LUNAR TRANSFER TRAJECTORY DESIGN AND THE FOUR BODY PROBLEM

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The existence of a ballistic trajectory from the Earth to orbit about the Moon was long considered to be impossible based on analysis of the three-body problem. In 1990 a ballistic trajectory from the Earth to lunar orbit was discovered while analyzing a plan to salvage the Muses A (Hiten) spacecraft. This trajectory utilized the Sun's gravity in conjunction with the Earth's and Moon's gravity and was thus the first example of a practical four-body trajectory design. This paper presents a review of lunar transfer trajectories that go beyond three-body theory and the Jacobi integral. These include Hiten, Lunar A and the Genesis return trajectory from the vicinity of the Moon to Earth. It is shown that these trajectories may be analyzed by piecing together segments where three-body motion dominates.

INTRODUCTION

An early investigation of flight to the Moon by V. A. Egorov in 1958 identified several problems relating to the design and navigation of trans-lunar trajectories. These included hitting the Moon, circumnavigation of the Moon with a return to Earth at a flat entry angle, using the Moon's gravity for assist in reaching the planets and the possibility of the Moon capturing a projectile launched from the Earth. Based on consideration of the three-body problem and its associated Jacoby integral, solutions can be demonstrated for these problems with the exception of the Moon capturing a projectile launched from Earth. For the problem of lunar capture, Egorov concluded that the Moon could not possibly capture a projectile launched from the Earth on the first circuit of the trajectory no matter what initial conditions are specified. This conclusion was based on analysis of the three-body problem and did not consider the Sun's gravity.

The first example of a ballistic trajectory of a spacecraft launched from the Earth into orbit about the Moon was discovered in 1990 while analyzing a plan to salvage the Muses A (Hiten) spacecraft in a highly eccentric orbit about the Earth. The key to the discovery was the utilization of the Sun's gravity to affect the transfer to a lunar capture orbit. The result was a numerical solution to the restricted four-body problem of the Earth, Moon, Sun and a point mass spacecraft. This paper presents a review of lunar transfer trajectories that require analysis that goes beyond that provided by three-body theory and the Jacobi integral. Examples are Hiten, Lunar A, and the Genesis return trajectory from the vicinity of the Moon to Earth. These trajectory designs cannot be fully explained or analyzed using three-body theory and the Jacobi integral. As is the case for the three-body problem, a complete analytic solution of the four-body problem has not been obtained. Furthermore, an integral relationship similar to the Jacobi integral has not been found for the four-body problem and the prospects for finding such an integral are dim. Current theories, such as Weak Stability Theory, are explanatory and not predictive and thus cannot be used for design of trajectories that require a simultaneous four-body solution without some intervention by the trajectory designer.

In the absence of a predictive four-body theory, the trajectory designer may use the existing solution of the two-body problem and the Jacobi integral to piece together trajectory segments and

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achieve the desired result. Indeed, most lunar transfer trajectory designs are obtained by patching together conic orbits where the Earth's gravity dominates to conic orbits where the Moon's gravity dominates. By extension, the trajectory segment dominated by the Earth, Moon and spacecraft Jacobi integral may be pieced together with the trajectory segment dominated by the Sun, Earth and spacecraft Jacobi integral to obtain continuous ballistic trajectories that connect Earth departure or arrival with capture orbits about the Moon and the nearby Lagrange points.

MOON CAPTURE OF PROJECTILE LAUNCHED FROM EARTH

A spacecraft in a lunar capture orbit will approach the Moon in a nearly circular orbit about the Earth that is just inside the Moon's orbit or just outside the Moon's orbit. As the spacecraft approaches the Moon, the Moon's gravity provides the necessary acceleration to slow down or speed up the spacecraft depending on whether the approach orbit is inside or outside the Moon's orbit. If the spacecraft has just the right approach velocity, it is drawn into orbit about the Moon. The orbital mechanics of capture orbits are well documented in the literature. In Ref. 1 a procedure is alluded to for generating capture orbits. A spacecraft is placed in an orbit that is loosely bound to the Moon and whose semi-major axis is just inside the Moon's sphere of influence as illustrated on Figure 12.3 of Reference 1. The lunar periapsis is directed toward the Earth and apoapsis is therefore directed away from the Earth. The orbit is integrated for several revolutions about the Moon and if it remains captured the apoapsis altitude is raised slightly. A convenient orbit parameter for raising apoapsis is the eccentricity which will tend to keep the energy of the orbit about the Moon constant. After several tries, the spacecraft will escape from the Moon and enter into an orbit about the Earth. Since the equations of motion are reversible, a capture trajectory can be obtained by repeating the above procedure only integrating the equations of motion backward.

The resulting capture orbits are generally nearly circular about the Earth and either inside or outside the Moon's orbit. For a critical value of the starting eccentricity of the orbit about the Moon, the spacecraft will just escape the Earth-Moon system and go into orbit about the Sun. Raising the eccentricity slightly will result in an eccentric orbit with a periapsis radius relative to the Earth that is inside the Moon's orbit. The results of generating several capture orbits are shown on Figure 1. For a range of starting eccentricities from 0.94151 to 0.943, most of the capture orbits either escape from the Earth-Moon system or fall into an uninteresting eccentric orbit about the Earth with periapsis radius less than that of the Moon's orbit. This behavior of capture orbits including the reduction in periapsis radius with respect to the Earth has been observed and is common knowledge.

A remarkable result was discovered during study of the Hiten trajectory during Memorial day weekend of 1990. If the starting eccentricity of the Moon's orbit was adjusted to 0.94171, the spacecraft falls into a highly eccentric orbit that returns to Earth as shown on Figure 1. This was a surprising result since previous studies of the possibility of the Moon capturing a projectile launched from the Earth indicated that this result was very improbable. Previous analysis by Fesenkov based on the Jacobi integral concluded that this result was impossible. Egorov introduced a term not considered by Fesenkov that opened the possibility of capture after more than one circuit. However, he acknowledged that the Sun may provide a perturbation that would probably forbid capture.

The discovery of a capture orbit was not made by systematically perturbing eccentricity, as suggested by Fig. 1, until the result was observed. The approach used in finding this orbit was from a different direction. In attempting to design a trajectory for the Hiten spacecraft to get to the Moon, a bielliptic transfer was attempted. The idea was to design a capture orbit that escapes from the Earth-Moon system and intersects a direct trajectory from Earth orbit. The capture trajectory was integrated backward and the trajectory from Earth orbit was integrated forward. At the intersection, a maneuver was performed to join the two trajectory segments. It was soon discovered that an escape trajectory would not work. The velocity correction required at the intersection was too big. It was also observed that the minimum velocity at the intersection point near the boundary of escape was about 250 m/s. While fine tuning the eccentricity of the capture orbit, it was observed that the velocity change began to drop precipitously. It became apparent that the minimum velocity change was zero and the result was an orbit similar to the orbit shown on Figure 1 which was obtained from Ref. 4.
Attempts to extend the result shown on Figure 1 to other initial orbit conditions revealed a strong dependence on the location of the Sun relative to the Earth and Moon. Clearly, the tidal acceleration of the Sun was the vehicle for transforming a nearly circular orbit coincident with the Moons orbit into a highly eccentric orbit that intersects the Earth. The affect of the Sun on the transfer trajectory can be seen from inspection of Figure 2. Shown is an Earth to Moon ballistic transfer trajectory with the orbit of the Sun in Earth centered inertial coordinates superimposed. The Sun's orbit has been reduced by a scale factor of 100. As the backward integrated trajectory spirals outward from the Moon, the Sun is on the opposite side of the Earth from the spacecraft most of the time. The spacecraft is in the second quadrant near apoapsis while the Sun is in the fourth quadrant. The net effect of the solar tide is to reduce the angular momentum sufficient to lower periapsis radius to the radius of the Earth. Reversing the direction of integration gives the desired lunar capture trajectory.

The lunar transfer trajectory from the Earth's surface to capture by the Moon may be modified slightly to enable transfer from a variety of Earth orbits to lunar capture. Also, Figure 1 suggests that capture orbits may be designed to escape from the Earth to the Sun-Earth Lagrange points.

Figure 1 Examples of Lunar Capture Orbits
With a little imagination, these capture orbits may be pieced together with the Earth transfer trajectory to design orbits that go from near Earth orbit to the Lagrange points briefly capturing the Moon along the way. An example of a lunar capture transfer trajectory with modified initial conditions near Earth orbit is shown on Figure 3. Figure 3 was obtained from Ref. 5 and was the initial trajectory design for the Japanese Lunar A mission. The spacecraft is launched into an elliptic staging orbit about the Earth with apoapsis radius that reaches the orbit of the Moon. The spacecraft remains in the staging orbit until the Sun is in the right position for a lunar capture orbit. The spacecraft is timed to arrive at the Moon for a gravity assists that places the spacecraft on the capture orbit. Figure 3 displays the characteristic kidney shape often associated with the 3 month variety of the capture orbit. A wide variety of examples of lunar capture orbits are given in Ref. 6.

**ANGULAR MOMENTUM MANAGEMENT**

An important tool for design of lunar capture orbits is the management of angular momentum. The raising of the periapsis radius of the Earth centered orbit from near the surface of the Earth to the radius of the Moon's orbit requires the addition of angular momentum to the orbit. This requires placing the spacecraft in a region of space where the angular momentum rate of increase from the solar tide can raise the angular momentum to that required for capture. The energy and angular momentum management is accomplished by starting from a lunar capture orbit with the correct angular momentum and energy (approximately the same as the Moon) and integrating backward to a region of space where the angular momentum is reduced to a small enough value to intersect the Earth's surface. The spacecraft then falls back to the Earth and the energy required is supplied by the launch vehicle when the direction of integration is reversed.

The angular momentum of the orbit relative to the Earth is given by

$$h = \sqrt{p G M_e}$$  \hspace{1cm} (1)
where $p$ is the parameter of orbit and $GM_e$ is the Earth's gravitational constant. For a spacecraft launched from the Earth that nearly escapes the Earth-Sun system, the orbit is nearly parabolic and $p$ is approximately twice the radius of the Earth (12,000 km). At lunar capture, $p$ is approximately the radius of the Moon's orbit (384,000 km). Equation 1 requires raising the angular momentum ($h$) from 69,000 km$^2$/s to 390,000 km$^2$/s, a net increase of 321,000 km$^2$/s.

The angular momentum orbit parameter ($h$) is the magnitude of the angular momentum vector given by,

$$h = r \times v \quad (2)$$

Consider an Earth centered rotating coordinate system with the $x$ axis pointing at the Sun and the $z$ axis in the direction of the orbit angular momentum vector. The geometry is shown on Fig. 4. Neglecting the rotation about the Sun and the tidal acceleration of the Moon, the rate of change of angular momentum is given by

$$\dot{h} = \dot{r} \times v + r \times \dot{v} \quad (3)$$

where

$$r = (x, y, 0)$$
$$\dot{r} = v = (\dot{x}, \dot{y}, \dot{z})$$
$$\dot{v} = (a_x, 0, 0)$$

Carrying out the indicated substitutions, the angular momentum rate is approximately

$$\dot{h} = -y \ a_x \quad (4)$$
The tidal acceleration \( a_x \) is approximately in the \( x \) direction since the Sun is far from the Earth at the scale shown on Figure 4. The tidal acceleration of the Sun is simply the difference between the acceleration of the spacecraft and the acceleration of the Earth caused by the Sun's gravity or

\[
a_x = \frac{GM_s}{(r_s + x)^2} - \frac{GM_s}{r_s^2}
\]

which may be approximated by

\[
a_x = a_x(x = 0) + \frac{da_x}{dx} \delta x
\]

\[
a_x = \frac{2GM_s}{r_s^3} x
\]

Substituting into the equation for angular momentum rate (Eq. 4) yields

\[
\dot{h} = \frac{2GM_s}{r_s^3} x y
\]

Eq. 8 is the equation for a hyperbola as a function of \( x \) and \( y \). A family of hyperbolas are plotted on Fig. 4 for various values of the angular momentum rate in the units of km\(^2\)/s\(^2\). As an example of the application of the angular momentum contours shown on Fig. 4, consider a spacecraft launched from Earth into the second or fourth quadrant of Figure 4 where the angular momentum rate attributable to the solar tide is positive. At coordinate \( x = 1,400,000\) km and \( y = -750,000\) km the angular momentum rate of increase is 0.084 km\(^2\)/s\(^2\). In order to raise the periapsis radius from the Earth surface to the radius of the Moon's orbit, an increase in angular momentum of 321,000 km\(^2\)/s.
is required. Thus, the spacecraft would need to dwell near the indicated coordinates for 3,821,000 seconds or about 44 days. The actual time required to achieve the required angular momentum increase can be obtained by performing a line integral along the actual flight path and include the tidal acceleration of the Moon. For an actual trajectory integration, the average value of the angular momentum rate would be about half the value used in this example and the Moon's tidal acceleration contribution would be small. The total flight time is therefore approximately 90 days.

**GENESIS EARTH RETURN TRAJECTORY**

The Genesis return trajectory starts from a Lagrange point and flies by the orbit of the Moon on a trajectory that is nearly captured and proceeds on a transfer orbit to the Earth. The portion of the orbit from near the Moon's orbit to the Earth is an example of a four body transfer that owes its inheritance to the earlier work of Ref. 4 and is the subject of this paper. The Genesis return trajectory is plotted on Figure 4 along with the ballistic capture trajectory from Ref. 4. The coordinate frame is the same as shown on Figure 3 with the Earth at the center and the Sun in the +z direction. Both trajectories go from the vicinity of the Moon's orbit to the Earth. In the rotating coordinate system, both trajectories execute a slow loop in the first quadrant where the maximum rate of angular momentum removal is about 0.1 km²/s² as indicated by the hyperbolic contours. The Genesis trajectory experiences a higher rate of angular momentum removal in the first quadrant which is partially restored in the first quadrant where the sign changes to positive. The total angular momentum removal is about the same for both trajectories which is characteristic of the four body transfer. For comparison, a transfer orbit with a greater rate of angular momentum removal in the first quadrant and a larger loop than the Genesis trajectory is shown on Fig. 22 of Ref. 6. The Genesis return trajectory is the average of the two trajectories given in Ref. 4 and Ref.6.

Since the trajectories shown on Figure 4 are initiated at different times, the position of the Moon in its orbit relative to the Genesis trajectory is not clear. The ballistic capture orbit originates at the Moon. The Genesis trajectory in the vicinity of the Moon's orbit comes under significant influence of the Moon's gravity. The boundary between domination by Earth-Sun gravity and domination by Earth-Moon gravity is a region of space that has been referred to as the Weak Stability Boundary, a term first coined in Ref. 4. Various theories have been put forward to explain the motion in this region with marginal success. However, no theory existed prior to the discovery of the ballistic capture trajectory⁴ that would completely predict this motion.

**Figure 4 Ballistic Capture and Angular Momentum Contours**
In order to gain some insight into the behavior of the trajectory dynamics near the Moon's orbit, a coordinate transformation is performed to a rotating coordinate system centered at the Moon with the \( +x \) axis in the direction of the Earth-Moon vector and the \( z \) axis in the direction of the angular momentum vector. The Genesis return trajectory and the ballistic capture orbit are plotted in this coordinate system as shown on Fig. 5. The departure from the vicinity of the Moon of the two trajectories are essentially the same. The motion near the Moon requires further investigation. The ballistic capture orbit enters into a close capture orbit of the Moon and the Genesis trajectory comes within 300,000 km of the Moon and executes a strange loop.

**JACOBI INTEGRAL AND CAPTURE**

When the spacecraft comes close to the Moon, the tidal perturbation from the Sun is small compared to the perturbations from the Moon and Earth. In this region of space, the trajectory may be analyzed using restricted three body theory. The geometry for a spacecraft or point mass perturbed by two massive bodies is shown on Figure 6. In the rotating primed coordinates, a certain integral relating to the energy of the point mass, referred to as the Jacobi Integral, is constant. The Jacobi integral in the rotating coordinate frame is given by

\[
\dot{x}'^2 + y'^2 + z'^2 = \omega^2 x'^2 + \omega^2 y'^2 + 2 \frac{GM_1}{r_1} + 2 \frac{GM_2}{r_2} - C
\]

Consider a point mass or spacecraft moving with zero velocity relative to the massive bodies. In inertial space, the rotation of both massive bodies about each other is \( \omega \). For zero velocity relative to the massive bodies, the Jacoby integral reduces to

\[
\omega^2 x'^2 + \omega^2 y'^2 + 2 \frac{GM_1}{r_1} + 2 \frac{GM_2}{r_2} = C
\]

A spacecraft moving with velocity or kinetic energy that is small compared to the gravitational potential energy will tend to move in a direction that keeps \( C \) constant. Thus, contours of constant
Figure 6 Restricted Two Body Geometry

$C$, will describe the motion in the rotating coordinate frame. Contours of constant $C$, referred to as Jacobi zero velocity contours or Hill’s surfaces, may be plotted in rotating coordinates as shown on Figure 7. The familiar zero velocity contours shown on Figure 7 are for two massive bodies that are of the same order of magnitude in mass. The five stable Lagrange points are labeled on Figure 7 as L1 through L5. A spacecraft placed at one of the stable Lagrange points will stay there unless perturbed by some external force. The zero velocity contours suggest other stable trajectories such as circular orbits about the massive bodies, a circular orbit about the center of mass and outside the orbits of the massive bodies and circumnavigation of one massive body and return to the other. It also appeared to Fesenkov and Egorov that a direct trajectory from one body to a close orbit about the other would not be possible because of the stricture near L1.

The zero velocity contours for the Earth-Moon system are highly distorted from the contours shown on Figure 7. Since the Earth is about 80 times more massive than the moon, the teardrop regions around L4 and L5 encircle the Earth and are joined through L3. A spacecraft in orbit near L4 or L5 can migrate back and forth between L4 and L5 through L3 without encountering the Moon. A trajectory of an asteroid in the Earth-Sun system has been recently discovered that exhibits this motion.

The actual zero velocity contours for the Earth-Moon system in the vicinity of the Moon are shown on Figure 8. Also plotted on Figure 8 are the Genesis return trajectory and the ballistic capture orbit of Ref. 4. As both orbits approach zero velocity relative to the Earth-Moon rotating frame, they fall onto the same Jacobi contour indicating that the orbits have essentially the same Jacobi constant or energy. The bifurcation that separates the two trajectories on departure from the Moon’s orbit is a property of chaotic trajectories.

CONCLUSION

A ballistic transfer orbit from Earth to orbit about the Moon was discovered in 1990 while analyzing a plan to salvage the Muses A spacecraft. This trajectory was the first practical example of a solution to the four-body problem of celestial mechanics. In the decade following this discovery, many missions have been proposed to take advantage of this discovery. Two examples are the Lunar A trajectory and the Genesis return trajectory. A careful examination of the Genesis return trajectory when compared to the Hiten trajectory has revealed an interesting bifurcation near the Moon’s orbit that is related to the Jacobi integral.

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Figure 7  Jacobi Zero Velocity Contours

Figure 8  Genesis and Ballistic Capture Orbits with Jacobi Contours
REFERENCES