{1}{\sigma_i^2 \sigma_j^2 \sigma_k^2} (s_i c_j - s_j c_i) [(s_i c_j - s_j c_i) + (s_j c_k - s_k c_j) + (s_k c_i - s_i c_k)], \\
\mathcal{B}_k^\circ &= \sum_{ij}^N \frac{1}{\sigma_i^2 \sigma_j^2 \sigma_k^2} (c_i - c_j) [(s_i c_j - s_j c_i) + (s_j c_k - s_k c_j) + (s_k c_i - s_i c_k)], \\
\mathcal{C}_k^\circ &= - \sum_{ij}^N \frac{1}{\sigma_i^2 \sigma_j^2 \sigma_k^2} (s_i - s_j) [(s_i c_j - s_j c_i) + (s_j c_k - s_k c_j) + (s_k c_i - s_i c_k)], \\
\mathcal{D}^\circ &= \sum_k^N \mathcal{A}_k^\circ = \frac{1}{3} \sum_{ijk}^N \frac{1}{\sigma_i^2 \sigma_j^2 \sigma_k^2} [(s_i c_j - s_j c_i) + (s_j c_k - s_k c_j) + (s_k c_i - s_i c_k)]^2. \quad (47)
\end{aligned}$$

Definitions for the quantities s_i and c_i , $s_i = \text{Im}\{\tilde{\mathcal{P}}_i\}$, $c_i = \text{Re}\{\tilde{\mathcal{P}}_i\}$, allow to present expressions (47) in a more convenient form. First, remember that complex matrix, $\tilde{\mathcal{P}}_i$, as any complex function, may be represented by its amplitude and its phase, namely

$$\tilde{\mathcal{P}}_j = \text{Re}\{\tilde{\mathcal{P}}_j\} + i \text{Im}\{\tilde{\mathcal{P}}_j\} = p_j e^{i\pi_j}, \quad (48)$$

where p_j and π_j are the amplitude and the phase of this complex matrix correspondingly and are given as follows:

$$p_j = \sqrt{\text{Re}^2\{\tilde{\mathcal{P}}_j\} + \text{Im}^2\{\tilde{\mathcal{P}}_j\}}, \quad \pi_j = \text{ArcTan}\left\{\frac{\text{Im}\{\tilde{\mathcal{P}}_j\}}{\text{Re}\{\tilde{\mathcal{P}}_j\}}\right\}, \quad (49)$$

with complex matrix $\tilde{\mathcal{P}}_{\ell i}$ is given by Eq. (33). These quantities allow presentation of

Eqs. (47) in the following form:

$$\begin{aligned}
\mathcal{A}_k^\circ &= \sum_{ij}^N \frac{p_i p_j \sin[\pi_i - \pi_j]}{\sigma_i^2 \sigma_j^2 \sigma_k^2} \times \\
&\quad \times \left(p_i p_j \sin[\pi_i - \pi_j] + p_j p_k \sin[\pi_j - \pi_k] + p_k p_i \sin[\pi_k - \pi_i] \right), \\
\mathcal{B}_k^\circ &= \sum_{ij}^N \frac{(p_i \cos \pi_i - p_j \cos \pi_j)}{\sigma_i^2 \sigma_j^2 \sigma_k^2} \times \\
&\quad \times \left(p_i p_j \sin[\pi_i - \pi_j] + p_j p_k \sin[\pi_j - \pi_k] + p_k p_i \sin[\pi_k - \pi_i] \right), \\
\mathcal{C}_k^\circ &= - \sum_{ij}^N \frac{(p_i \sin \pi_i - p_j \sin \pi_j)}{\sigma_i^2 \sigma_j^2 \sigma_k^2} \times \\
&\quad \times \left(p_i p_j \sin[\pi_i - \pi_j] + p_j p_k \sin[\pi_j - \pi_k] + p_k p_i \sin[\pi_k - \pi_i] \right), \\
\mathcal{D}^\circ &= \sum_k^N \mathcal{A}_k^\circ = \\
&= \frac{1}{3} \sum_{ijk}^N \frac{1}{\sigma_i^2 \sigma_j^2 \sigma_k^2} \left(p_i p_j \sin[\pi_i - \pi_j] + p_j p_k \sin[\pi_j - \pi_k] + p_k p_i \sin[\pi_k - \pi_i] \right)^2. \quad (50)
\end{aligned}$$

At this point we have all the expressions necessary to present the optimally-weighted solution for the polychromatic phasors.

C. Photon Noise-Optimized Solution for Polychromatic Phasors

An optimally-weighted solution for the quantities X^α may be obtained directly now from Eq.(42) with the help of expressions (46)-(47) in the following compact form:

$$\begin{aligned}
\mathcal{I}_0^\circ &= \frac{1}{\mathcal{D}^\circ} \sum_k^N \bar{N}_k \mathcal{A}_k^\circ, \\
\mathcal{I}_0^\circ V_0^\circ \cos \bar{\phi}^\circ &= \frac{1}{\mathcal{D}^\circ} \sum_k^N \bar{N}_k \mathcal{B}_k^\circ, \\
\mathcal{I}_0^\circ V_0^\circ \sin \bar{\phi}^\circ &= \frac{1}{\mathcal{D}^\circ} \sum_k^N \bar{N}_k \mathcal{C}_k^\circ. \quad (51)
\end{aligned}$$

with coefficients of $\mathcal{A}_k^\circ, \mathcal{B}_k^\circ, \mathcal{C}_k^\circ$ and \mathcal{D}° are given by Eqs.(47) and (50).

The obtained solution for the polychromatic visibility phasors given by Eq. (51) is given in the form of a linear combination of weighted photon counts recorded during a particular integration period. This form turned out to be very when analyzing contributions of CCD pixels that are systematically biased. The obtained result may be used to de-weight 'bad' pixels (in a statistical sense) and, thus, to reduce the problem of biases while estimating fringe parameters.

This form allows to express an optimally-weighted solution for visibility, phase and the constant intensity terms in a familiar compact form:

$$\begin{aligned}
 V_0^{\circ 2} &= \frac{\left(\sum_k^N \bar{N}_k \mathcal{B}_k^{\circ}\right)^2 + \left(\sum_k^N \bar{N}_k \mathcal{C}_k^{\circ}\right)^2}{\left(\sum_k^N \bar{N}_k \mathcal{A}_k^{\circ}\right)^2}, \\
 \bar{\phi}^{\circ} &= \text{ArcTan}\left[\frac{\sum_k^N \bar{N}_k \mathcal{C}_k^{\circ}}{\sum_k^N \bar{N}_k \mathcal{B}_k^{\circ}}\right], \quad \mathcal{I}_0^{\circ} = \frac{\sum_k^N \bar{N}_k \mathcal{A}_k^{\circ}}{\sum_k^N \mathcal{A}_k^{\circ}}. \quad (52)
 \end{aligned}$$

The form of the obtained solution is simple to understand and it is straightforward to implement in the software codes. All the information necessary to calculate the $3N$ coefficients of \mathcal{A}_k° , \mathcal{B}_k° , \mathcal{C}_k° and \mathcal{D}° is presumed to be known before the experiment. Thus, for the case when $N = 8$ one would have to calculate only 24 numbers from Eq. (47). These numbers correspond to 8 numbers of \mathcal{A}_k° , 8 numbers of \mathcal{B}_k° and 8 numbers of \mathcal{C}_k° . Then, by taking the data and estimating variances σ_i one may process the data with the help of Eqs. (51) or directly Eqs. (52). This approach is currently being utilized and corresponding results will be reported elsewhere.¹⁸

4. Rectangular Bandpass Filter

In this Section we will discuss the properties of rectangular bandpass filter, which is, due to its analytical simplicity, is the most known construction in Fourier optics. This analysis will allow us to establish correspondence with the previously obtained results both for monochromatic and polychromatic light.

To take advantage of the results derived in the previous section, we must first decide on the properties of the bandpass filter. This decision in return will affect the properties of the envelop function. Below we shall develop a model for a special case of the bandpass filter – a rectangular bandpass filter denoted here as \mathcal{F}_ℓ , which is done analytically in the following form

$$\mathcal{F}(k) = \sum_{\ell=1}^L \mathcal{F}_\ell(k), \quad \text{where} \quad \mathcal{F}_\ell(k) = \begin{cases} \mathcal{F}_{0\ell} = \text{const}, & k \in [k_\ell^-, k_\ell^+], \\ 0, & k \notin [k_\ell^-, k_\ell^+]. \end{cases} \quad (53)$$

We can also assume that the width of a spectral channel is small, so that both intensity of incoming radiation, $\mathcal{I}_0(k)$, and apparent visibility, $V(k)$, do not change within the spectral channel (in particular, this leads to $\hat{V}_{0\ell}(k) \equiv 1$ in Eqs. (10) and (11)). Therefore, the following conditions are satisfied with a particular spectral channel, ℓ :

$$\mathcal{I}_0(k) = \text{const}, \quad V(k) = \text{const}, \quad \mathcal{F}_{0\ell} = \text{const}, \quad (54)$$

$$\phi(k) - \phi(k_\ell) = d_{0\ell}(k - k_\ell) + \mathcal{O}\left(\frac{\partial^2 \phi}{\partial k_\ell^2}\right), \quad (55)$$

where $d_{0\ell} = \frac{\partial \phi}{\partial k_\ell}$ is the delay within the ℓ -th channel.

One may perform integration of the fringe envelope function $\tilde{W}_\ell[x(t)]$ which is given by Eq. (17) (or use equation (91) for the unperturbed envelope function and then apply iterative procedure outlined in Appendix B.). To the second order in phase

variation (i.e. $\mathcal{O}(\frac{\partial^2 \phi}{\partial k_\ell^2})$), the resulted envelope function has following properties:

$$\begin{aligned}\tilde{W}_\ell[\Delta k_\ell, \phi(k_\ell), x_i] &= \frac{1}{\mathcal{I}_{0\ell} V_{0\ell} \Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \mathcal{F}(k) \mathcal{I}_0(k) V(k) e^{j((k-k_\ell)x(t) + \phi(k) - \phi(k_\ell))} dk = \\ &= \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} e^{j(k-k_\ell)(x(t) + d_{0\ell})} dk + \mathcal{O}(\frac{\partial^2 \phi}{\partial k_\ell^2}).\end{aligned}\quad (56)$$

Let us introduce a convenient variable, $\kappa = k - k_\ell$, and remember that $\Delta k_\ell = k_\ell^+ - k_\ell^-$ and $k_\ell = \frac{1}{2}(k_\ell^+ + k_\ell^-)$ are the width of the spectral channel and the mean wavenumber. This allows us to integrate expression (56) over the wavenumber space

$$\tilde{W}_\ell[\Delta k_\ell, \phi(k_\ell), x_i] = \frac{1}{\Delta k_\ell} \int_{-\frac{1}{2}\Delta k_\ell}^{+\frac{1}{2}\Delta k_\ell} d\kappa e^{j\kappa(d_{0\ell} + x(t))} = \frac{\sin[\frac{1}{2}\Delta k_\ell(d_{0\ell} + x(t))]}{\frac{1}{2}\Delta k_\ell(d_{0\ell} + x(t))}. \quad (57)$$

We can now present Eq. (33) for matrix $\tilde{\mathcal{P}}_{\ell i}$ as follows:

$$\tilde{\mathcal{P}}_{\ell i} = \frac{1}{\Delta \tau_i} \int_{t_i^-}^{t_i^+} dt e^{jk_\ell x(t)} \frac{\sin[\frac{1}{2}\Delta k_\ell(d_{0\ell} + x(t))]}{\frac{1}{2}\Delta k_\ell(d_{0\ell} + x(t))}. \quad (58)$$

At this moment, we introduce another convenient variable, $\tau = t - t_i$. Analogously, $\Delta \tau_i = t_i^+ - t_i^-$ and $t_i = \frac{1}{2}(t_i^+ + t_i^-)$ are the duration of the temporal integration within the i -th bin and the mean time for this bin correspondingly. This result is used to transform Eq. (58) as below:

$$\tilde{\mathcal{P}}_{\ell i} = e^{jk_\ell x(t_i)} \delta \tilde{\mathcal{P}}_{\ell i} \quad (59)$$

with coefficient $\delta \tilde{\mathcal{P}}_{\ell i}$ given by

$$\delta \tilde{\mathcal{P}}_{\ell i} = \frac{1}{\Delta \tau_i} \int_{-\frac{1}{2}\Delta \tau_i}^{+\frac{1}{2}\Delta \tau_i} d\tau e^{jk_\ell(x(t_i + \tau) - x(t_i))} \frac{\sin[\frac{1}{2}\Delta k_\ell(d_{0\ell} + x(t_i + \tau))]}{\frac{1}{2}\Delta k_\ell(d_{0\ell} + x(t_i + \tau))}. \quad (60)$$

The obtained result explicitly depends on the functional form of the OPD modulation, $x(t)$. To integrate this equation one first needs to make certain assumptions

on the temporal behavior of $x(t)$, which will be done in the following Sections. At this moment we present Eq. (29) in the following form:

$$N_{\ell i} = \mathcal{I}_{0\ell} \left(1 + V_{0\ell} \sin [\phi(k_\ell) + k_\ell x(t_i)] \cdot \text{Re}\{\delta\tilde{\mathcal{P}}_{\ell i}\} + V_{0\ell} \cos [\phi(k_\ell) + k_\ell x(t_i)] \cdot \text{Im}\{\delta\tilde{\mathcal{P}}_{\ell i}\} \right), \quad (61)$$

where the complex matrix $\delta\tilde{\mathcal{P}}_{\ell i}$ is given by Eq. (60). The importance of separating terms with $\delta\tilde{\mathcal{P}}_{\ell i}$ is in the fact that one can establish clear correspondence with monochromatic light, for which $\delta\tilde{\mathcal{P}}_{\ell i} = I_{\ell i}$, the identity matrix.

In the next two subsections we will study two different special cases of OPD modulation, namely the stepping and ramping modulations of the optical path difference.

A. Stepping Phase Modulation

The stepping phase modulation realized when the pathlength difference is changes as a set of discrete values corresponding to a number of steps in the OPD space. mathematically this process represented as follows:

$$x(t) = \sum_{i=1}^{N=8} x(t_i), \quad \text{where} \quad x(t_i) = \begin{cases} x_i, & t \in [t_i^-, t_i^+], \\ 0, & t \notin [t_i^-, t_i^+], \end{cases} \quad (62)$$

with $t_i = \frac{1}{2}(t_i^+ + t_i^-)$. This procedure defines the temporal bins that will be used to modulate the interferometric pattern.

Conditions (62) allow for a significant simplification of temporal integration in Eq. (60). It simply is leading to a substitution $x(t) \rightarrow x_i$ in Eq. (21), and matrix $\tilde{\mathcal{P}}_{\ell i}$ takes the following form

$$\tilde{\mathcal{P}}_{\ell i} = e^{j k_\ell x(t_i)} \frac{\sin[\frac{1}{2}\Delta k_\ell (d_{0\ell} + x_i)]}{\frac{1}{2}\Delta k_\ell (d_{0\ell} + x_i)}, \quad (63)$$

As a result, to the second order in the phase variation (i.e. $\phi(k_\ell) \approx \phi_\ell + \mathcal{O}(\frac{\partial^2 \phi}{\partial k_\ell^2} \Big|_{k_\ell})$), the observational equation Eq. (61) is taking form as below:

$$N_{\ell i} = \mathcal{I}_{0\ell} \left(1 + V_{0\ell} \cdot \text{sinc} \left[\frac{1}{2} \Delta k_\ell (x_i + d_{0\ell}) \right] \cdot \sin \left[\phi(k_\ell) + k_\ell x_i \right] \right). \quad (64)$$

The obtained result Eq. (64) clearly depends on the particular form of the envelope function. As such, it has most of the parameters that are necessary for the phase estimation purposes in the case of wide bandwidth.

For the most practical cases the value of the sinc function will be close to $\text{sinc} \sim 1$. Indeed, let us analyze the argument of this function, $\frac{1}{2} \Delta k_\ell (x_i + d_{0\ell})$. Thus, one might expect that within the spectral channel the phase will stay constant, hence $d_{0\ell} = \frac{\partial \phi}{\partial k_\ell} \approx 0$. Furthermore, for the estimation purposes let us assume that all the step-sizes x_i are essentially $x_i = i \frac{\lambda_0}{N}$, where n is the total number of temporal integration bins, i is the number of a particular temporal bin, $i \in 1, N$, and λ_0 is the modulation wavelength or $\lambda_0 = \frac{2\pi}{k_0}$, where k_0 is the corresponding modulation wavenumber. Also remember that width of a spectral channel is related to the total SIM bandwidth as $\Delta k_\ell = \frac{\Delta k_{\text{SIM}}}{L}$, where Δk_{SIM} is the total SIM bandwidth and L is the total number of spectral channels used for the white light fringe detection. Therefore, one has

$$\frac{1}{2} \Delta k_\ell (x_i + d_{0\ell}) \approx \frac{1}{2} \Delta k_\ell x_i = \frac{\pi i}{LN} \frac{\Delta k_{\text{SIM}}}{k_0}. \quad (65)$$

Assuming $\lambda_{\text{SIM}}^- = 450$ nm and $\lambda_{\text{SIM}}^+ = 900$ nm, and $\lambda_0 = 900$ nm, thus yielding $\frac{\Delta k_{\text{SIM}}}{k_0} = 1$. The maximal value for the expression (65) is realized when $i = N$, thus

$$\frac{\pi i}{LN} \frac{\Delta k_{\text{SIM}}}{k_0} \leq \frac{\pi}{L}. \quad (66)$$

Currently, there are different numbers of spectral channels used to process data from our testbeds. This number may be as large as $L = 80$ and as small as $L = 4$. Of course, when $L = 80$, the ratio π/L becomes $\pi/L = 0.03927$ and, thus, $\text{sinc}[\frac{1}{2}\Delta k_\ell x_i]|_{L=80} = 0.99974$, and similarly for $L = 4$ the sinc function becomes $\text{sinc}[\frac{1}{2}\Delta k_\ell x_i]|_{L=4} = 0.90032$. We will address the issue of phase estimation sensitivity to the width of a spectral channel Δk_ℓ at a later time.¹⁸

This observation allows us to present the sinc function as a series with respect to the small parameter $\Delta k_\ell z$ (with z being defined as $z = x_i + d_{0\ell}$) as

$$\begin{aligned} \text{sinc}\left[\frac{\Delta k_\ell z}{2}\right] &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left[\frac{\Delta k_\ell z}{2}\right]^{2n} \\ &= 1 - \frac{1}{3!} \left[\frac{\Delta k_\ell z}{2}\right]^2 + \frac{1}{5!} \left[\frac{\Delta k_\ell z}{2}\right]^4 + \mathcal{O}\left(\frac{1}{7!} \left[\frac{\Delta k_\ell z}{2}\right]^6\right), \end{aligned} \quad (67)$$

one can present the expression Eq. (64) in the following form:

$$\begin{aligned} N_{\ell i} &= \mathcal{I}_{0\ell} \left[1 + \right. \\ &\quad \left. + V_{0\ell} \cdot \left(1 - \frac{\Delta k_\ell^2 (x_i + d_{0\ell})^2}{24} + \frac{\Delta k_\ell^4 (x_i + d_{0\ell})^4}{1920} \right) \cdot \sin [\phi(k_\ell) + k_\ell x_i] \right]. \end{aligned} \quad (68)$$

The obtained expression models the expected number of photons detected at the CCD for the rectangular bandpass filter and stepping phase modulation. It extends the results obtained for the monochromatic case on the finite size spectral bandwidth. This fact is indicated by the explicit dependency of the obtained result on the width of a spectral channel Δk_ℓ . (For the most of the interesting practical applications, the size of the delay within a particular spectral channel is very small $d_{0\ell} = \frac{\partial \phi}{\partial k_\ell} \approx 0$, which further simplifies Eq. (68)).

B. Ramping Phase Modulation

In this Section we will discuss another type of phase modulation – the case when the phase is linearly changes with time. This modulation utilizes the phase ramping technique.(For more details, see Refs.3-8.) To develop analytical solution we will be using the system equations developed above, specifically Eqs. (29) and (33).

The optical path difference for the case of ramping phase modulation is modeled as a continuous function of time as follows:

$$x(t) = x_0 + v \cdot t, \quad (69)$$

where x_0 is the initial PZT position and v is the instantaneous velocity of PZT motion.

Remembering the definition for τ as $\tau = t - t_i$, and $\Delta\tau_i = t_i^+ - t_i^-$ and $t_i = \frac{1}{2}(t_i^+ + t_i^-)$,

Eq. (60) takes the form:

$$\tilde{\mathcal{P}}_{\ell i} = e^{j k_{\ell} x(t_i)} \delta \tilde{\mathcal{P}}_{\ell i}, \quad (70)$$

with coefficient $\delta \tilde{\mathcal{P}}_{\ell i}$ given by

$$\delta \tilde{\mathcal{P}}_{\ell i} = \frac{1}{\Delta\tau_i} \int_{-\frac{1}{2}\Delta\tau_i}^{+\frac{1}{2}\Delta\tau_i} d\tau e^{j k_{\ell} v \tau} \cdot \frac{\sin[\frac{1}{2}\Delta k_{\ell} z(\tau)]}{\frac{1}{2}\Delta k_{\ell} z(\tau)} \quad (71)$$

and $z(\tau) = d_{0\ell} + x(t_i) + v \tau$. This allows us to present Eq. (61) in the following form:

$$N_{\ell i} = \mathcal{I}_{0\ell} \left(1 + V_{0\ell} \sin [\phi(k_{\ell}) + k_{\ell} x(t_i)] \cdot \text{Re}\{\delta \tilde{\mathcal{P}}_{\ell i}\} + \right. \\ \left. + V_{0\ell} \cos [\phi(k_{\ell}) + k_{\ell} x(t_i)] \cdot \text{Im}\{\delta \tilde{\mathcal{P}}_{\ell i}\} \right), \quad (72)$$

where the complex matrix of additional rotation in the phase space, $\delta \tilde{\mathcal{P}}_{\ell i}$, is given by

Eq. (60). Equation (72) may equivalently be presented in a matrix form as below

$$N_{\ell i} = \begin{pmatrix} 1; & \sin k_{\ell} x(t_i); & \cos k_{\ell} x(t_i) \end{pmatrix} \times \\ \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & \operatorname{Re}\{\delta\tilde{\mathcal{P}}_{\ell i}\} & -\operatorname{Im}\{\delta\tilde{\mathcal{P}}_{\ell i}\} \\ 0 & \operatorname{Im}\{\delta\tilde{\mathcal{P}}_{\ell i}\}; & \operatorname{Re}\{\delta\tilde{\mathcal{P}}_{\ell i}\} \end{pmatrix} \begin{pmatrix} \mathcal{I}_{0\ell} \\ \mathcal{I}_{0\ell} V_{0\ell} \cos \phi(k_{\ell}) \\ \mathcal{I}_{0\ell} V_{0\ell} \sin \phi(k_{\ell}) \end{pmatrix}. \quad (73)$$

The result of integration of Eq.(71) may not be presented in a compact analytical form. It rather could be expressed in the form of two functions defined as **SinIntegral** and **CosIntegral**. To simplify the analysis, the sinc function may be given in the form of power series expansion with respect to the small parameter $\Delta k_{\ell} z(\tau)$ as given by Eq. (67). This expansion allows us to present Eqs. (71) in the following form:

$$\delta\tilde{\mathcal{P}}_{\ell i} = \frac{1}{\Delta\tau_i} \int_{-\frac{1}{2}\Delta\tau_i}^{+\frac{1}{2}\Delta\tau_i} d\tau e^{j k_{\ell} v \tau} \cdot \frac{\sin[\frac{1}{2}\Delta k_{\ell} z(\tau)]}{\frac{1}{2}\Delta k_{\ell} z(\tau)} = \\ = \frac{1}{\Delta\tau_i} \int_{-\frac{1}{2}\Delta\tau_i}^{+\frac{1}{2}\Delta\tau_i} d\tau e^{j k_{\ell} v \tau} \cdot \left(1 - \frac{\Delta k_{\ell}^2 z(\tau)^2}{24} + \frac{\Delta k_{\ell}^4 z(\tau)^4}{1920} + \mathcal{O}\left(\frac{\Delta k_{\ell}^6 z^6}{7! \cdot 2^6}\right) \right), \quad (74)$$

where $z(\tau) = d_{0\ell} + x(t_i) + v\tau = z_i + v\tau$ with $z_i = d_{0\ell} + x(t_i)$. This equation, (74), was integrated to obtain the following result for $\delta\tilde{\mathcal{P}}_{\ell i}$:

$$\delta\tilde{\mathcal{P}}_{\ell i} = \frac{\sin[\frac{1}{2}k_{\ell} v \Delta\tau_i]}{\frac{1}{2}k_{\ell} v \Delta\tau_i} + \tilde{\mathcal{A}}_{\ell i} \frac{\Delta k_{\ell}^2}{24k_{\ell}^2} + \tilde{\mathcal{B}}_{\ell i} \frac{\Delta k_{\ell}^4}{1920 k_{\ell}^4} + \mathcal{O}\left(\frac{\Delta k_{\ell}^6}{7! \cdot 2^6 k_{\ell}^6}\right), \quad (75)$$

where complex coefficients $\mathcal{A}_{\ell i}$ is given as follows:

$$\tilde{\mathcal{A}}_{\ell i} = \left[1 + (1 - j k_{\ell} z_i)^2 - \left(\frac{1}{2}k_{\ell} v \Delta\tau_i\right)^2 \right] \frac{\sin[\frac{1}{2}k_{\ell} v \Delta\tau_i]}{\frac{1}{2}k_{\ell} v \Delta\tau_i} - \\ - 2\left(1 - j k_{\ell} z_i\right) \cos\left[\frac{1}{2}k_{\ell} v \Delta\tau_i\right] \quad (76)$$

and $\mathcal{B}_{\ell i}$ is computed as follows:

$$\begin{aligned} \tilde{\mathcal{B}}_{\ell i} = & \left(\left[1 + (1 - jk_{\ell}z_i)^2 - \left(\frac{1}{2}k_{\ell}v\Delta\tau_i\right)^2 \right]^2 + 4(2 - jk_{\ell}z_i)^2 + \right. \\ & \left. + 4\left(1 - \left(\frac{1}{2}k_{\ell}v\Delta\tau_i\right)^2\right) \frac{\sin\left[\frac{1}{2}k_{\ell}v\Delta\tau_i\right]}{\frac{1}{2}k_{\ell}v\Delta\tau_i} - \right. \\ & \left. - 4\left(5 + (1 - jk_{\ell}z_i)\left((1 - jk_{\ell}z_i)^2 - \left(\frac{1}{2}k_{\ell}v\Delta\tau_i\right)^2\right)\right) \cos\left[\frac{1}{2}k_{\ell}v\Delta\tau_i\right]. \right. \end{aligned} \quad (77)$$

The obtained expressions may be used to simplify the results of temporal integration Eq. (72). As a result, the coefficients $\text{Re}\{\delta\tilde{\mathcal{P}}_{\ell i}\}$ and $\text{Im}\{\delta\tilde{\mathcal{P}}_{\ell i}\}$ in the fringe equation Eq. (72) may be written in the following form:

$$\begin{aligned} \text{Re}\{\delta\tilde{\mathcal{P}}_{\ell i}\} = & \frac{\sin\left[\frac{1}{2}k_{\ell}v\Delta\tau_i\right]}{\frac{1}{2}k_{\ell}v\Delta\tau_i} \left[1 + \left(2 - k_{\ell}^2z_i^2 - \left(\frac{1}{2}k_{\ell}v\Delta\tau_i\right)^2\right) \frac{\Delta k_{\ell}^2}{24k_{\ell}^2} \right] - \\ & - 2 \cos\left[\frac{1}{2}k_{\ell}v\Delta\tau_i\right] \frac{\Delta k_{\ell}^2}{24k_{\ell}^2} + \mathcal{O}\left(\frac{\Delta k_{\ell}^4}{5!2^4 k_{\ell}^4}\right), \end{aligned} \quad (78)$$

$$\text{Im}\{\delta\tilde{\mathcal{P}}_{\ell i}\} = 2k_{\ell}z_i \left(-\frac{\sin\left[\frac{1}{2}k_{\ell}v\Delta\tau_i\right]}{\frac{1}{2}k_{\ell}v\Delta\tau_i} + \cos\left[\frac{1}{2}k_{\ell}v\Delta\tau_i\right] \right) \frac{\Delta k_{\ell}^2}{24k_{\ell}^2} + \mathcal{O}\left(\frac{\Delta k_{\ell}^4}{5!2^4 k_{\ell}^4}\right), \quad (79)$$

with $z_i = d_{0\ell} + x(t_i) \equiv d_{0\ell} + x_0 + vt_i$.

The obtained expression models the photon flux detected at the CCD for the case of rectangular bandpass filter and ramping phase modulation. It extends the results obtained for the monochromatic case on the finite size of spectral bandwidth. This fact is indicated by the explicit dependency of the obtained result on the width of a spectral channel Δk_{ℓ} .

5. Discussion and Future Plans

The main objective of this paper has been to introduce the reader to the concepts and the instrumental logic of the SIM astrometric observations, especially as they relate to

estimation of the white light fringe parameters. The set of formulae described herein will serve as the kernel for the future mission analysis and simulations. We have also developed a set of expressions that may be used for fringe visibility and phase extraction for both SIM science and guide interferometers. The obtained expressions depend on the effective operational wavelength of OPD modulation, the width of a particular spectral channel Δk_ℓ with the mean wavenumber k_ℓ and corresponding wavelength λ_ℓ . Our model accounts for a number of instrumental and physical effects and is able to compensate for a number of operational regimes.

The logic of our method is straightforward: one first assumes the desirable properties of the bandpass filter, then finds the corresponding envelope function, and then applies the obtained expressions (which are valid for a generic case). The obtained solutions for the envelope function W and, most specifically, $\delta\tilde{\mathcal{P}}_{\ell i}$ may be directly substituted either in the expression for the complex visibility phasors Eq. (50) and (51), or into equations for the visibility, amplitude and phase of the fringe, given by Eqs.(52). We applied this formalism to the case of a rectangular bandpass (the obtained results are given by Eqs. (97)-(100)). While the complex visibility phasors are linear with respect to photon counts, the explicit expressions for the fringe parameters are non-linear. This fact may be used to design specific properties of unbiased fringe estimators for processing the white light data. In our further work we will numerically address the problem of unbiased estimators for the fringe phase, visibility and group delay. This effort is currently underway and results will be reported elsewhere.¹⁸

Our simulations show¹⁸ that, while the model of the rectangular bandpass filter is working quite, for the ‘real life’ one must account for the effect of leakage of light.

This effect is concerned the leakage of light onto the studied spectrometer pixel of the detector from the adjacent pixels with different wavenumbers. At this moment, it seems more appropriate that a combination of the rectangular bandpass filter and the additional effect of light leakage from the adjacent pixels that must be included into the model of a CCD detector. The corresponding analysis, simulation results and implications for the instrument design will be reported elsewhere.¹⁸

A. Two Definitions for the Fringe Phase

In this Appendix we will address the issue of the mean phase definition which requires some additional work. It is tempting to define the mean phase as

$$\phi_\ell = \frac{1}{\Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \hat{\mathcal{I}}_{0\ell}(k) \phi(k) dk \equiv \frac{1}{\mathcal{I}_{0\ell} \Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \mathcal{F}(k) \mathcal{I}_0(k) \phi(k) dk. \quad (80)$$

However, one needs to relate this expression to the quantity $\phi(k_\ell)$, which is the phase value at a particular wavenumber. Assuming that phase $\phi(k)$ is a slow varying function of k and, as such, it may be expanded in a Taylor series as follows:

$$\phi(k) = \phi(k_\ell) + \left. \frac{\partial \phi}{\partial k} \right|_{k_\ell} (k - k_\ell) + \frac{1}{2} \left. \frac{\partial^2 \phi}{\partial k^2} \right|_{k_\ell} (k - k_\ell)^2 + \mathcal{O}(\Delta k_\ell^3). \quad (81)$$

We can now substitute this formula directly in Eq.(80), which results in

$$\phi_\ell = \phi(k_\ell) + \left. \frac{1}{2} \frac{\partial^2 \phi}{\partial k^2} \right|_{k_\ell} \frac{1}{\mathcal{I}_{0\ell} \Delta k_\ell} \int_{k_\ell^-}^{k_\ell^+} \mathcal{F}(k) \mathcal{I}_0(k) (k - k_\ell)^2 dk + \mathcal{O}(\Delta k_\ell^3). \quad (82)$$

Or in other words, the phase value $\phi(k_\ell)$ at a particular wavenumber k_ℓ is related to the mean phase ϕ_ℓ within the spectral channel with width Δk_ℓ (i.e. Eq.(80)) by the following expression

$$\phi(k_\ell) = \phi_\ell - \left. \frac{1}{2} \frac{\partial^2 \phi}{\partial k^2} \right|_{k_\ell} \Delta k_\ell^2 \mu_\ell^{(2)} + \mathcal{O}(\Delta k_\ell^3), \quad (83)$$

where $\mu_\ell^{(2)}$ is the dimension-less second-order moment of wavenumber distribution within the spectral channel of interest:

$$\mu_\ell^{(2)} = \frac{1}{\mathcal{I}_{0\ell}\Delta k_\ell^3} \int_{k_\ell^-}^{k_\ell^+} \mathcal{F}(k)\mathcal{I}_0(k)(k - k_\ell)^2 dk. \quad (84)$$

In the general case, when the higher order moments are considered, this expression takes the following form:

$$\phi(k_\ell) = \phi_\ell - \sum_{p=2}^P \frac{1}{p!} \frac{\partial^p \phi}{\partial k^p} \Big|_{k_\ell} \Delta k_\ell^p \mu_\ell^{(p)} + \mathcal{O}(\Delta k_\ell^p). \quad (85)$$

with moments $\mu_\ell^{(p)}$ given as follows:

$$\mu_\ell^{(0)} = 1, \quad \mu_\ell^{(1)} = 0, \quad \mu_\ell^{(2)} = \frac{1}{\mathcal{I}_{0\ell}\Delta k_\ell^3} \int_{k_\ell^-}^{k_\ell^+} \mathcal{F}(k)\mathcal{I}_0(k)(k - k_\ell)^2 dk, \quad (86)$$

$$\mu_\ell^{(p)} = \frac{1}{\mathcal{I}_{0\ell}\Delta k_\ell^{p+1}} \int_{k_\ell^-}^{k_\ell^+} \mathcal{F}(k)\mathcal{I}_0(k)(k - k_\ell)^p dk, \quad 0 < |\mu_\ell^{(p)}| < 1, \quad \forall p. \quad (87)$$

Note that the wavenumbers within a spectral channel may be considered uniformly distributed, thus prompting to use $\phi(k_\ell) = \phi_\ell + \mathcal{O}(\Delta k_\ell^2)$. However, the knowledge of the second moment $\mu_\ell^{(2)}$ may be important in combining the fringe solution for the whole operational bandwidth. This question will be addressed elsewhere.

B. Approximation for the Complex Fringe Envelope Function

Expression for the fringe envelope function (17), contains terms that are of the first and higher orders of phase variation within the ℓ -th spectral channel. We shall separate these terms by expanding phase $\phi(k)$ in the Taylor series around the mean wavenumber k_ℓ as given by Eq. (81). This transforms the argument in Eq. (17) as:

$$[(k - k_\ell)x(t) + \phi(k) - \phi(k_\ell)] = (k - k_\ell)[x(t) + d_{0\ell}] + \mathcal{O}(\Delta k_\ell^2), \quad (88)$$

where we $d_{0\ell} = \left. \frac{\partial \phi}{\partial k} \right|_{k_\ell}$ in the group delay within the ℓ -th channel. In the regime of small phase variations within the spectral channel $\Delta k_\ell \left. \frac{\partial \phi}{\partial k} \right|_{k_\ell} \equiv \Delta k_\ell d_{0\ell} \ll 1$, we can expand the exponential argument in the expression Eq. (17) as given below

$$\begin{aligned} \exp\left\{j \left[(k - k_\ell) \left(x(t) + d_{0\ell} \right) + \mathcal{O}(\Delta k_\ell^2) \right] \right\} &= \\ &= \left\{ 1 + j (k - k_\ell) d_{0\ell} + \mathcal{O}(\Delta k_\ell^2) \right\} \cdot \exp\left\{ j (k - k_\ell) x(t) \right\}. \end{aligned} \quad (89)$$

This last expression may be used to re-write the phase-dependent envelope function from Eq.(17) as

$$\tilde{W}_\ell[\Delta k_\ell, \phi_\ell, x(t)] = \left\{ 1 + d_{0\ell} \frac{\partial}{\partial x(t)} + \mathcal{O}(\Delta k_\ell^2) \right\} \int_{-\infty}^{+\infty} \hat{V}_{0\ell}(k) e^{j(k-k_\ell)x(t)} dk. \quad (90)$$

Defining the unperturbed fringe envelope function (i.e. that is un-affected by the phase variations inside the spectral channel) as below

$$\tilde{W}_\ell[\Delta k_\ell, x(t)] = \int_{-\infty}^{+\infty} \hat{V}_{0\ell}(k) e^{j(k-k_\ell)x(t)} dk. \quad (91)$$

We may present expression (90) for envelope function in the following form:

$$\tilde{W}_\ell[\Delta k_\ell, \phi_\ell, x(t)] = \tilde{W}_\ell[\Delta k_\ell, x(t)] + d_{0\ell} W'_\ell[\Delta k_\ell, x(t)] + \mathcal{O}(\Delta k_\ell^2), \quad (92)$$

where superscript ' denotes partial derivative with respect to OPD $\delta/\delta x(t)$.

At this point we have established the functional dependency of the envelope function, but for the immediate purposes we will be using a generic form for this function, $\tilde{W}_\ell[\Delta k_\ell, x(t)]$ presenting it only by it's amplitude and phase:

$$\tilde{W}_\ell[\Delta k_\ell, x(t)] = \mathcal{E}_\ell[\Delta k_\ell, x(t)] \exp\left\{ j\Omega_\ell[\Delta k_\ell, x(t)] \right\} \equiv \mathcal{E}_\ell \exp\left\{ j\Omega_\ell \right\}. \quad (93)$$

Similarly to the expression (92) we re-write this result in the following form

$$\tilde{W}_\ell[\Delta k_\ell, \phi_\ell, x(t)] = \mathcal{E}_\ell \exp\left\{ i\Omega_\ell \right\} + d_{0\ell} \left(\mathcal{E}'_\ell + j\mathcal{E}_\ell \Omega'_\ell \right) \exp\left\{ j\Omega_\ell \right\} + \mathcal{O}(\Delta k_\ell^2). \quad (94)$$

At this moment we show the functional form of real and imaginary components of the envelope function. Thus, from Eq. (94) one immediately has

$$\operatorname{Re}\left\{\tilde{W}_\ell[\Delta k_\ell, \phi_\ell, x_i]\right\} = \mathcal{E}_\ell \cos \Omega_\ell + d_{0\ell} \left(\mathcal{E}'_\ell \cos \Omega_\ell - \mathcal{E}_\ell \Omega'_\ell \sin \Omega_\ell \right) + \mathcal{O}(\Delta k_\ell^2), \quad (95)$$

$$\operatorname{Im}\left\{\tilde{W}_\ell[\Delta k_\ell, \phi_\ell, x_i]\right\} = \mathcal{E}_\ell \sin \Omega_\ell + d_{0\ell} \left(\mathcal{E}'_\ell \sin \Omega_\ell + \mathcal{E}_\ell \Omega'_\ell \cos \Omega_\ell \right) + \mathcal{O}(\Delta k_\ell^2). \quad (96)$$

The obtained equation exhibits explicit dependence on the phase variation inside the spectral channel represented by the term $d_{0\ell} = \partial\phi/\partial k|_{k_\ell}$. This issue will be addressed further.

C. Solution for Rectangular Bandpass and Stepping OPD Modulation

In consideration of completeness, we present here a general case solution for an optimally-weighted visibility phasor for a rectangular bandpass and stepping OPD modulation. In the previous Section we obtained this solution in a general case, therefore, the desired solution may be obtained directly with the help of expressions (51). Corresponding optimally-weighted solution may be presented in the form of Eqs. (51) and (52) with coefficients $\mathcal{A}_k^\circ, \mathcal{B}_k^\circ, \mathcal{C}_k^\circ$ and \mathcal{D}° depend only on the size of modulation steps x_i , mean wavenumber \bar{k} , width of a spectral channel Δk , variances of the data σ_i^2 in a particular temporal bin. These coefficients are given as follows:

$$\begin{aligned} \mathcal{A}_k^\circ = & \sum_{ij}^N \frac{1}{\sigma_i^2 \sigma_j^2 \sigma_k^2} \operatorname{sinc}\left[\frac{\Delta k x_i}{2}\right] \operatorname{sinc}\left[\frac{\Delta k x_j}{2}\right] \sin[\bar{k}(x_i - x_j)] \times \\ & \times \left[\operatorname{sinc}\left[\frac{\Delta k x_i}{2}\right] \operatorname{sinc}\left[\frac{\Delta k x_j}{2}\right] \sin[\bar{k}(x_i - x_j)] + \right. \\ & + \operatorname{sinc}\left[\frac{\Delta k x_j}{2}\right] \operatorname{sinc}\left[\frac{\Delta k x_k}{2}\right] \sin[\bar{k}(x_j - x_k)] + \\ & \left. + \operatorname{sinc}\left[\frac{\Delta k x_k}{2}\right] \operatorname{sinc}\left[\frac{\Delta k x_i}{2}\right] \sin[\bar{k}(x_k - x_i)] \right], \quad (97) \end{aligned}$$

$$\begin{aligned}
\mathcal{B}_k^\circ &= \sum_{ij}^N \frac{1}{\sigma_i^2 \sigma_j^2 \sigma_k^2} \left(\text{sinc}\left[\frac{\Delta k x_i}{2}\right] \cos \bar{k} x_i - \text{sinc}\left[\frac{\Delta k x_j}{2}\right] \cos \bar{k} x_j \right) \times \\
&\quad \times \left[\text{sinc}\left[\frac{\Delta k x_i}{2}\right] \text{sinc}\left[\frac{\Delta k x_j}{2}\right] \sin[\bar{k}(x_i - x_j)] + \right. \\
&\quad + \text{sinc}\left[\frac{\Delta k x_j}{2}\right] \text{sinc}\left[\frac{\Delta k x_k}{2}\right] \sin[\bar{k}(x_j - x_k)] + \\
&\quad \left. + \text{sinc}\left[\frac{\Delta k x_k}{2}\right] \text{sinc}\left[\frac{\Delta k x_i}{2}\right] \sin[\bar{k}(x_k - x_i)] \right], \tag{98}
\end{aligned}$$

$$\begin{aligned}
\mathcal{C}_k^\circ &= - \sum_{ij}^N \frac{1}{\sigma_i^2 \sigma_j^2 \sigma_k^2} \left(\text{sinc}\left[\frac{\Delta k x_i}{2}\right] \sin \bar{k} x_i - \text{sinc}\left[\frac{\Delta k x_j}{2}\right] \sin \bar{k} x_j \right) \times \\
&\quad \times \left[\text{sinc}\left[\frac{\Delta k x_i}{2}\right] \text{sinc}\left[\frac{\Delta k x_j}{2}\right] \sin[\bar{k}(x_i - x_j)] + \right. \\
&\quad + \text{sinc}\left[\frac{\Delta k x_j}{2}\right] \text{sinc}\left[\frac{\Delta k x_k}{2}\right] \sin[\bar{k}(x_j - x_k)] + \\
&\quad \left. + \text{sinc}\left[\frac{\Delta k x_k}{2}\right] \text{sinc}\left[\frac{\Delta k x_i}{2}\right] \sin[\bar{k}(x_k - x_i)] \right], \tag{99}
\end{aligned}$$

$$\begin{aligned}
\mathcal{D}^\circ &= \sum_k^N \mathcal{A}_k^\circ = \\
&= \frac{1}{3} \sum_{ijk}^N \frac{1}{\sigma_i^2 \sigma_j^2 \sigma_k^2} \left[\text{sinc}\left[\frac{\Delta k x_i}{2}\right] \text{sinc}\left[\frac{\Delta k x_j}{2}\right] \sin[\bar{k}(x_i - x_j)] + \right. \\
&\quad + \text{sinc}\left[\frac{\Delta k x_j}{2}\right] \text{sinc}\left[\frac{\Delta k x_k}{2}\right] \sin[\bar{k}(x_j - x_k)] + \\
&\quad \left. + \text{sinc}\left[\frac{\Delta k x_k}{2}\right] \text{sinc}\left[\frac{\Delta k x_i}{2}\right] \sin[\bar{k}(x_k - x_i)] \right]^2, \tag{100}
\end{aligned}$$

in consideration of brevity we omitted index ℓ denoting a particular spectral channel.

The obtained result Eqs. (100) clearly depends on the sinc envelope function and thus it has all the information that is necessary for the phase estimation purposes in the case of the wide band-pass. Note, that this result assumes that the phase does not change inside the spectral channel. Also, this result directly corresponds to the result obtained for the monochromatic case. This may be demonstrated by taking the

limit $\Delta k/\bar{k} \rightarrow 0$, which will lead to recovering the familiar form of monochromatic fringe with coefficients \mathcal{A}_k° , \mathcal{B}_k° , \mathcal{C}_k° and \mathcal{D}° given as follows:

$$\mathcal{A}_k^\circ = \sum_{ij}^N \frac{1}{\sigma_i^2 \sigma_j^2 \sigma_k^2} \sin[k(x_i - x_j)] \times \\ \times \left[\sin[k(x_i - x_j)] + \sin[k(x_j - x_k)] + \sin[k(x_k - x_i)] \right], \quad (101)$$

$$\mathcal{B}_k^\circ = \sum_{ij}^N \frac{1}{\sigma_i^2 \sigma_j^2 \sigma_k^2} (\cos kx_i - \cos kx_j) \times \\ \times \left[\sin[k(x_i - x_j)] + \sin[k(x_j - x_k)] + \sin[k(x_k - x_i)] \right], \quad (102)$$

$$\mathcal{C}_k^\circ = - \sum_{ij}^N \frac{1}{\sigma_i^2 \sigma_j^2 \sigma_k^2} (\sin kx_i - \sin kx_j) \times \\ \times \left[\sin[k(x_i - x_j)] + \sin[k(x_j - x_k)] + \sin[k(x_k - x_i)] \right], \quad (103)$$

$$\mathcal{D}^\circ = \sum_k^N \mathcal{A}_k^\circ = \\ = \frac{1}{3} \sum_{ijk}^N \frac{1}{\sigma_i^2 \sigma_j^2 \sigma_k^2} \left[\sin[k(x_i - x_j)] + \sin[k(x_j - x_k)] + \sin[k(x_k - x_i)] \right]^2. \quad (104)$$

This form demonstrates that only the terms with $i \neq j \neq k$ are producing a non-zero contributions to the result, while the terms where at least two of the indexes are equal (i.e. $i = j$ or $i = k$ or $j = k$) will vanish from the sum.

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