Accumulate Repeat Accumulate Codes

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December 2, 2003

Abstract

In this paper we propose an innovative channel coding scheme called "Accumulate Repeat Accumulate codes" (ARA). This class of codes can be viewed as turbo-like codes, namely a double serial concatenation of a rate-1 accumulator as an outer code, a regular or irregular repetition as a middle code, and a punctured accumulator as an inner code. More important is the fact that the ARA codes can also be considered as a subclass of Low Density Parity Check (LDPC) codes, thus belief propagation (message passing algorithm) can be used for iterative decoding of ARA codes on a graph. The structure of encoder for this class can be viewed as precoded Repeat Accumulate (RA) code or as precoded Irregular Repeat Accumulate (IRA) code, where simply an accumulator is chosen as a precoder. Thus ARA codes have simple, and very fast encoder structure when they representing LDPC codes. Based on density evolution for LDPC codes through some examples for ARA codes, we show that for maximum variable node degree 5 a minimum bit SNR as low as 0.08 dB from channel capacity for rate 1/2 can be achieved as the block size goes to infinity . Thus based on fixed low maximum variable node degree, its threshold outperforms not only the RA and IRA codes but also the best known LDPC codes with the same maximum node degree. Furthermore by puncturing the accumulators any desired high rate codes close to code rate 1 can be obtained with thresholds that stay close to the channel capacity thresholds uniformly. Iterative decoding simulation results are also provided that shows comparable performance with the best LDPC performance reported in the literature. The ARA codes also have projected graph or protograph representation, that allows for high speed decoder implementation.

1 Introduction and motivation of the work

Low Density Parity Check (LDPC) codes were proposed by Gallager [1] in 1962. After introduction of turbo codes by Berrou et al [2] in 1993, researchers revisited the LDPC codes, and extended the work of Gallager. During 1962 to 1993, Tanner [3] in 1981 was the only one that paid some attention to the work of Gallager and made some contribution. After 1993 lots of contributions have been made to LDPC codes which due to the space limitation we can not acknowledge all of them, but we will do so in the expanded version of our paper.

Recently RA [5] and IRA [6] codes as a simple subclass of LDPC codes with fast encoder structure were proposed. Classical RA codes in addition to simplicity have reasonable good performance and with iterative decoding threshold within 1 dB from capacity for rates less than or equal to 1/3. RA codes use fixed repetition for input bits. On the other hand IRA codes inspired by RA and irregular LDPC [4] codes have irregular repetition for input bits. For IRA codes node degree distribution can be optimized to achieve low thresholds. To achieve very low threshold for IRA, as for LDPC codes, maximum repetition for some portion of input bits can be very high. These recent results on RA and IRA which have fast encoders, motivated us to find a way to enhance the performance of RA codes in particular. Researchers in [8], tried to improve the input-output extrinsic SNR behavior of the outer convolutional codes in serial concatenation in low extrinsic SNR region to

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†Jet Propulsion Laboratory - Caltech USA. This research was carried out at the Jet Propulsion Laboratory, California Institute of Technology, under a contract with NASA.

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lower SNR threshold of serial concatenation by using repetition of certain bits of the outer code. On the other hand if a repetition code is used as an outer code such as in RA codes, one should try to improve the input-output extrinsic SNR behavior at the high extrinsic SNR region, since the input-output extrinsic SNR behavior of repetition codes are excellent in the low extrinsic SNR region. We discovered that a rate-1 precoder such as accumulator applied before the repetition code will do the job. Before elaborating on the role of accumulator as a precoder for RA codes and graph representation of ARA codes, we use the definition of projected graph introduced in [9], [10], [11] for implementation of decoder for LDPC codes which shows that if an LDPC code can be represented by a smallest base-graph (projected graph) then high speed implementation of decoder will be more feasible. Similar definition also is provided in [7] for base-graph which was called protograph. This definition also facilitates the minimal graph representation for overall graph description of LDPC codes. In fact we will show that ARA codes have such projected graph or protograph representation which is another advantage. Moreover Jeremy Thorpe [7] of JPL and Caltech provided us with an efficient density evolution program on such protographs to obtain iterative decoding thresholds for infinite block size.

A protograph [7] is a Tanner graph with a relatively small number of nodes. A protograph $G = (V, C, E)$ consists of a set of variable nodes V , a set of check nodes C, and a set of edges E. Each edge $e \in E$ connects a variable node $v_e \in V$ to a check node $c_e \in C$. Parallel edges are permitted, so the mapping $e \rightarrow (v_e, c_e) \in V \times C$ is not necessarily 1:1. As a simple example, we consider the protograph shown in Fig. 1 (b). This graph consists of $|V| = 4$ variable nodes and $|C| = 3$ check nodes, connected by $|E| = 9$ edges. The four variable nodes in the protograph are denoted by 0, 1, 2, 3, and the three check nodes by 0, 1, 2. By itself, this graph may be recognized as the Tanner graph of an (n = 3, k = 1) LDPC code (in this case, a Repeat and Accumulate code Fig. 1 (a)). In Fig. 1 (b), the variable nodes connected to the channel are shown with dark filled circles. Blank circles are those variable nodes not connected to the channel (i.e. punctured). Check nodes are circles with inside plus sign. We can obtain a larger graph by a copy-and-permute operation. For the details on protograhs see [7]. The resulting larger graph is called the derived graph, and the corresponding LDPC code is a protograph code. In general, we can apply the copy-and-permute operation to any protograph to obtain derived graphs of different sizes. This operation consists of first making T copies of the protograph, and then permuting the endpoints of each edge among the T variable and T check nodes connected to the set of T edges copied from the same edge in the protograph. In Fig. 1 (b), the minimum E_b/N_o threshold of RA code with iterative decoding is also shown.

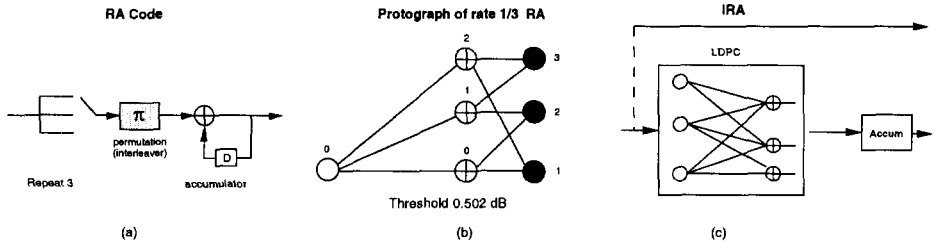


Figure 1: (a) RA code, (b) Protograph of rate 1/3 RA code with repetition 3, (c) IRA code

Hui Jin et al., [6] proposed Irregular RA (IRA) code which is a serial concatenation of a simple LDPC code with different degree variable nodes (irregular repetition) as an outer code and an accumulator as an inner code to generalize the RA code. The encoder can be implemented by repetition codes, Exclusive OR's, and an accumulator as shown in Fig. 1 (c).

2 Punctured RA and alternative encoding of IRA codes

Rate 1/2 classical RA code has high threshold of 3.01 dB. Lower threshold for rate 1/2 RA can be obtained if we puncture the accumulator, provided that the repetition is greater than or equal to 3, and the systematic bits are transmitted through the channel i.e. systematic punctured RA. Based on an equivalent graph of a punctured accumulator, we obtain the protograph of systematic punctured RA with threshold 1.116 dB. This is an improvement close to 2 dB. The systematic punctured RA is shown in Fig. 2.

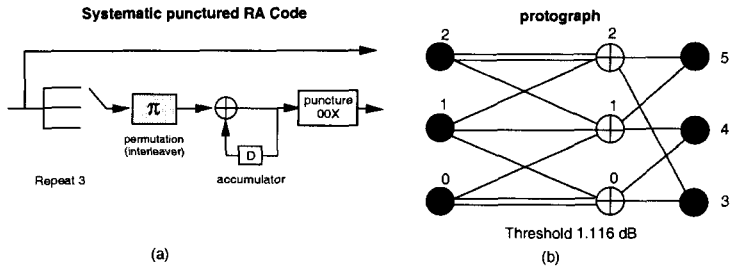


Figure 2: Systematic punctured RA code

The immediate consequence of this observation suggests an alternative encoding structure for systematic IRA codes by just using irregular repetition, permutation, and punctured accumulator. An example of rate 1/2 systematic IRA, the encoder and the corresponding protograph is shown in Fig. 3. The threshold for this example is 0.990 dB.

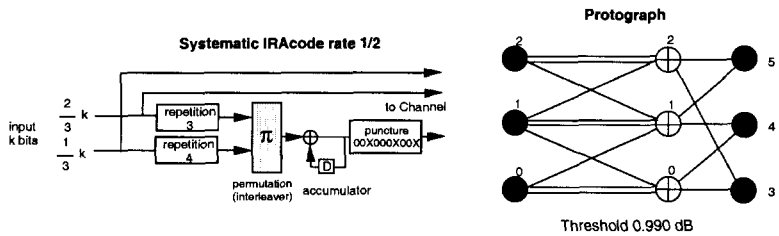


Figure 3: Systematic IRA

3 Accumulate-Repeat-Accumulate codes

Lets consider a rate 1/3 serial concatenated code where the outer code is a repetition 3 code. Assume the systematic bits are transmitted to the channel. Alternatively consider the same outer code but the repetition 3 is precoded by an accumulator. Lets compare the extrinsic SNR behavior of these two outer codes using Gaussian density evolution as shown in Fig. 4 (a). As the Gaussian density evolution analysis shows the use of a rate-1 accumulator dramatically improve the extrinsic SNR behavior of repetition 3 at high extrinsic SNR region. However it slightly deteriorates the behavior of repetition code at very low extrinsic SNR region.

Now lets use a punctured accumulator as an inner code. The periodic puncturing pattern in this example is X0X where 0's indicate the puncturing positions. Since the serial concatenation consists of outer accumulator, middle repetition, and inner accumulator, we call it Accumulate-Repeat-Accumulate (ARA) code. The rate 1/3 ARA, and the extrinsic input-output SNR curves using Gaussian density evolution are shown in Fig. 4 (b).

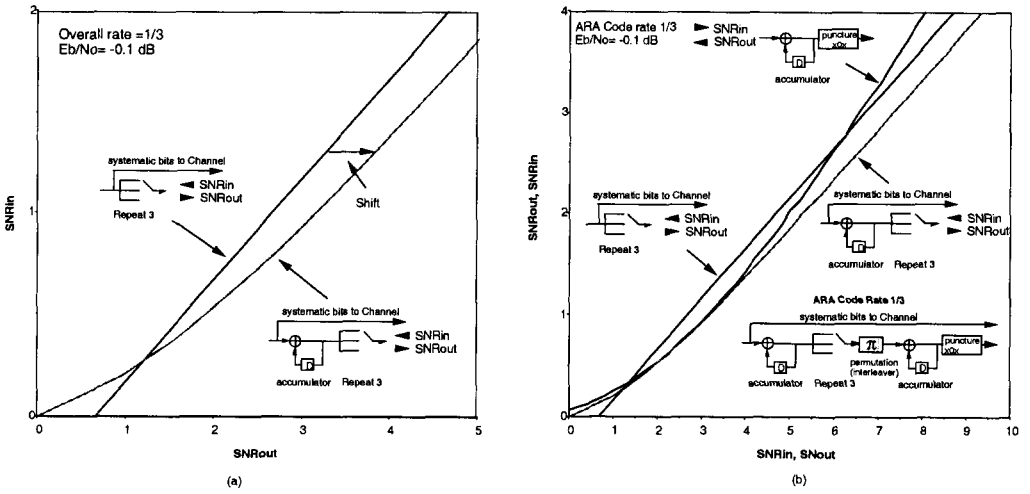


Figure 4: Gaussian density evolution: (a) repeat 3 w/o precoder, (b) rate 1/3 ARA

Gaussian density evolution provide an approximate threshold. Next we present the protograph for this rate 1/3 ARA code, and compute its threshold using density evolution. The protograph and computed threshold of -0.048 dB are shown in Fig. 5. This threshold shows 0.55 dB improvement over classical RA code. If we remove the precoder the threshold will be 0.73 dB. These comparisons will be fair if we fix the maximum variable node degree. Shortly we will show such comparisons with rate 1/2 LDPC codes.

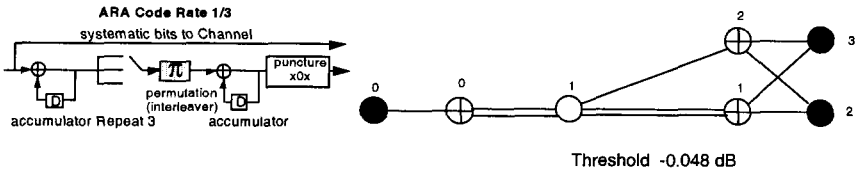


Figure 5: Rate 1/3 ARA, and the corresponding protograph representation

In a similar way we can construct rate 1/2 ARA code. However due to more puncturing of the inner accumulator, some portion of input before repetition should not be passed through the precoder to allow the iterative decoding to start. An example of a simple rate 1/2 ARA, its protograph, and the corresponding threshold are shown in Fig. 6.

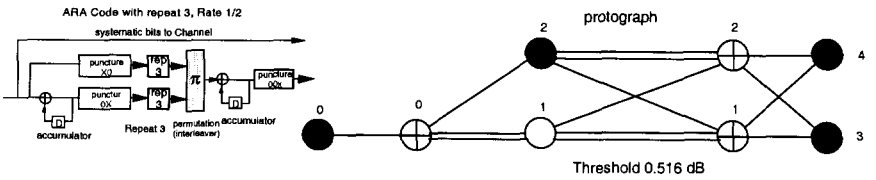
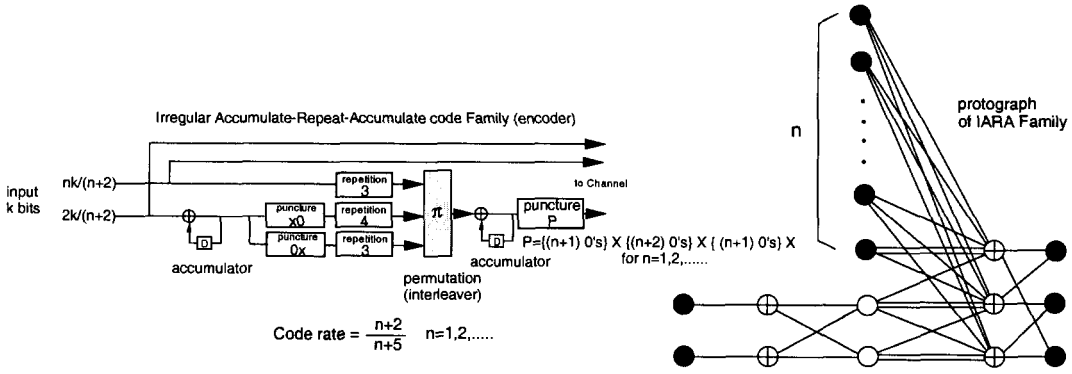


Figure 6: Rate 1/2 ARA, and the corresponding protograph representation

Instead of using regular repetition in ARA if we use irregular repetition then we refer to it as Irregular ARA or simply IARA code. A simple example of rate 1/2 IARA is the precoded version of the IRA example given previously. The threshold for that IRA example was 0.990 dB. The precoded version has threshold of 0.364 dB.

Higher code rates by just puncturing the IARA example is shown in Fig. 7. .



Code Rate	Proto graph Threshold	Capacity Threshold	Differ -ence	Puncturing Patterns P
1/2	0.364	0.187	0.177	00X000X00X
4/7	0.700	0.530	0.170	000X0000X000X
5/8	1.006	0.815	0.191	0000X00000X0000X
2/3	1.272	1.059	0.213	00000X000000X00000X
7/10	1.506	1.272	0.234	000000X0000000X000000X
8/11	1.710	1.459	0.251	0000000X00000000X0000000X
3/4	1.894	1.626	0.268	00000000X000000000X00000000X
10/13	2.057	1.777	0.280	000000000X0000000000X000000000X
11/14	2.204	1.914	0.290	0000000000X00000000000X0000000000X
4/5	2.338	2.040	0.298	00000000000X000000000000X00000000000X
13/16	2.461	2.156	0.305	000000000000X0000000000000X000000000000X
14/17	2.573	2.263	0.310	0000000000000X00000000000000X0000000000000X
5/6	2.681	2.362	0.319	00000000000000X000000000000000X00000000000000X
16/19	2.777	2.455	0.322	000000000000000X0000000000000000X000000000000000X
17/20	2.868	2.543	0.325	0000000000000000X00000000000000000X0000000000000000X
6/7	2.953	2.625	0.328	00000000000000000X000000000000000000X00000000000000000X
19/22	3.033	2.702	0.331	000000000000000000X0000000000000000000X000000000000000000X
20/23	3.108	2.775	0.333	0000000000000000000X00000000000000000000X0000000000000000000X
7/8	3.178	2.845	0.333	00000000000000000000X000000000000000000000X00000000000000000000X
22/25	3.245	2.910	0.335	000000000000000000000X0000000000000000000000X000000000000000000000X
23/26	3.309	2.973	0.336	0000000000000000000000X00000000000000000000000X0000000000000000000000X
8/9	3.370	3.033	0.337	00000000000000000000000X000000000000000000000000X00000000000000000000000X

Figure 7: IARA code family for rates 1/2 to 8/9, encoder, protographs, and table of thresholds in dB

An example of low threshold of 0.264 dB for rate 1/2 ARA code is shown in Fig. 8 (a). The protograph has maximum degree 5. The best rate 1/2 LDPC with maximum degree 5 in [4] has threshold 0.72 dB. There are few reasons for such difference. In [4] degree of variable nodes are greater or equal to 2, and punctured variable nodes were not allowed. If we look at protographs of ARA it contains degree 1 variable nodes and punctured variable nodes. But this is not the main reason. In fact, later Richardson and Urbanke [10], [11] mentioned that the degree 1 variable nodes and punctured variable nodes also can be used in LDPC code design but more than that they mentioned that so called "multi edge" representation can be used in the LDPC design. The "multi edge" in ARA code design simply means that the structure of interleaver in ARA codes should be based on edge connections in the ARA protograph between inner variable and inner check nodes (right most check nodes), i.e. between repetition and accumulator. Otherwise a finite threshold may not exist. Finally in this paper preliminary simulation results are shown in Fig. 8 (b) for two examples of rate 1/2 ARA codes and compared with a rate 1/2 turbo code.

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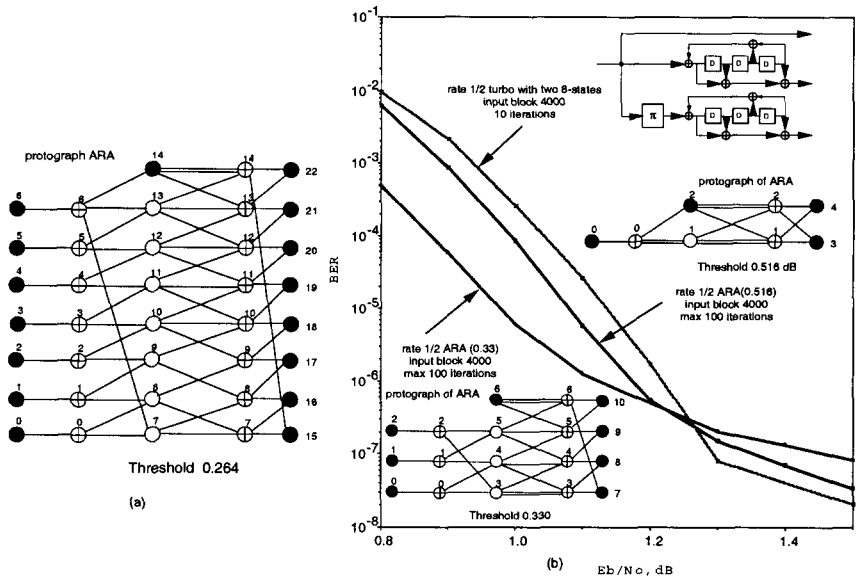

Figure 8: (a) rate 1/2 low threshold ARA protograph, (b) simulation results for two examples of rate 1/2 ARAs

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