rmation

rgou,

367
echnology
sadena,

Outline

- How does spin-spin entanglement transform under Lorentz transformations?
  - Massive case
  - Massless case

Spin Entanglement

- Many models for quantum computation use spin degrees of freedom for qubit
- When we use spin we are implicitly tracing out the momentum part of the wavefunction, e.g.:

$$|\psi\rangle = \int f(\mathbf{p})|\mathbf{p}\rangle \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) d^3\mathbf{p}$$

$$\rho_{\text{spin}} = \text{Tr}_{\mathbf{p}}(|\psi\rangle\langle\psi|)$$

- We answer how ρ_{spin} transforms

Some Definitions

- We define $|p_s\rangle$ as the standard momentum state, and $|\lambda\rangle$ to be some representation of the spin

$$P^\mu|\mathbf{p_s}\lambda\rangle = p_s^\mu|\mathbf{p_s}\lambda\rangle$$

- H(p) (nearly arbitrary function of p) is a Lorentz transformation that takes p_s to p:

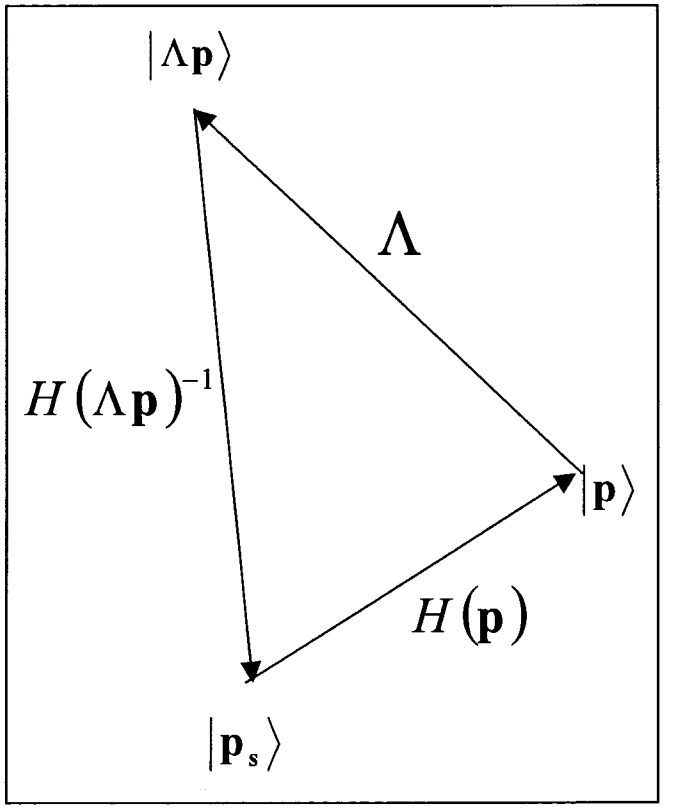
$$|\mathbf{p}\lambda\rangle \equiv H(\mathbf{p})|\mathbf{p}_s\lambda\rangle$$

- H(p) sets the "meaning" of $|\lambda\rangle$ (rest-state spin, helicity, etc.)

Wigner Rotations

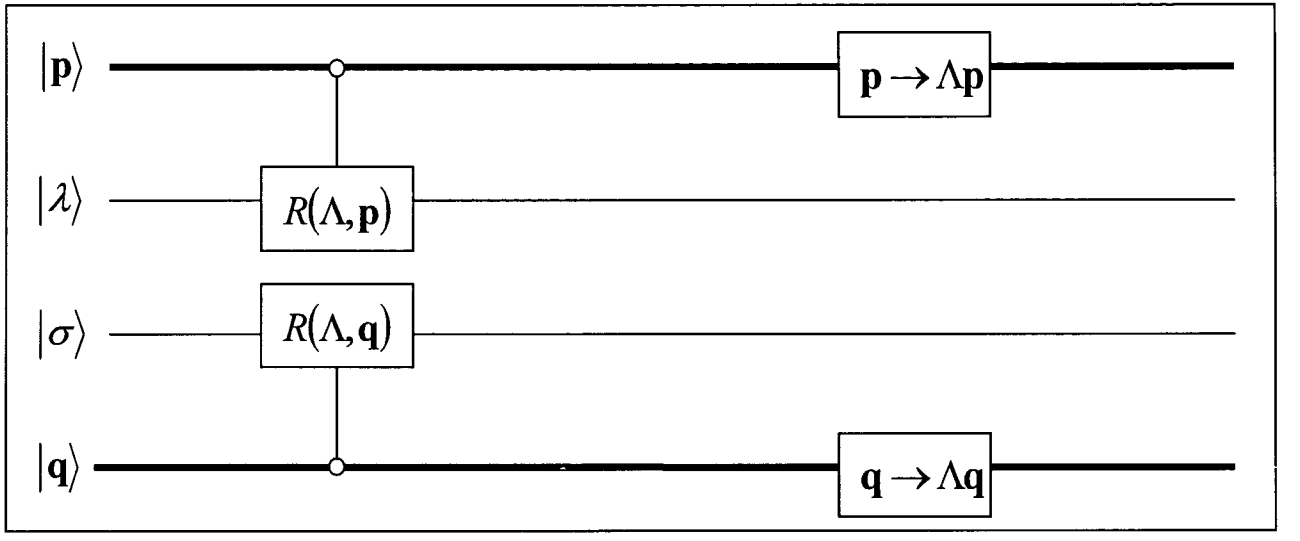
- Lorentz transformation acting on state:

$$\Lambda|\mathbf{p}\lambda\rangle = \Lambda H(\mathbf{p})|\mathbf{p}_s\lambda\rangle$$
$$= H(\Lambda\mathbf{p})\underbrace{H(\Lambda\mathbf{p})^{-1}\Lambda H(\mathbf{p})}_{R(\Lambda,\mathbf{p})}|\mathbf{p}_s\lambda\rangle$$



- R(Λ,p) is Wigner rotation – member of the little-group of $|p_s\rangle$
  - group of Lorentz transformations that leave $|p_s\rangle$ invariant

Circuit Diagram Representation



- Lorentz transformation (on two particles) represented as a quantum circuit
- Controlled-U gate between spin and momentum states "mixes" momentum and spin entanglement

Massive Particles

- We use $|p_s\rangle$ as the rest momentum and H(p):

$$H(\mathbf{p}) \equiv L(\xi_{\mathbf{p}})$$

$$\mathbf{p}_s \equiv \langle m \quad 0 \quad 0 \quad 0 \rangle$$

- Corresponds to a little group SO(3), and $|\lambda\rangle$ "rest-state spin"
  - For spin-½ Wigner rotation becomes:

$$p^\mu = (E, p\sin\phi\cos\theta, p\sin\phi\sin\theta, p\cos\phi)$$

$$R(\vec{p}, \xi) = \begin{bmatrix} \alpha & \beta e^{i\theta} \\ -\beta e^{-i\theta} & \alpha \end{bmatrix}$$

$$\alpha = \sqrt{\frac{E+m}{E'+m}}\left(\cosh\left(\frac{\xi}{2}\right) + \frac{p\cos\phi}{E+m}\sinh\left(\frac{\xi}{2}\right)\right)$$

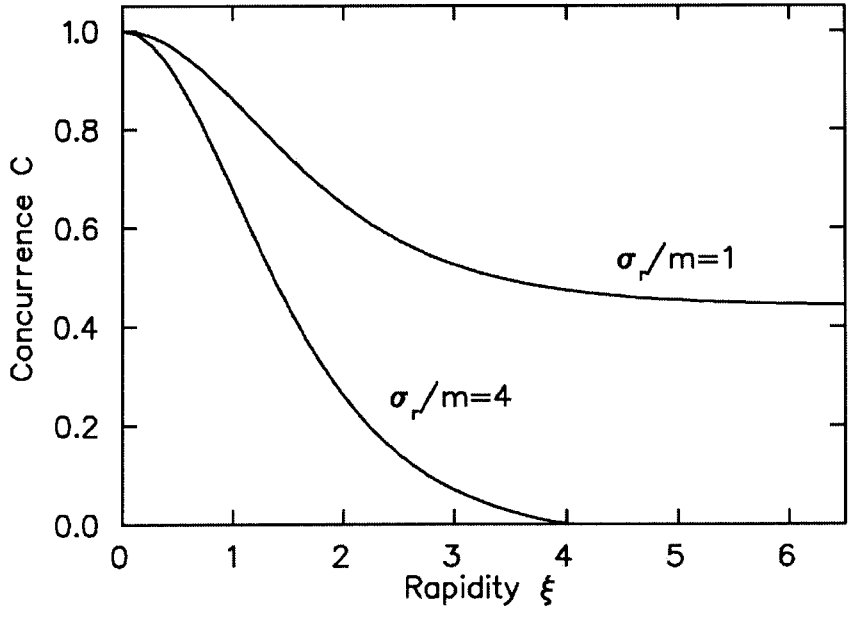
$$\beta = \frac{p\sin\phi}{\sqrt{(E+m)(E'+m)}}\sinh\left(\frac{\xi}{2}\right)$$

$$E = \sqrt{p^2 + m^2}$$

$$E' = E\cosh\left(\frac{\xi}{2}\right) + p\cos\phi\sinh\left(\frac{\xi}{2}\right)$$

Massive Spin-½ Entanglement

• We transform a state, fully entangled in spin, in a Gaussian momentum distribution (centered about zero):



$$|\psi\rangle = \frac{1}{\sqrt{2}} \int\int \sum \delta_{\lambda\sigma} f(\mathbf{p}) f(\mathbf{q}) |\mathbf{p}\lambda\rangle |\mathbf{q}\sigma\rangle \mathrm{d}\tilde{\mathbf{p}}\mathrm{d}\tilde{\mathbf{q}}$$

$$f(\mathbf{p}) = \frac{1}{N(\sigma_r)} e^{-\left(\frac{p}{2\sigma_r}\right)^2}$$

- Above is concurrence (C) as a function of rapidity (ξ) for two different spreads.

Massless Particles

- We use $|p_s\rangle$ and H(p) (θ, φ are polar, azimuthal angles of p):

$$H(\mathbf{p}) \equiv R_z(\theta) R_y(\phi) L_z(|\xi_{\mathbf{p}}|)$$

$$\mathbf{p}_s \equiv \langle 1 \quad 0 \quad 0 \quad 1 \rangle$$

- Corresponds to a little group SO(2)×T(2), and $|\lambda\rangle$ helicity
- The T(2) portion of the little group disappears, so states transform as:

$$\Lambda|\mathbf{p}\lambda\rangle = e^{-i\lambda\Theta(p,\Lambda)}|\mathbf{p}\lambda\rangle$$

- Not yet a very useful form because helicity is meaningless without momentum (so can't just easily trace it out)

Massless Particles

- Define components of helicity states:

$$|\mathbf{p}\lambda\rangle \rightarrow \mathbf{p} \otimes \varepsilon_{\mathbf{p}}^{\pm}$$

$$\varepsilon_{\mathbf{p}_s}^{\pm} \equiv \langle 0 \quad 0 \quad 1 \quad \pm i \rangle$$

- The requirements on helicity states:

$$\varepsilon_{\mathbf{p}} \cdot \mathbf{p} = \mathbf{0}$$

$$\varepsilon^0 = 0$$

- By reducing the states to "x, y, z polarization" we can have a meaningful reduced density matrix
  - x, y, z polarization is analogous to rest state spin
- With this new notation/interpretation can define nice succinct transformation rule for helicity vectors:

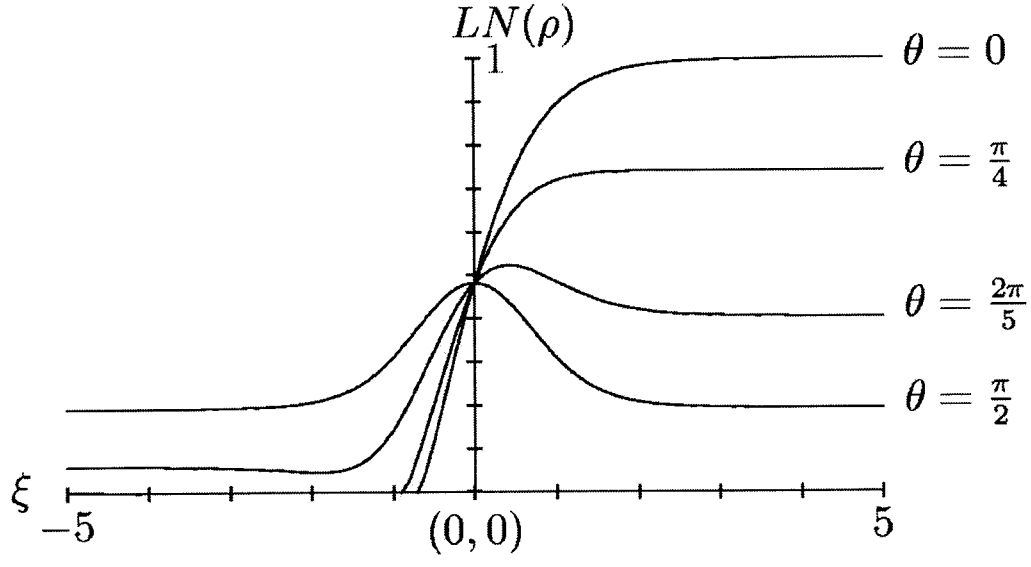
$$\varepsilon_{\Lambda\mathbf{p}} = \Lambda\varepsilon_{\mathbf{p}} - \frac{(\Lambda\varepsilon_{\mathbf{p}})^0}{(\Lambda\mathbf{p})^0}\Lambda\mathbf{p}$$

Negativity

- So how do we measure the entanglement of a 3-level system?
- Negativity, $N(\rho) \equiv$ sum of the magnitude of the eigenvalues of the partial-transpose of the density matrix
- Log Negativity, $LN(\rho) \equiv Log_2(N(\rho))$
- $LN(\rho)$ is a measure of the entanglement of distillation
  - Based on the Peres criterion
  - Range is 0 to 1
  - Useful because it is valid for multi-level systems.

Photon Entanglement Results

Helicity entangled photon pair with a Gaussian spread in direction of motion:

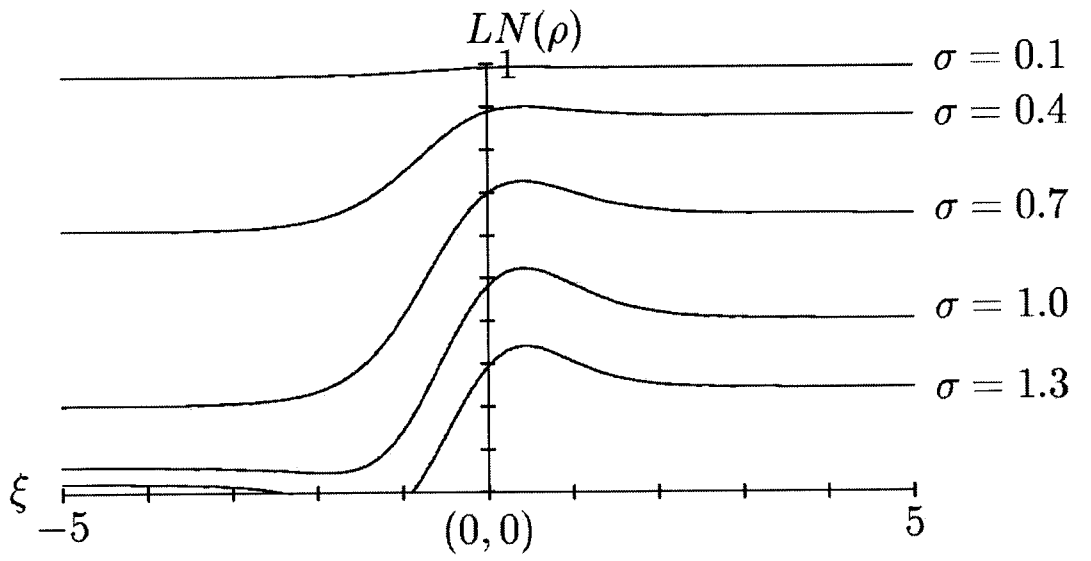


$$|\psi\rangle = \frac{1}{\sqrt{2}} \int\int \sum \delta_{\lambda\sigma} f(\mathbf{p}) f(\mathbf{q}) |\mathbf{p}\lambda\rangle |\mathbf{q}\sigma\rangle d\tilde{\mathbf{p}} d\tilde{\mathbf{q}} \qquad f(\mathbf{p}) = \frac{1}{N(\sigma)} e^{-\left(\frac{\theta'}{\sigma}\right)^2} e^{i\phi} \delta(p - p_0)$$

- Log Negativity as a function of rapidity, shown for various boost directions (σ=1.0 , p_0=1.0)

Photon Entanglement Results

Helicity entangled photon pair with a Gaussian spread in direction of motion :



$$|\psi\rangle = \frac{1}{\sqrt{2}} \iint \sum \delta_{\lambda\sigma} f(\mathbf{p}) f(\mathbf{q}) |\mathbf{p}\lambda\rangle |\mathbf{q}\sigma\rangle d\tilde{\mathbf{p}} d\tilde{\mathbf{q}} \qquad f(\mathbf{p}) = \frac{1}{N(\sigma)} e^{-\left(\frac{\theta'}{\sigma}\right)^2} e^{i\phi} \delta(p - p_0)$$

- Log Negativity as a function of rapidity, shown for various beam spreads ($\theta = 2\pi / 5$, $p_0 = 1.0$)

Summary

- Spin behaves differently in relativistic QM
- Spin and momentum degrees of freedom can exchange entanglement when boosted