Geostrophic Balance with a full Coriolis Force: Implications for low Latitude Studies.

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Abstract: In its standard form, geostrophic balance uses a partial representation of the Coriolis force. The resulting formulation has a singularity at the equator, and violates mass and momentum conservation. When the horizontal projection of the planetary rotation vector is considered, the singularity at the equator disappears, continuity can be preserved, and quasigeostrophy can be formulated at planetary scale. At the same time, the predicted geostrophic winds can differ significantly from the standard approach. Similarities and differences between both approaches to wind diagnostics are shown in an application example.

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Partial and full Coriolis force

$$\boxed{f\mathbf{e}_z \times \mathbf{v}_{gp} = -\frac{\nabla p}{\rho} + \mathbf{g}}, \; f = 2\Omega \sin\lambda \quad \Big| \quad \boxed{2\Omega\mathbf{k} \times \mathbf{v}_g = -\frac{\nabla p}{\rho} + \mathbf{g}}$$

$$\begin{aligned}
-2\Omega(v\sin\lambda - w\cos\lambda) &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\
2\Omega u \sin\lambda &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \\
-2\Omega u \cos\lambda &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g
\end{aligned}$$

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- Hydrostatic: $(\nabla p + \rho g) \perp \mathbf{e}_z$
- Singular at the Equator:
  $\mathbf{v} = \mathbf{e}_z \times \frac{\nabla p - \rho \mathbf{g}}{2\Omega \sin\lambda\rho} + v_z \mathbf{e}_z$

$$\left\{\begin{array}{lll} u_{gp} & = & -\frac{1}{f\rho}\frac{\partial p}{\partial y} \\ v_{gp} & = & \frac{1}{f\rho}\frac{\partial p}{\partial x} \\ w_{gp} & = & 0 \end{array}\right.$$

$$\boxed{2\Omega\mathbf{k} \times \mathbf{v}_g = -\frac{\nabla p}{\rho} + \mathbf{g}}$$

- Non-hydrostatic: $(\nabla p + \rho g) \perp \mathbf{k}$.
  (de Verdière & Schopp 1994).

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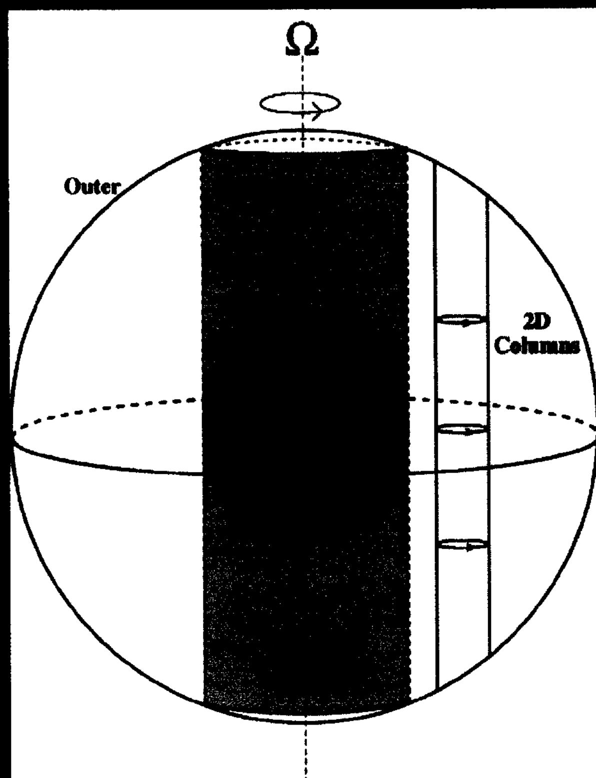
- Non-singular and 3-dimensional
  $\mathbf{v} = \mathbf{k} \times \frac{\nabla p - \rho \mathbf{g}}{2\Omega\rho} + v_Z \mathbf{k}$ (Veronis 1968)

$$\begin{cases} u_g &= -\frac{\sin\lambda}{2\Omega\rho}\frac{\partial p}{\partial y} + \frac{\cos\lambda}{2\Omega\rho}\left(\rho g + \frac{\partial p}{\partial z}\right) \\ v_g &= \frac{\sin\lambda}{2\Omega\rho}\frac{\partial p}{\partial x} \\ w_g &= -\frac{\cos\lambda}{2\Omega\rho}\frac{\partial p}{\partial x} \end{cases}$$

Taylor columns with full Coriolis

Chandrasekhar 1962, F.H. Busse 1970, (figure by F.H. Busse):



- Inner boundary acts as obstacle to flow
- Taylor column-like structure forms around inner shell
- Referred to as the tangent cylinder (TC)

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- Hydrostatic: $(\nabla p + \rho g) \perp \mathbf{e}_z$
- Singular at the Equator
- Divergent in incompressible and compressible flows. Pedlosky 2.9c
  $\nabla \cdot (\rho \mathbf{v}_{gp}) = -\frac{v \cos\lambda}{r \sin\lambda} \neq 0$
  Vertical velocity obtained from divergence in diagnostics.

$\boxed{2\Omega\mathbf{k} \times \mathbf{v}_g = -\frac{\nabla p}{\rho} + \mathbf{g}}$

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- Non-hydrostatic: $(\nabla p + \rho g) \perp \mathbf{k}$.
- Non-singular and 3-dimensional
- Non-divergent in incompressible flows and under constraints in compressible flows. (de la Torre et al. 2002)

$$2\Omega\nabla\cdot(\rho\mathbf{v}_g) = \frac{g\epsilon_\rho}{r\epsilon}\frac{\partial\rho}{\partial\varphi} + 2\Omega\frac{\partial(\rho v_Z)}{\partial Z} =$$

$$= g\frac{\epsilon_\rho}{\epsilon}\frac{\partial\rho}{\partial x}\cos\lambda + 2\Omega\frac{\partial(\rho v_Z)}{\partial Z} = 0$$

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- Singular 2-dimensional thermal winds:

$$\begin{cases} \dfrac{\partial u_{gp}}{\partial z^*} = -\dfrac{R}{fM_r}\left(\dfrac{\partial T}{\partial y}\right)_p \\ \dfrac{\partial v_{gp}}{\partial z^*} = \dfrac{R}{fM_r}\left(\dfrac{\partial T}{\partial x}\right)_p \end{cases}$$

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Partial and full Coriolis force

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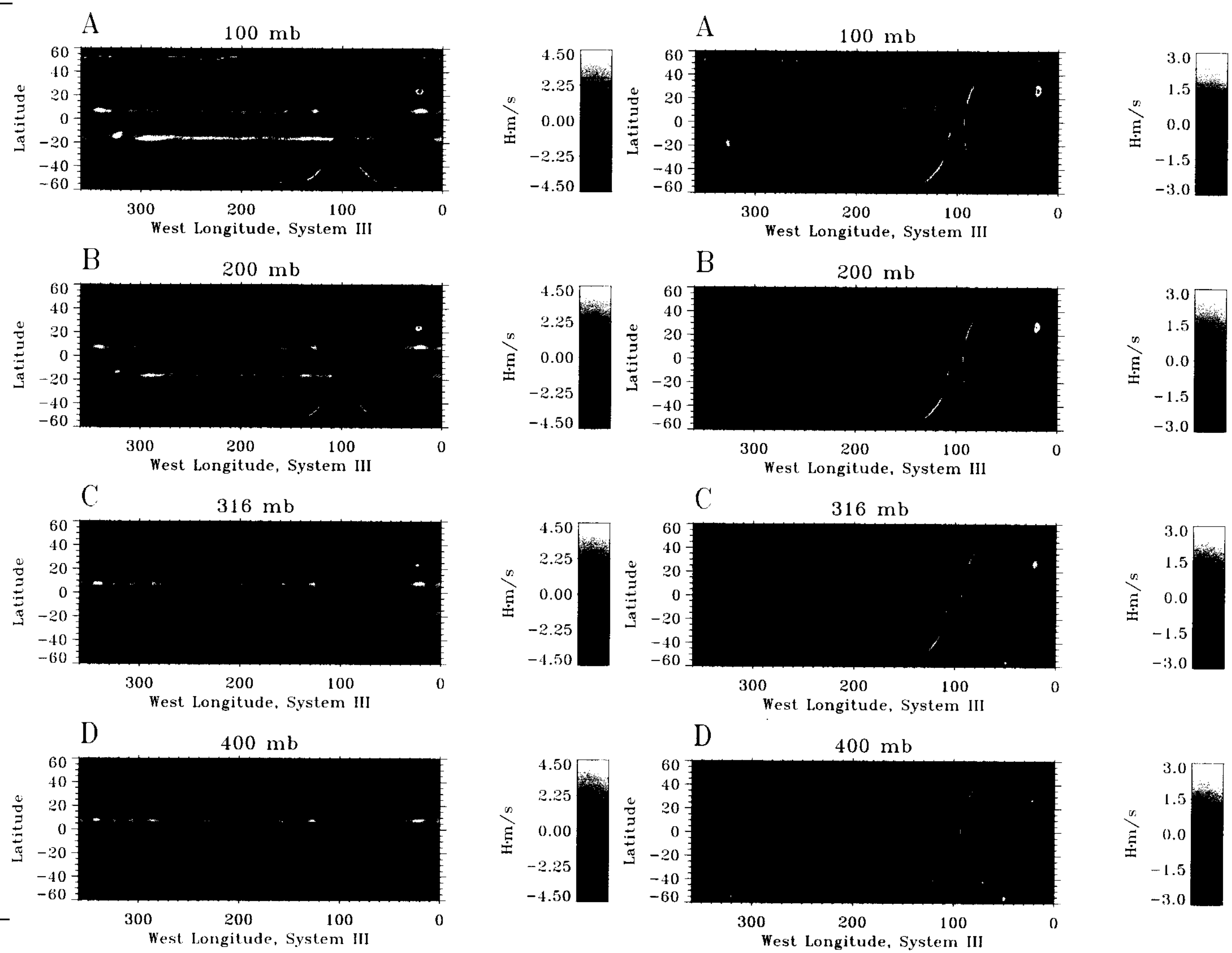
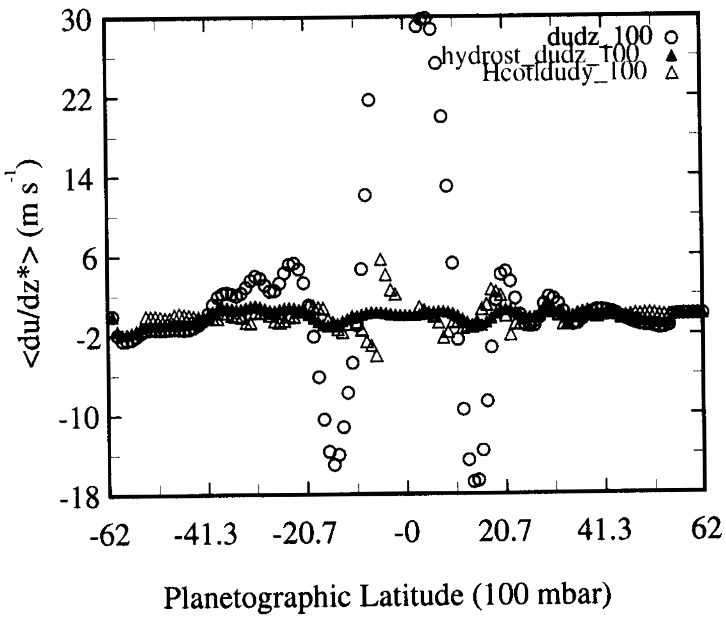
- Non-divergent in incompressible flows and for zonal density in anelastic flows.
- 3-dimensional non-singular (!) thermal winds (de la Torre 2002)

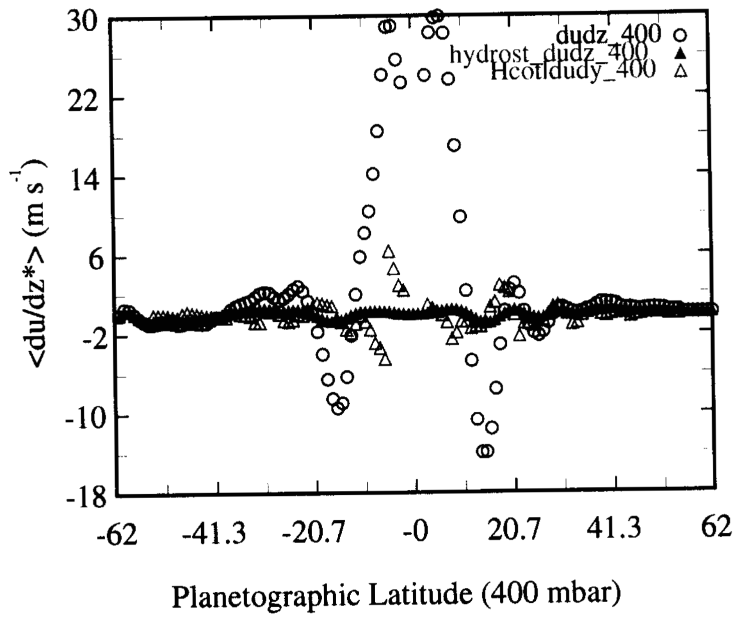
$$\begin{cases} \dfrac{\partial u_g}{\partial z^*} = -\dfrac{R}{2\Omega \sin\lambda M_r}\left(\dfrac{\partial T}{\partial y}\right)_p - H\cot\lambda \dfrac{\partial u_g}{\partial y} \\ \dfrac{\partial v_g}{\partial z^*} = \dfrac{R}{2\Omega \sin\lambda M_r}\left(\dfrac{\partial T}{\partial x}\right)_p - H\cot\lambda \dfrac{\partial v_g}{\partial y} \\ \dfrac{\partial w_g}{\partial z^*} = -\cot\lambda H \dfrac{\partial w_g}{\partial y} \end{cases}$$

Jovian thermal winds



Jovian thermal winds





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- Angular momentum violation.

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- Ertel's potential vorticity conservation only on f-plane

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Ertel's Potential vorticity conservation

$\nabla \times (2\boldsymbol{\Omega} \times \mathbf{v}) = 2\boldsymbol{\Omega}(\nabla \cdot \mathbf{v}) - \mathbf{v}(\nabla \cdot 2\boldsymbol{\Omega}) + (\mathbf{v} \cdot \nabla)2\boldsymbol{\Omega} - (2\boldsymbol{\Omega} \cdot \nabla)\mathbf{v}$,
Absolute vorticity: $\omega_a \equiv \omega + 2\boldsymbol{\Omega}$
Continuity $\rightarrow \frac{d}{dt}\left(\frac{\omega_a}{\rho}\right) - \left(\frac{\omega_a}{\rho} \cdot \nabla\right)\mathbf{v} = \frac{\nabla\rho \times \nabla p}{\rho^3} + \frac{\nabla \times \mathbf{F}}{\rho}$.
For any scalar field λ: $\left(\frac{\omega_a}{\rho} \cdot \nabla\right)\frac{d\lambda}{dt} = \frac{\omega_a}{\rho} \cdot \frac{d\nabla\lambda}{dt} + \left[\left(\frac{\omega_a}{\rho} \cdot \nabla\right)\mathbf{v}\right] \cdot \nabla\lambda$.
If the dot product of $\nabla\lambda$ is taken with the vorticity equation, and if $\Psi = \frac{d\lambda}{dt}$:
$\nabla\lambda \cdot \frac{d}{dt}\left(\frac{\omega_a}{\rho}\right) = \nabla\lambda \cdot \left[\left(\frac{\omega_a}{\rho} \cdot \nabla\right)\mathbf{v} + \frac{\nabla\rho \times \nabla p}{\rho^3} + \frac{\nabla \times \mathbf{F}}{\rho}\right]$, which leads to
$\frac{d}{dt}\left(\frac{\omega_a}{\rho} \cdot \nabla\lambda\right) = \frac{\omega_a}{\rho} \cdot \nabla\Psi + \nabla\lambda \cdot \left[\frac{\nabla\rho \times \nabla p}{\rho^3} + \frac{\nabla \times \mathbf{F}}{\rho}\right]$.
Thus: if λ is a conserved quantity for every fluid element (i.e.$\Psi = 0$), if the bulk forces are negligible or potential (i.e. $\mathbf{F} = -\nabla V$), and if either λ is a function of ρ, and p only, or the fluid is barotropic (i.e. $\rho = \rho(p)$), then the potential vorticity PV $q \equiv \left(\frac{\omega_a}{\rho} \cdot \nabla\lambda\right)$ is a conserved quantity.
This proof is valid for both, incompressible and compressible flows. which is the form that we will use below.
Incompressible case: λ can be the density and $\dot{\rho} = 0$.

$$\dot{q} = \frac{1}{\rho}\frac{d(\omega_a \cdot \nabla\lambda)}{dt} = 0$$

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