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SERIES SOLUTION FOR MOTION IN AN ARBITRARY POTENTIAL FIELD

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Abstract

In this paper, the equations of motion for a test particle in an arbitrary potential field is solved as a formal power series. This solution is in terms of the time derivatives of the motion of the test particle. These derivative may be used to construct a Taylor series for the motion of the particle in the vicinity of given initial conditions.

1 Introduction

This paper applies a method previously presented at the August 2018 Astrodynamics Specialist Conference [1] to solve the equations of motion of a particle in an arbitrary potential field. This is found in terms of recurrence relations for the derivatives of the solution. These derivatives can then be used to generate a formal power series (i.e. a Taylor Series) representation of the solution similar to previous series solutions to the 3-body and n-body problems [2, 3, 4]. However, unlike those approaches, this new approach is easily generalized to other equations of motion. This paper's problem of motion in an arbitrary potential field, demonstrates this flexibility.

To employ this method, a new notation system is presented to simplify the manipulation of series expansions with nested summations, recursively-dependent coefficients, and other complexities. This notation system can then be used to develop relations that facilitate addition, subtraction, multiplication, division, exponentiation, differentiation, and integration of power series. This then allows direct manipulation of Taylor series expansions without resorting to either a computer algebra system or pages upon pages of nested sums and new variables for intermediate series expansions. Using these techniques, it is straightforward to solve a system of differential equations using the method of Frobenius (i.e. substituting a Taylor series for the solution into a differential equation and then solving for a recurrence relation for the coefficients of that Taylor series).

This section will present the notation system and the series algebra relations needed to solve for the motion of a test particle in a potential field. Derivation of these series algebra relations along with additional relations for algebraic manipulation of infinite power series may be found in the Appendix.

1.1 Notation System

As in tensor algebra, indices will be both subscripted and superscripted. I.e., the expression: x^i, i shall denote an index rather than a power, i.e. the ith element of x as opposed to x to the ith power. A power of x will be denoted by parentheses, e.g. $(x)^i$. This allows the use of the Einstein summation convention where indices repeated both as subscripts and superscripts imply summation over that index:

$$a_n(x)^n = \sum_{n=0}^{\infty} a_n(x)^n = a_0 + a_1 x + a_2(x)^2 + \dots \tag{1.1}$$

This summation convention allows complicated sums to be written much more compactly:

$$\begin{aligned} a_n^i b_i c^j d_j (x)^n &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{\infty} a_n^i b_i \right) \left(\sum_{j=0}^{\infty} c^j d_j \right) (x)^n \\ &= \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_n^i b_i c^j d_j (x)^n \end{aligned} \tag{1.2}$$

A repeated index is a placeholder for a sum (i.e. a "dummy index") and can be replaced without changing the meaning of a relation:

$$a_n^i b_i c^j d_j (x)^n = a_p^q b_q c^r d_r (x)^p \tag{1.3}$$

A convenient convention is to think of upstairs indices as rows of a matrix and downstairs as columns. This then allows easy transcription between this notation system and matrix notation, for example:

$$\mathbf{AB} = a_j^i b_k^j \tag{1.4}$$

$$\vec{x}^T \mathbf{A} \, \vec{x} = x_i a_j^i x^j \tag{1.5}$$

The Kronecker delta provides the Identity transformation, and is defined as:

$$\delta^{\nu_1 \nu_2 \ldots \nu_n}_{\rho_1 \rho_2 \ldots \rho_m} \equiv \begin{cases} 1 & \text{if all } \nu_n \text{ and } \rho_m \text{ equal} \\ 0 & \text{otherwise} \end{cases} \tag{1.6}$$

Where there are n upstairs indices and m downstairs indices.

We will also use the Levi-Civita symbol, which for n indices is defined as:

$$\epsilon^{\rho_1 \ldots \rho_n} \equiv \begin{cases} 1 & \text{if indices are an even permutation} \\ -1 & \text{if indices are an odd permutation} \\ 0 & \text{if any index is repeated} \end{cases} \tag{1.7}$$

For three indices the even permutations are: (1,2,3), (2,3,1), (3,1,2), and the odd permutations are: (1,3,2), (2,1,3), (3,2,1). The Levi-Civita symbol then allows us to write the vector cross product for arbitrary-dimension vectors:

$$\vec{c} = \vec{a} \times \vec{b} = c^i = \epsilon^{ijk} a_j b_k \tag{1.8}$$

1.1.1 Sigma

The Σ symbol, defined as:

$$\Sigma^{\rho_1 \rho_2 \ldots \rho_m}_{\mu_1 \mu_2 \ldots \mu_n} \equiv 1 \tag{1.9}$$

can be used indicate sums when indices are not repeated as subscripts and superscripts. For example, the geometric series may be described as:

$$\Sigma_n (-x)^n = 1 - x + (x)^2 - (x)^3 + \ldots \tag{1.10}$$

Since Σ is always equal to one, multiplying or dividing Σ with itself yields the following identities:

$$\Sigma^n \Sigma^n = \Sigma^n \tag{1.11}$$

$$\frac{\Sigma^n}{\Sigma^n} = \Sigma^n = \frac{1}{\Sigma^n} \tag{1.12}$$

When multiplying or dividing Σ coefficients with different indices, the indices may be combined:

$$\Sigma^m \Sigma^n \Sigma_p = \Sigma_p^{mn} \tag{1.13}$$

$$\frac{\Sigma_m}{\Sigma^n} = \Sigma_m^n = \frac{1}{\Sigma_m^n} \tag{1.14}$$

1.1.2 Range of Indices

When not specified, it will be assumed that indices can have any non-negative integer value and that sums run from zero to infinity. When other ranges are needed, we will say indices take on values from the set $\mathscr{I}$ (i.e. $i, j, k, etc. \in \mathscr{I}$) and define the set $\mathscr{I}$ used in a given case. To denote that indices could only have non-negative integer values, we may use: $\mathscr{I} \subseteq \mathbb{N}_0$. When needed, $\mathscr{I}_n$ can be used to denote values allowed by lowercase Latin indices, $\mathscr{I}_N$ for uppercase Latin indices, and $\mathscr{I}_\alpha$ for Greek indices. This allows us to specify limited ranges for special indices when we want to use some indices for infinite sums and others for finite ranges, for example: $\mathscr{I}_n = \mathbb{N}_0, \mathscr{I}_\alpha = \{0, 1, \ldots, 6\}$.

Sometimes it will be necessary to have some sums in an expression that start at different initial indices. For entities that have only one superscript or subscript, we maydenote a special starting index with a prepended superscript notation:

$$^{j}a_n \implies n \geq j \tag{1.15}$$

That is, in (1.15), n can take on values from $\mathscr{I}$ such that $n \geq j$. This restriction will affect anything summed with $^{j}a_n$, but not other terms. For example, consider the following (with $\mathscr{I} = \mathbb{N}_0$):

$$c_n = b^n a_n + b_n\, {}^{3}a_n \tag{1.16}$$

In (1.16), the first term is summed from 0 to ∞ and the second is summed from 3 to ∞. To denote a lower and an upper bound on an index, we may use two semicolon-separated prepended superscripts, like this:

$$^{j;k}a_n \implies j \leq n \leq k \tag{1.17}$$

And to put limits on a particular index when an entity has multiple indices, we may use a Σ symbol. For example, the following:

$$c_n = {}^{3;6}\Sigma^m a_{mn} \tag{1.18}$$

would sum a_{mn} over the first index from 3 to 6.

A principal utility of this notation is that it allows us to partition infinite sums when using the summation convention:

$$\begin{aligned} a_n(x)^n &= a_0 + {}^{1}a_n(x)^n \\ &= a_0 + a_1(x) + {}^{2}a_n(x)^n \\ &= a_0 + a_1(x) + a_2(x)^2 + {}^{3}a_n(x)^n \\ &= a_0 + a_1(x) + {}^{2;99}a_n(x)^n + {}^{100}a_n(x)^n \end{aligned} \tag{1.19}$$

This notation is also useful as a shorthand to specify a values for range of coefficients when defining a series. For example, the coefficients of an infinite series could be specified by the following:

$$a_0 = 2 \qquad a_1 = 0 \qquad {}^{2}a_n = 1/n \tag{1.20}$$

1.2 The Bracket Power Function & The Series Derivative

This section will introduce a modified power function, the *bracket power*, defined below, instead of the standard power function, $(x)^n$. This notation allows us to work more directly with the derivatives in Taylor series expansions and helps to reduce visual noise in complicated sums.

Definition 1.1. Let $x, \tilde{x} \in \mathbb{C}$ and $n \in \mathscr{I} \subseteq \mathbb{N}_0$, then the *bracket power* function is defined as:

$$[x]_n \equiv [x]^n \equiv \frac{1}{n!}(x - \tilde{x})^n \tag{1.21}$$

Where $\tilde{x}$ is the bracket power's *center point*. Alternatively, the center point can be explicitly specified as a second argument to the bracket power:

$$[x, \tilde{x}_i]_n \equiv [x, \tilde{x}_i]^n \equiv \frac{1}{n!}(x - \tilde{x}_i)^n \tag{1.22}$$

When $\tilde{x}$ is not specified it is assumed to be the same for all bracket powers and other entities using the center point in a given expression.

$$f^n(\tilde{x})[x]_n = f^n(\tilde{x})[x,\tilde{x}]_n = \Sigma_n f^n(\tilde{x})\frac{1}{n!}(x-\tilde{x})^n \tag{1.23}$$

Usually there is no need to explicitly specify $\tilde{x}$, even when it is an argument to functions other than the bracket power.

$$f^n[x]_n + g^m[x]_m = f^n(\tilde{x})[x,\tilde{x}]_n + g^m(\tilde{x})[x,\tilde{x}]_m \tag{1.24}$$

Relations derived with $[x]^n$ or $[x]_n$ will hold for any allowable value of $\tilde{x}$. The goal in treating the center point as arbitrary is to allow series manipulations and the solution of a differential equation without choosing any particular point about which to expand the series until after the desired form of a series is found.

The definition of the bracket power in (1.21) was chosen because it provides the following property when differentiated:

$$\begin{aligned}\frac{d}{dx}[x]_n &= \frac{n}{n!}(x-\tilde{x})^{n-1}\\ &= \frac{1}{(n-1)!}(x-\tilde{x})^{n-1}\\ &= [x]_{(n-1)}\end{aligned} \tag{1.25}$$

From (1.25), we see that the s-th order derivative of the bracket power is given by simply:

$$\frac{d^s}{dx^s}[x]_n = [x]_{(n-s)} \tag{1.26}$$

However, the price for this simple differentiation rule is that the multiplication of bracket powers of x is less straightforward than just adding exponents with the regular power function because of the $1/n!$ terms:

$$[x]_m[x]_n = \frac{1}{m!n!}(x-\tilde{x})^{(m+n)} = \frac{(m+n)!}{m!n!}[x]_{(m+n)} = \binom{m+n}{n}[x]_{(m+n)} \tag{1.27}$$

Similarly, the division of two power brackets of x is given by:

$$\frac{[x]_n}{[x]_m} = \frac{m!}{n!}\frac{(x-\tilde{x})^n}{(x-\tilde{x})^m} = \frac{m!(n-m)!}{n!}[x]_{(n-m)} = \frac{1}{\binom{n}{m}}[x]_{(n-m)} \tag{1.28}$$

The factorial terms must also be dealt with when raising to a power:

$$([x]_m)^n = \left(\frac{1}{m!}(x-\tilde{x})^m\right)^n = \frac{(mn)!}{(m!)^n}\frac{1}{(mn)!}(x-\tilde{x})^{(mn)} = \frac{(mn)!}{(m!)^n}[x]_{(mn)} \tag{1.29}$$

1.2.1 Differentiation and Integration of Series Using Bracket Powers

In the following sections I will derive relations for the manipulation of infinite series of bracket powers without regard for convergence. That is, I will treat them as formal power series.

Proposition 1.2. Consider the infinite series $a^n[x]_n$ with index $n \in \mathscr{I} \subseteq \mathbb{N}_0$ and each $a^n, x \in \mathbb{C}$. If the a^n are not functions of x, then the sth-derivative of the series with respect to x is given by:

$$\frac{d^s}{dx^s}a^n[x]_n = a^{(n+s)}[x]_n \tag{1.30}$$

Proof. $\frac{d^s}{dx^s}a_n[x]^n = \frac{d^s}{dx^s}\left({}^{0;k}a_n[x]^n + {}^{k+1}a_n[x]^n\right) = \frac{d^s}{dx^s}{}^{0;k}a_n[x]^n + \frac{d^s}{dx^s}{}^{k+1}a_n[x]^n$. The sth-derivative of the finite sum: ${}^{0;k}a_n[x]^n$ may be taken term by term and then summed. Since the a_n are not functions of x, by (1.25), the derivative of each term is ${}^{0;k}a_n[x]^{n-s}$ (here there is no summation and $n \geq s$). This may then be summed over n to give the sth-derivative of the sum, ${}^{0;k}a_n[x]^n$, as the sum, ${}^{s;k}a_n[x]^{n-s}$. We may then make the index substitution $m = n-s$ and write ${}^{s;k}a_n[x]^{n-s} ={}^{s;k} a_{(m+s)}[x]^m$. Since $(m+s) \geq s$ for all non-negative m, and since m is a dummy index and we are free to change its symbol, so: ${}^{s;k}a_{(m+s)}[x]^m ={}^{0;k} \Sigma_n a_{(n+s)}[x]^n$. Finally, since $\frac{d^s}{dx^s}a_n[x]^n ={}^{0;k} \Sigma_n a_{(n+s)}[x]^n + \frac{d^s}{dx^s}{}^{k+1}a_n[x]^n$ for all k, by induction then: $\frac{d^s}{dx^s}a_n[x]^n = a_{n+s}[x]^n$. ☐

Proposition 1.3. Consider the infinite series $a^n[x]_n$ with index $n \in \mathscr{I} \subseteq \mathbb{N}_0$ and each $a^n, x \in \mathbb{C}$. If the a^n are not functions of x, then the sth indefinite integral over x of the series is given by:

$$\int \dots \int a^n[x]_n (dx)^s = {}^{0,s-1}A^n[x]_n + a^n[x]_{(n+s)} \tag{1.31}$$

where the ${}^{0,s-1}A_n$ are constants of integration.

Proof. By Proposition 1.2: $\frac{d^s}{dx^s} a_{n-s}[x]^n = a_n[x]^n$. Therefore, by the Fundamental Theorem of Calculus, $\int \dots \int a_n[x]^n (dx)^s = a_{n-s}[x]^n$. □

In Proposition 1.3 we run into a subtle consequence of the allowed index ranges. In (1.31) the index n is restricted to $\mathscr{I} \subseteq \mathbb{N}_0$. If we subtracted s from the subscript for the integral, e.g. $a^{(n-s)}$, then we imply that the index could take on negative values outside of $\mathscr{I}$. To avoid this, Proposition 1.3 puts the $n+s$ on the bracket power and partitions the sum with the ${}^{0,s-1}A^n$ coefficients for the constants of the integration.

1.2.2 Series Derivative

If we construct a Taylor series such that the point about which the series is expanded is a free parameter, then we can make a univariate function from C^∞ into a bivariate function that describes all possible Taylor series expansions of that function. We can also use bracket powers to simply the Taylor series expansion so that the f^n coefficients are explicitly the nth-derivatives of $f(x)$ evaluated at $\tilde{x}$. I will call this Taylor series with the more explicit use of derivatives as series coefficients and the hidden center point, the *series derivative*:

Definition 1.4. Let $\mathscr{I} \subseteq \mathbb{N}_0$, $x, \tilde{x} \in X \subseteq \mathbb{C}$, and $f(x) \in C^\infty(X)$ (i.e., $f(x)$ is infinitely differentiable throughout X), then the *series derivative* of $f(x)$ is:

$$f^n(\tilde{x})[x]_n = f^n[x]_n \tag{1.32}$$

where:

$$f^n(\tilde{x}) = f^n = \frac{d^n}{dx^n} f(x)\Big|_{x=\tilde{x}} \tag{1.33}$$

We may choose to explicitly show $\tilde{x}$ in a series derivative as $f^n(\tilde{x})[x]_n$. Or we may omit it and use $f^n[x]_n$ with the convention that all center points in a given expression are consistent. We can keep $\tilde{x}$ hidden away until we need it. The $\tilde{x}$ doesn't affect the derivatives or algebraic operations we perform on a series derivative.

Proposition 1.5. Consider the series derivative $f^n(\tilde{x})[x]_n$ with index $n \in \mathscr{I} \subseteq \mathbb{N}_0$. Then the sth-derivative of the series respect to $\tilde{x}$ is:

$$\frac{d^s}{d\tilde{x}^s} f^n(\tilde{x})[x]_n = 0 \tag{1.34}$$

Proof. Since $f^n(\tilde{x}) = \frac{d^n}{dx^n} f(x)\big|_{x=\tilde{x}}$ then $\frac{d}{d\tilde{x}} f^n(\tilde{x}) = f^{(n+1)}$. But $\frac{d}{d\tilde{x}}[x]_n = -[x]_{n-1}$ and $\frac{d}{d\tilde{x}} f^n(\tilde{x})[x]_n = f^{(n+1)}(\tilde{x})[x]_n - {}^1f^n(\tilde{x})[x]_{n-1} = f^{(n+1)}(\tilde{x})[x]_n - f^{(n+1)}(\tilde{x})[x]_n = 0$. Since $\frac{d}{d\tilde{x}} f^n(\tilde{x})[x]_n = 0$, all higher derivatives are also zero. □

If the starting function is analytic everywhere with a radius of convergence large enough to cover the whole domain of the starting function, then the bivariate function from the Taylor series is always equivalent to the univariate function for any expansion center point, $\tilde{x}$. But what if the radius of convergence is smaller? What if the function is somewhere non-analytic and the radius of convergence is zero? Well, even then there is an $\tilde{x}$ for any x that lets us get $f(x)$ from the series derivative:

Theorem 1.6. $\exists(\tilde{x}, x) \in X : \frac{d^s}{dx^s} f^n(\tilde{x})[x]_n = \frac{d^s}{dx^s} f(x), \forall s \in \mathbb{N}_0$

Proof. If $\tilde{x} = x$, then by Prop. 1.2, $\frac{d^s}{dx^s} f^n(\tilde{x})[x]_n\big|_{x=\tilde{x}} = f^s = \frac{d^s}{dx^s} f(x)$ □

For any x there is at least one $\tilde{x}$ that returns values for $f(x)$ and all of its derivatives from $f^n(\tilde{x})[x]_n$. This means we can manipulate a functions series derivative like a formal power series, without regard to convergence, and at the very least, we will have relations valid for the function's derivatives at $\tilde{x} = x$ for any x in the function's domain.

1.3 Multiplication and Exponentiation of Series Derivatives

Two power series may be multiplied using the Cauchy Product:

Proposition 1.7. Let $\mathscr{I} \subseteq \mathbb{N}_0$, then two differential series may be multiplied using the binomial coefficient, $\binom{p}{m} = p!/(n!(n-m)!)$, with this relation:

$$(f^m[x]_m)(g^n[x]_n) = \Sigma_m \binom{p}{m} f^m g^{(p-m)}[x]_p \tag{1.35}$$

Proof. When we multiply $(f_m[x]^m)(g_n[x]^n)$, $[x]^m[x]^n = \binom{m+n}{m}[x]^{m+n}$ in each term of the sum. Therefore: $(f_m[x]^m)(g_n[x]^n) = \binom{m+n}{m} f_m g_n [x]^{m+n}$. We may then make the change of indices $p = m+n$ to arrive at (1.35). □

In the appendix, I generalize this product using the β-coefficient:

$$\beta^n_{pq} = \begin{cases} \binom{n}{p} & \text{if } p+q=n \\ 0 & \text{if } p+q \neq n \end{cases} \tag{1.36}$$

Then:

$$\beta^n_{pq} f^p g^q = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \begin{cases} \binom{n}{p} f^p g^q & \text{if } p+q=n \\ 0 & \text{if } p+q \neq n \end{cases} \tag{1.37}$$

and since $p+q=n$ for nonzero terms, we may substitute $q = n-p$:

$$\beta^n_{pq} f^p g^q = \sum_{p=0}^{\infty} \binom{n}{p} f^p g^{(n-p)} = \Sigma_p \binom{n}{p} f^p g^{(n-p)} \tag{1.38}$$

This then enables an alternate form of the Cauchy product:

$$(f^p[x]_p)(g^q[x]_q) = \beta^n_{pq} f^p g^q [x]_n \tag{1.39}$$

To raise a series to a power, I will use this *dot power* notation that groups all of the products from the exponentiation of a series into coefficients of like powers:

Definition 1.8. Provided $\mathscr{I} \subseteq \mathbb{N}_0$ and $x, z \in \mathbb{C}$, then:

$$(f^n[x]_n)^z = f^{n\bullet(z)}[x]_n \tag{1.40}$$

or

$$(f_n[x]^n)^z = f_n^{\bullet(z)}[x]^n \tag{1.41}$$

where $f^{n\bullet(z)}$ and $f_n^{\bullet(z)}$ are the *dot powers* of f^n and f_n respectively.

in the appendix, I develop several ways to find the dot product coefficients. The most powerful is using Newton's generalized binomial expansion. This lets us raise a series to any complex number power, z:

$$a_n^{\bullet(z)} = \Sigma_s \binom{z}{s} (a_0)^{(z-s)}\ {}^1a_n^{\bullet(s)} \tag{1.42}$$

For the above $\mathscr{I} \subseteq \mathbb{N}_0$, $a_0 \neq 0$, and the a_n,$z \in \mathbb{C}$.

1.4 Bracket Powers of Series Derivatives

To solve the problem posed in this paper, we will need to substitute series derivatives of univariate functions such as $x(t)$ into series derivatives of multivariate functions such as $\Phi(x, y, z, t)$. To do this we need to define a bracket power of a series:

Definition 1.9. Provided $\mathscr{I} \subseteq \mathbb{N}_0$, $x \in \mathbb{C}$, and $m \in \mathbb{N}_0$ then:

$$[f^n[x]_n]^m = f^{n\bullet[m]}[x]_n \tag{1.43}$$

or

$$[f_n[x]^n]^m = f_n^{\bullet[m]}[x]^n \tag{1.44}$$

or

$$[f_n[x]^n]_m = f^n_{\bullet[m]}[x]^n \tag{1.45}$$

or

$$[f_n[x]^n]_m = f_{n\bullet[m]}[x]^n \tag{1.46}$$

where $\tilde{f} = f(\tilde{x})$, $f^{n\bullet[m]} = f^n_{\bullet[m]}$, and $f_n^{\bullet[m]} = f_{n\bullet[m]}$. These $f^{n\bullet[m]}$, etc. are the *bracket dot powers* of f^n and f_n.

From the Definition of the bracket power, Definition 1.1:

$$f_n^{\bullet[m]}[t]^n = \frac{1}{m!}\left(f_n[t]^n - f_n[\tilde{t}]\right)^m = \frac{1}{m!}\,{}^1f_n^{\bullet(m)} \tag{1.47}$$

similarly:

$$f^n_{\bullet[m]}[t]^n = \frac{1}{m!}\,{}^1f^{n\bullet(m)} \tag{1.48}$$

Given a multivariate function, $\Phi(x, y, z, t)$, with the Taylor series:

$$\Phi(x, y, z, t) = \Phi_{pqrs}[x]^p[y]^q[z]^r[t]^s \tag{1.49}$$

Using the bracket dot power, we may re-write this in terms of the $x_a[t]^a, y_b[t]^b, z_c[t]^c$ series as:

$$\Phi_{pqrs}[x]^p[y]^q[z]^r[t]^s = \Phi_{pqrs}x_a^{\bullet[p]}[t]^a[y]_b^{\bullet[q]}[t]^b[z]_c^{\bullet[r]}[t]^c[t]^s = \beta_n^{abcs}\Phi_{pqrs}x_a^{\bullet[p]}[y]_b^{\bullet[q]}[z]_c^{\bullet[r]}[t]^n \tag{1.50}$$

2 Motion in a Potential Field

Now that we have developed the tools, we may apply them to the motion of a particle in a potential field.

Consider a a potential field, $\Phi(x, y, z, t)$, such that it induces an acceleration on a test particle in the x, y, z directions of:

$$\ddot{x} = \frac{\partial}{\partial x}\Phi(x, y, z, t) \tag{2.1}$$

$$\ddot{y} = \frac{\partial}{\partial y}\Phi(x, y, z, t) \tag{2.2}$$

$$\ddot{z} = \frac{\partial}{\partial z}\Phi(x, y, z, t) \tag{2.3}$$

This acceleration potential is equivalent to a force potential divided by the mass of the test particle. For gravitational motion, this specific force relation is convenient as the force is also potential to the test particle mass. If the x, y, z were in an inertial frame or there were no other forces, the above would be sufficient for the equations of motion. But if we were to add additional accelerations from either additional forces or motion of the reference frame, the equations of motion would be:

$$\ddot{x} = \frac{\partial}{\partial x}\Phi(x, y, z, t) + u(x, y, z, t) \tag{2.4}$$

$$\ddot{y} = \frac{\partial}{\partial y}\Phi(x, y, z, t) + v(x, y, z, t) \tag{2.5}$$

$$\ddot{z} = \frac{\partial}{\partial z}\Phi(x, y, z, t) + w(x, y, z, t) \tag{2.6}$$

The partials of $\Phi(x, y, z)$ may be expressed as:

$$\Phi_{ijkl} = \left(\frac{\partial}{\partial x}\right)^i \left(\frac{\partial}{\partial y}\right)^j \left(\frac{\partial}{\partial z}\right)^k \left(\frac{\partial}{\partial t}\right)^l \Phi(x, y, z, t) \tag{2.7}$$

Series algebra methods may be used to find the total nth-order derivative of $\Phi(x, y, z)$ along some trajectory $x(t), y(t), z(t)$ as:

$$\left(\frac{d}{dt}\right)^n \Phi(x(t), y(t), z(t), t) = \beta_n^{abcd} \Phi_{ijkd} x_a^{\bullet[i]} y_b^{\bullet[j]} z_c^{\bullet[k]} \tag{2.8}$$

where:

$$x_a = \left(\frac{d}{dt}\right)^a x(t) \qquad y_b = \left(\frac{d}{dt}\right)^b y(t) \qquad z_c = \left(\frac{d}{dt}\right)^c z(t) \tag{2.9}$$

We may use a similar approach to take the partial derivatives of the additional forces/accelerations. If we do this and substitute in the series derivatives of $x(t), y(t), z(t)$ into the $[x]^i, [y]^j, [z]^k$ we arrive at a recurrence relation for the time derivatives of $x(t), y(t), z(t)$ from the equations of motion:

$$x_{(n+2)} = \beta_n^{abcd} \Phi_{(i+1)jkd} x_a^{\bullet[i]} y_b^{\bullet[j]} z_c^{\bullet[k]} + \beta_n^{abcd} u_{ijkd} x_a^{\bullet[i]} y_b^{\bullet[j]} z_c^{\bullet[k]} \tag{2.10}$$

$$y_{(n+2)} = \beta_n^{abcd} \Phi_{i(j+1)kd} x_a^{\bullet[i]} y_b^{\bullet[j]} z_c^{\bullet[k]} + \beta_n^{abcd} v_{ijkd} x_a^{\bullet[i]} y_b^{\bullet[j]} z_c^{\bullet[k]} \tag{2.11}$$

$$z_{(n+2)} = \beta_n^{abcd} \Phi_{ij(k+1)d} x_a^{\bullet[i]} y_b^{\bullet[j]} z_c^{\bullet[k]} + \beta_n^{abcd} w_{ijkd} x_a^{\bullet[i]} y_b^{\bullet[j]} z_c^{\bullet[k]} \tag{2.12}$$

If we drop the u, v, w accelerations, this then simplifies to:

$$x_{(n+2)} = \beta_n^{abcd} \Phi_{(i+1)jkd} x_a^{\bullet[i]} y_b^{\bullet[j]} z_c^{\bullet[k]} \tag{2.13}$$

$$y_{(n+2)} = \beta_n^{abcd} \Phi_{i(j+1)kd} x_a^{\bullet[i]} y_b^{\bullet[j]} z_c^{\bullet[k]} \tag{2.14}$$

$$z_{(n+2)} = \beta_n^{abcd} \Phi_{ij(k+1)d} x_a^{\bullet[i]} y_b^{\bullet[j]} z_c^{\bullet[k]} \tag{2.15}$$

For integer dot powers the indices of the coefficients in the products will sum to the power (see Proposition A.9 in the Appendix). This means that the $x_a^{\bullet[j]}$ will only need x_a coefficients where $a \leq i$. Similarly, the β_n^{abcd} require that $a \leq n$. Therefore in (2.13)-(2.15), the $x_{(n+2)}, y_{(n+2)}, z_{(n+2)}$ depend only on the n or lower terms of x_n, y_n, z_n. So in the above relations, given the initial x, y, z position and velocity we could find all of the derivative of the motion of a test particle in the potential field, provided we had the Φ_{ijk} partials.

2.1 Partial Derivatives of N-Body Potential Field

Consider the case in which the potential field is induced by N gravitating particles, i.e.:

$$\Phi(x, y, z, t) = \frac{\Sigma_i \mu^i}{\sqrt{(x(t) - x^i(t))^2 + (y(t) - y^i(t))^2 + (z(t) - z^i(t))^2}} \tag{2.16}$$

Where $i = 1 \ldots N$, μ^i is the gravitational parameter (i.e. GM) of each body, and the x^i, y^i, z^i are each body's coördinates.

If we were to directly take the partials of (2.23) to find the Φ_{ijkd}, we would quickly find ourselves overwhelmed with terms from successive applications of the chain rule to the square root in the denominator. However, the series algebra relations are equivalent to the chain rule and we may use them to simplify the process of taking these derivatives. To do this we introduce auxiliary functions for each gravitating body:

$$a^i(x, y, z, x^i, y^i, x^i) = (x - x^i)^2 + (y - y^i)^2 + (z - z^i)^2 \tag{2.17}$$

The partials of these auxiliary functions quickly terminate:

$$a^i_{100000} = 2(\tilde{x} - \tilde{x}^i) \quad a^i_{010000} = 2(\tilde{y} - \tilde{y}^i) \quad a^i_{001000} = 2(\tilde{y} - \tilde{y}^i) \tag{2.18}$$

$$a^i_{000100} = 2(\tilde{x}^i - \tilde{x}) \quad a^i_{000010} = 2(\tilde{y}^i - \tilde{y}) \quad a^i_{000001} = 2(\tilde{y}^i - \tilde{y}) \tag{2.19}$$

$$a^i_{100100} = -2 \qquad a^i_{010010} = -2 \qquad a^i_{001001} = -2 \tag{2.20}$$

$$a^i_{200000} = 2 \qquad a^i_{020000} = 2 \qquad a^i_{002000} = 2 \tag{2.21}$$

$$a^i_{000200} = 2 \qquad a^i_{000020} = 2 \qquad a^i_{000002} = 2 \tag{2.22}$$

and all other a^i_{bcdefg} are then zero.

We may then re-write the potential as:

$$\Phi(x,y,z,t) = \Sigma_i \mu^i \left(a^i(t)\right)^{(-1/2)} = \mu_i \left(a^i\right)^{(-1/2)} \tag{2.23}$$

There is a slight complexity now when we try to find the partials of $\Phi(x,y,z,t)$ for (2.13)-(2.15), they don't match the indices of the a^i_{bcdefg} partials that so conveniently terminate. The b,c,d indices provide the x,y,z partials needed, the the d,e,f partials are for x^i,y^i,z^i which are different for each of the a^i, and don't math the time partial derivative needed for $\Phi(x,y,z,t)$. So we need to collapse all of the x^i,y^i,z^i derivatives into time derivatives by using the series derivatives of the x^i,y^i,z^i. That is:

$$\Phi_{pqrs} = \mu_i a^i_{pqrefg}\, \beta^{lmn}_s x^{i\bullet[e]}_l y^{i\bullet[f]}_m z^{i\bullet[g]}_n \tag{2.24}$$

where:

$$x^i_l = \left(\frac{d}{dt}\right)^l x^i(t) \qquad y^i_m = \left(\frac{d}{dt}\right)^m y^i(t) \qquad z^i_n = \left(\frac{d}{dt}\right)^n z^i(t) \tag{2.25}$$

We may now use (2.13)-(2.15) to find all of the derivatives of the position, x,y,z, in terms of the derivatives of the positions of the gravitating bodes, x^i,y^i,z^i. If these x^i,y^i,z^i were functions, we could use series algebra techniques to build expansions of the derivatives. However, the most likely scenario is that these positions are found via a Chebyshev or spline fits to ephermerides of the gravitating bodies. In this case the x^i_s,y^i_s,z^i_s derivatives may be obtained directly from these polynomials, provided the time step does not exceed the time span of the ephemeris polynomial fit.

3 Conclusion

The relations provided in the paper may be used to find the time derivatives for the motion of a test particle induced by a potential field. These derivatives may then be used in a Taylor Series to compute the motion of the particle in the vicinity of some given initial conditions. This Taylor Series may then be used to calculate new initial conditions for an analytic continuation of the original Taylor Series to expand its region of convergence.

However, using these methods for numerical evaluation of the equations of motion may not offer much beyond what we can already do with numerical integration techniques. Rather, I think the main value of this approach is it gives a symbolic representation of the solution that can be manipulated to provide additional insights into the problem.

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A Additional Detail on Series Algebra Techniques

This section builds a unified framework that will facilitate the algebraic manipulation of series derivatives and to make it more convenient than previous approaches to the algebraic manipulation of Taylor series expansions. Although the application here is with series derivatives and bracket powers, all of these methods could be re-written for the manipulation of generic power series. In the following, relations for algebraic manipulation of entities with upstairs indices also hold for downstairs indices and vice versa. However, this isn't always explicitly shown to avoid unnecessary tedium. (I.e., I have tried to limit myself to just necessary tedium.)

A.1 Compound-Index Notation and Implied Multiplication

In the definitions for the Kronecker delta and the Levi-Civita symbol in (1.6) and (1.7), I used a subscripted indices with ellipses such as "$\rho_1 \dots \rho_n$" to indicate an arbitrary number of indices. This case happens often enough that a more convenient shorthand is needed. Compound indices inside of angle brackets will allow us to indicate a set of indices:

$$A^{\langle \rho_i \rangle} = A^{\rho_0 \rho_1 \cdots \rho_n} \tag{A.1}$$

In the above case, the subscript of the compound index is not assumed to be a member of the set $\mathscr{I}$ (i.e. a compound index's subscript does not have the same range as that index or other indices in an expression). Rather, the compound subscript is used to denote an arbitrary number of indices. If the compound subscript appears elsewhere in an expression, it will be assumed to run from 0 to the one less than the value of that index in other subscripts. E.g.: $a_i b^{<\rho_i>} = a_i b^{\rho_1 \cdots \rho_{(i-1)}}$.

Although having $\langle \rho_i \rangle$ run from ρ_0 to $\rho_{(i-1)}$ instead of ρ_i is counter-intuitive, this convention pays off in simplifying complex relations that arise in later sections. This is primarily because we will want expressions like $\langle a^{\rho_i} \rangle$ to denote that a has i indices, but we will also want to use the i index elsewhere in an expression where indices start at 0. The only way to be consistent with both desires is to run i in $\langle a^{\rho_i} \rangle$ from 0 to $i-1$.

This compound-index notation also allows us to denote repeated multiplications when angle brackets are used on factors instead of indices. For example:

$$A^{\langle \rho_i \rangle} \langle b_{\rho_i} \rangle = A^{\rho_0 \rho_1 \cdots \rho_{(i-1)}} b_{\rho_0} b_{\rho_1} \dots b_{\rho_{(i-1)}} \tag{A.2}$$

or:

$$A^{\langle \rho_i \rangle} \langle (b_{\rho_i})^i \rangle = A^{\rho_0 \rho_1 \cdots \rho_{(i-1)}} (b_{\rho_0})^0 (b_{\rho_1})^1 \dots (b_{\rho_{(i-1)}})^{(i-1)} \tag{A.3}$$

This angle bracket shorthand allows definition of operations and symbols for arbitrary dimension. For example, the matrix determinant may be written for an $n \times n$ square matrix as:

$$\det \left| a^i_j \right| = \epsilon^{\langle \rho_i \rangle} \langle a^i_{\rho_i} \rangle = \epsilon^{\rho_0 \rho_1 \cdots \rho_{(n-1)}} a^0_{\rho_0} a^1_{\rho_1} \dots a^{(n-1)}_{\rho_{(n-1)}} \tag{A.4}$$

where the range for each of the ρ_i indices goes from 0 to $n-1$. Notice that the i superscript of $a^i_{\rho_i}$ does not imply a summation with the i subscripts in the ρ_i indices. Instead, the index matching inside of angle brackets is used to describe the pattern in the implied multiplication.

We may also use a superscript outside of the angle brackets, analogous to a power, to denote when the same index is specific number of times:

$$A^{\langle \rho \rangle^s} = A^{\rho\rho\cdots\rho} \quad \text{where there are } s \text{ of the } \rho \text{ superscripts} \tag{A.5}$$

This outside superscript matches with the same index inside of the angle brackets. This allows for more complex indexing when needed, e.g.:

$$A^{\langle \rho_s \rangle^s} = A^{\langle \rho_0 \rangle^0 \langle \rho_1 \rangle^1 \langle \rho_2 \rangle^2 \cdots} = A^{\rho_1 \rho_2 \rho_2 \rho_3 \rho_3 \rho_3 \cdots} \tag{A.6}$$

In the case of nested angle brackets, the inner angle brackets are evaluated first:

$$\langle A^{\nu_j}_{\langle \rho_s \rangle} \rangle = \langle A^{\nu_j}_{\rho_0 \cdots \rho_{(s-1)}} \rangle = A^{\nu_0}_{\rho_0 \cdots \rho_{(s-1)}} A^{\nu_i}_{\rho_0 \cdots \rho_{(s-1)}} \dots A^{\nu_{(j-1)}}_{\rho_0 \cdots \rho_{(s-1)}} \tag{A.7}$$

A.2 Addition and Subtraction

Due to the distributive law of elementary algebra, series derivatives may be added and subtracted term by term by combining the coefficients of like terms:

$$f(x) + g(x) = f^m[x]_m + g^n[x]_n = (f^m + g^m)[x]_m \tag{A.8}$$

$$f(x) - g(x) = f^m[x]_m - g^n[x]_n = (f^m - g^m)[x]_m \tag{A.9}$$

A.3 Multiplication

Proposition A.1. Let $\mathscr{I} \subseteq \mathbb{N}_0$, then two differential series may be multiplied using the binomial coefficient, $\binom{p}{m} = p!/(n!(n-m)!)$, with this relation:

$$(f^m[x]_m)(g^n[x]_n) = \Sigma_m \binom{p}{m} f^m g^{(p-m)}[x]_p \tag{A.10}$$

Proof. When we multiply $(f_m[x]^m)(g_n[x]^n)$, $[x]^m[x]^n = \binom{m+n}{m}[x]^{m+n}$ in each term of the sum. Therefore: $(f_m[x]^m)(g_n[x]^n) = \binom{m+n}{m} f_m g_n [x]^{m+n}$. We may then make the change of indices $p = m + n$ to arrive at (A.10). ☐

The above is merely a restatement of the Cauchy product for power series in terms of bracket powers. But in this form, it also reminds us of the general Leibniz Rule for the pth order derivative of the product of $f(x)$ and $g(x)$. It is also evocative of the binomial theorem.

Let's explore this connection with the binomial series further. We start by introducing the β-coefficient as:

$$\beta^n_{pq} = \begin{cases} \binom{n}{p} & \text{if } p + q = n \\ 0 & \text{if } p + q \neq n \end{cases} \tag{A.11}$$

Then:

$$\beta^n_{pq} f^p g^q = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \begin{cases} \binom{n}{p} f^p g^q & \text{if } p + q = n \\ 0 & \text{if } p + q \neq n \end{cases} \tag{A.12}$$

and since $p + q = n$ for nonzero terms, we may substitute $q = n - p$:

$$\beta^n_{pq} f^p g^q = \sum_{p=0}^{\infty} \binom{n}{p} f^p g^{(n-p)} = \Sigma_p \binom{n}{p} f^p g^{(n-p)} \tag{A.13}$$

This then enables an alternate form of (A.10):

$$(f^p[x]_p)(g^q[x]_q) = \beta^n_{pq} f^p g^q [x]_n \tag{A.14}$$

If we instead define this β-coefficient for an arbitrary number of indices using the multinomial coefficient $\left(\frac{(\rho_0+\rho_1+\ldots+\rho_{(i-1)})!}{\rho_0!\rho_1!\ldots\rho_{(i-1)}!}\right)$, we can then use it to multiply an arbitrary number of differential series together.

Definition A.2. Let $\mathscr{I} \subseteq \mathbb{N}_0$, then the β-coefficient is defined as:

$$\beta_n^{\langle\rho_i\rangle} \equiv \beta^n_{\langle\rho_i\rangle} \equiv \begin{cases} \frac{n!}{\rho_0!\rho_1!\ldots\rho_{(i-1)}!} & \text{if } \Sigma^i \rho_i = n \\ 0 & \text{if } \Sigma^i \rho_i \neq n \end{cases} \tag{A.15}$$

Theorem A.3. *For $\mathscr{I} \subseteq \mathbb{N}_0$, the product of an arbitrary number of bracket series is given by:*

$$(f^0_{\rho_0}[x]^{\rho_0})(f^1_{\rho_1}[x]^{\rho_1})\ldots(f^{(i-1)}_{\rho_{(i-1)}}[x]^{\rho_{(i-1)}}) = \beta_n^{\langle\rho_i\rangle}\langle f^i_{\rho_i}\rangle[x]^n \tag{A.16}$$

or:

$$(f_0^{\rho_0}[x]_{\rho_0})(f_1^{\rho_1}[x]_{\rho_1})\ldots(f_{(i-1)}^{\rho_{(i-1)}}[x]_{\rho_{(i-1)}}) = \beta^n_{\langle\rho_i\rangle}\langle f_i^{\rho_i}\rangle[x]_n \tag{A.17}$$

Proof. First we prove (A.16). From the definition of the bracket power in (1.21), the product of i bracket powers is:

$$[x]^{\rho_0}[x]^{\rho_1}\ldots[x]^{\rho_{(i-1)}} = \frac{1}{\rho_0!\rho_1!\ldots\rho_{(i-1)}!}(x-\tilde{x})^{\rho_0+\rho_1+\ldots+\rho_{(i-1)}}$$
$$= \frac{(\rho_0+\rho_1+\ldots+\rho_{(i-1)})!}{\rho_0!\rho_1!\ldots\rho_{(i-1)}!}[x]^{\rho_0+\rho_1+\ldots+\rho_{(i-1)}} \tag{A.18}$$

where $\frac{(\rho_0+\rho_1!+\ldots+\rho_{(i-1)})!}{\rho_0!\rho_1!\ldots\rho_{(i-1)}!}$ is the multinomial coefficient for the ρ_i. If we wish to replace the $[x]^{\rho_0+\rho_1+\ldots+\rho_{(i-1)}}$ with $[x]^n$, we could do that unless we were also summing over the ρ_i. But this is exactly what we are doing with the $f^i_{\rho_i}$ when we multiply $(f^0_{\rho_0}[x]^{\rho_0})(f^1_{\rho_1}[x]^{\rho_1})\ldots(f^{(i-1)}_{\rho_{(i-1)}}[x]^{\rho_{(i-1)}})$. Because of this, when we write each $[x]^n$ term we must ensure that it is multiplied by all combinations of the $f^i_{\rho_i}$ such that the ρ_i sum to n and that each product of the $f^i_{\rho_i}$ have the proper multinomial coefficient. Summing with $\beta^{\langle\rho_i\rangle}_n$ as it is defined in Definition A.2 achieves this:

$$f^0_{\rho_0}[x]^{\rho_0}\ldots f^{(i-1)}_{\rho_{(i-1)}}[x]^{\rho_{(i-1)}} = f^0_{\rho_0}\ldots f^{(i-1)}_{\rho_{(i-1)}}\frac{(\rho_0+\ldots+\rho_{(i-1)})!}{\rho_0!\ldots\rho_{(i-1)}!}[x]^{\rho_0+\ldots+\rho_{(i-1)}}$$
$$= \begin{cases} f^0_{\rho_0}\ldots f^{(i-1)}_{\rho_{(i-1)}}\frac{n!}{\rho_0!\ldots\rho_{(i-1)}!}[x]^n & \text{if } \Sigma^i\rho_i = n \\ 0 & \text{if } \Sigma^i\rho_i \neq n \end{cases}$$
$$= \beta^{\langle\rho_i\rangle}_n\langle f^i_{\rho_i}\rangle[x]^n \tag{A.19}$$

The above proof also holds for (A.17) when repeated with superscripted and subscripted indices swapped. □

A.3.1 Properties of the β-coefficients

These β-coefficients have many interesting properties that will be useful will when deriving relations in the following sections.

First, the β-coefficient with two indices, β^n_m, reduces to the Kronecker delta.

Identity A.4. If $\mathscr{I} \subseteq \mathbb{N}_0$, then:

$$\beta^n_m = \delta^n_m \tag{A.20}$$

Proof. Per Definition A.2, if β^n_m is interpreted as $\beta^n_{\langle\rho_m\rangle}$:

$$\beta^n_m = \begin{cases} \frac{n!}{m!} & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{A.21}$$

Since $\frac{n!}{m!} = 1$ when $n = m$, this is equivalent to the Kronecker Delta, δ^n_m. The same also holds if the if β^n_m is interpreted as $\beta^{\langle\rho_n\rangle}_n$. □

The β-coefficient multiplication identities below for the $\beta^n_{\langle\rho_m\rangle}$ form also hold for the $\beta^{\langle\rho_m\rangle}_n$ form.

Identity A.5. Provided $\mathscr{I} \subseteq \mathbb{N}_0$, then:

$$\beta^{nj}_q\beta^{\langle\rho_m\rangle}_n\beta^{\langle\nu_i\rangle}_j = \beta^{\langle\rho_m\rangle\langle\nu_i\rangle}_q \tag{A.22}$$

Proof. The left side is only nonzero when all of the β-coefficients in the product are nonzero. This happens when $\Sigma^m\rho_m = n$, $\Sigma^i\nu_i = j$, and $n + j = q$. I.e., when $\Sigma^m\rho_m + \Sigma^i\nu_i = q$. In those cases the product of $\beta^{nj}_q\beta^{\langle\rho_m\rangle}_n\beta^{\langle\rho_m\rangle}_n\beta^{\langle\nu_i\rangle}_j$ is $\frac{q!}{n!j!}\frac{n!}{\rho_0!\rho_1!\ldots\rho_{(m-1)}!}\frac{j!}{\nu_1 0!\nu_1!\ldots\nu_{(i-1)}!}$, which, per Definition A.2, is equivalent to $\beta^{\langle\rho_m\rangle\langle\nu_i\rangle}_q$. □

Identity A.6. Provided $\mathscr{I} \subseteq \mathbb{N}_0$, then:

$$\beta^{j\langle\rho_m\rangle}_n\beta^{\langle\nu_i\rangle}_j = \beta^{\langle\rho_m\rangle\langle\nu_i\rangle}_n \tag{A.23}$$

Proof. By Definition A.2, $n = j + \Sigma^m\rho_m = \Sigma^i\nu_i + \Sigma^m\rho_m$ for the β-coefficients to be non-zero, and $\frac{n!}{j!\rho_0!\rho_1!\ldots\rho_{(m-1)}!}\frac{j!}{\nu_0!\nu_1!\ldots\nu_{(i-1)}!} = \frac{n!}{\rho_0!\rho_1!\ldots\rho_{(m-1)}!\nu_0!\nu_1!\ldots\nu_{(i-1)}!}$ □

Identity A.7. Provided $\mathscr{I} \subseteq \mathbb{N}_0$, then:

$$\beta_n^{\langle\rho_m\rangle}\langle\beta_{\rho_m}^{\langle\nu_i\rangle}\rangle = \beta_n^{\langle\nu_i\rangle^m} \tag{A.24}$$

Proof. Recall that $\beta_n^{\langle\rho_m\rangle}\langle\beta_{\rho_m}^{\langle\nu_i\rangle}\rangle$ denotes $\beta_n^{\rho_1\ldots\rho_{(m-1)}}(\beta_{\rho_0}^{\langle\nu_i\rangle}\beta_{\rho_1}^{\langle\nu_i\rangle}\ldots\beta_{\rho_{(m-1)}}^{\langle\nu_i\rangle})$. So $\Sigma^m\rho_m = n$ for $\beta_n^{\langle\rho_m\rangle} \neq 0$, and for each $\beta_{\rho_m}^{\langle\nu_i\rangle} \neq 0$ the $\Sigma^i\nu_i = \rho_m$. This means $(\Sigma^i\nu_i)^m = n$ is required for the expression in (A.24) to be nonzero. In that case it is equal to $\frac{n!}{\rho_0!\ldots\rho_{(m-1)}!}\frac{\rho_0!}{\nu_0!\ldots\nu_{(i-1)}!}\cdots\frac{\rho_{(m-1)}!}{\nu_1!\ldots\nu_{(i-1)}!} = \frac{n!}{(\nu_1!)^m\ldots(\nu_{(i-1)}!)^m} = \beta_n^{\langle\nu_i\rangle^m}$ □

A.4 Raising to a Positive Integer Power

We can use Theorem A.3 to develop relations for raising a differential series to a power. Let's start with a definition:

Definition A.8. Provided $\mathscr{I} \subseteq \mathbb{N}_0$ and $x, z \in \mathbb{C}$, then:

$$(f^n[x]_n)^z = f^{n\bullet(z)}[x]_n \tag{A.25}$$

or

$$(f_n[x]^n)^z = f_n^{\bullet(z)}[x]^n \tag{A.26}$$

where $f^{n\bullet(z)}$ and $f_n^{\bullet(z)}$ are the *dot powers* of f^n and f_n respectively.

For dot powers with exponents in $\mathbb{N}_0$, we can use β-coefficients with implied multiplication to write an expression for these dot power coefficients:

Proposition A.9. Provided $\mathscr{I} \subseteq \mathbb{N}_0$ and $m \in \mathbb{N}_0$, then:

$$f^{n\bullet(m)} = \beta^n_{\langle\rho_m\rangle}\langle f^{\rho_m}\rangle \tag{A.27}$$

and

$$f_n^{\bullet(m)} = \beta_n^{\langle\rho_m\rangle}\langle f_{\rho_m}\rangle \tag{A.28}$$

Proof. We may raise a series to a positive integer power by performing successive multiplications on the series using Theorem A.3:

$$(f^n[x]_n)^m = \beta^n_{\langle\rho_m\rangle}\langle f^{\rho_m}\rangle[x]_n = f^{n\bullet(m)} = f^n_{\bullet(m)} \tag{A.29}$$

and

$$(f_n[x]^n)^m = \beta_n^{\langle\rho_m\rangle}\langle f_{\rho_m}\rangle[x]^n = f_n^{\bullet(m)} = f_{n\bullet(m)} \tag{A.30}$$

□

In the following sections, for brevity, I will present relations for dot powers using only the $f_n^{\bullet(m)}$ form, but these also hold for other forms.

Now let's derive a few useful identities. As with scalar powers, we can add positive integer dot powers when multiplying:

Identity A.10. Provided $\mathscr{I} \subseteq \mathbb{N}_0$ and $s, t \in \mathbb{N}_0$, then:

$$\beta_q^{mn}a_m^{\bullet(s)}a_n^{\bullet(t)} = a_q^{\bullet(s+t)} \tag{A.31}$$

Proof. Employing Proposition A.9 with Identity A.5:

$$\begin{aligned}
\beta_q^{mn}a_m^{\bullet(s)}a_n^{\bullet(t)}[x]^q &= \beta_q^{mn}\beta_m^{\langle\rho_s\rangle}\langle a_{\rho_s}\rangle\beta_n^{\langle\nu_t\rangle}\langle a_{\nu_t}\rangle[x]^q \\
&= \beta_q^{\langle\rho_s\rangle\langle\nu_t\rangle}\langle a_{\rho_s}\rangle\langle a_{\nu_t}\rangle[x]^q \\
&= \beta_q^{\langle\gamma_{s+t}\rangle}\langle a_{\gamma_{s+t}}\rangle[x]^q \\
&= a_q^{\bullet(s+t)}[x]^q
\end{aligned} \tag{A.32}$$

By comparing like terms in $\beta_q^{mn}a_m^{\bullet(s)}a_n^{\bullet(t)}[x]^q = a_q^{\bullet(s+t)}[x]^q$, we arrive at (A.31). □

The dot powers of zero and one behave as expected:

Identity A.11. Provided $\mathscr{I} \subseteq \mathbb{N}_0$, then:

$$a_m^{\bullet(1)} = a_m \tag{A.33}$$

Proof. By Proposition A.9: $a_m^{\bullet(1)} = \beta_n^m a_n$. By Identity A.4: $\beta_n^m a_n = \delta_n^m a_n = a_m$. □

Identity A.12.

$$a_0^{\bullet(0)} = 1 \quad \text{and} \quad {}^1a_n^{\bullet(0)} = 0 \tag{A.34}$$

Proof. By Identities A.10 and A.11: $\beta_q^{mn} a_m^{\bullet(0)} a_n^{\bullet(1)} = a_q^{\bullet(0+1)} = a_q$. But by Identity A.11: $\beta_q^{mn} a_m^{\bullet(0)} a_n = a_q$. This can be true for all x iff $a_0^{\bullet(0)} = 1$ and ${}^1a_n^{\bullet(0)} = 0$. □

Remark A.13. Identities A.10 and A.11 provide a recursive formula for calculating dot-powers. Setting $t = p - 1$ and $s = 1$ in (A.31) yields the following relation for $p \in \mathbb{N}_0$ and $\mathscr{I} \subseteq \mathbb{N}_0$:

$$a_q^{\bullet(p)}[x]^q = \beta_q^{nm} a_n a_m^{\bullet(p-1)}[x]^q \tag{A.35}$$

A.4.1 Binomial Expansions of Dot-Powers

We may use the pre-pended superscript notation to break apart a sum into two parts, for example:

$$a_n[x]^n = a_0 + {}^1a_n[x]^n \tag{A.36}$$

We may then raise $a_n[x]^n$ to the positive power s using a binomial expansion:

$$\begin{aligned}
(a_n[x]^n)^s &= (a_0 + {}^1a_n[x]^n)^s \\
&= {}^{0,s}\Sigma_m \binom{s}{m} (a_0)^m ({}^1a_n[x]^n)^{s-m} \\
&= {}^{0,s}\Sigma_m \binom{s}{m} (a_0)^m \; {}^1a_n^{\bullet(s-m)}[x]^n \\
&= \beta_{ij}^s (a_0)^i \; {}^1a_n^{\bullet(j)}[x]^n
\end{aligned} \tag{A.37}$$

Now let's generalize this to ${}^ka_n^{\bullet(s)}$:

Lemma A.14. *Provided $\mathscr{I} \subseteq \mathbb{N}_0$ and $s \in \mathbb{N}_0$, then:*

$${}^ka_n^{\bullet(s)} = \beta_{ij}^s \beta_n^{q\langle r\rangle^i} \delta_k^r (a_k)^i \; {}^{(k+1)}a_q^{\bullet(j)} \tag{A.38}$$

Proof. Applying the Binomial Theorem as in (A.37):

$$\begin{aligned}
({}^ka_n[x]^n)^s &= (a_k[x]_k + {}^{(k+1)}a_q[x]^q)^s \\
&= {}^{0,s}\Sigma_i \binom{s}{i} (a_k[x]_k)^i ({}^{(k+1)}a_q[x]^q)^{s-i}
\end{aligned} \tag{A.39}$$

We then simplify with the formulas for raising bracket powers to a power, (1.29), and for multiplying bracket powers, (1.27):

$$\begin{aligned}
({}^ka_n[x]^n)^s &= {}^{0,s}\Sigma_i \frac{s!}{i!(s-i)!} \frac{(ki)!}{(k!)^i} (a_k)^i [x]_{(ki)} \; {}^{(k+1)}a_q^{\bullet(s-i)}[x]^q \\
&= {}^{0,s}\Sigma_i \frac{s!}{i!(s-i)!} \frac{(ki)!}{(k!)^i} \frac{(ki+q)!}{(ki)!q!} (a_k)^i \; {}^{(k+1)}a_q^{\bullet(s-i)} \Sigma^q [x]_{(ki+q)} \\
&= {}^{0,s}\Sigma_i \frac{s!}{i!(s-i)!} \frac{(ki+q)!}{(k!)^i q!} (a_k)^i \; {}^{(k+1)}a_q^{\bullet(s-i)} \Sigma^q [x]_{(ki+q)}
\end{aligned} \tag{A.40}$$

Next we replace indices with $n = ki + q$ and $j = s - i$ and introduce β-coefficients:

$$({}^ka_n[x]^n)^s = \Sigma^q \beta_{ij}^s \beta_{q\langle k\rangle^i}^n (a_k)^i \; {}^{(k+1)}a_q^{\bullet(j)}[x]_n \tag{A.41}$$

There are only sums in i, j, q and n above, not in k or s. In order to keep the n as a subscript to match the left hand side, while still avoiding a sum in k, we may use the Kronecker Delta, δ_k^r from (1.6):

$$({}^k a_n[x]^n)^s = \beta_{ij}^s \beta_n^{q\langle r\rangle^i} \delta_k^r (a_k)^i \ {}^{(k+1)}a_q^{\bullet(j)}[x]_n \tag{A.42}$$

□

Next we need the following identities that will allow us to simplify the ${}^k a_n^{\bullet(s)}$ type factors.

Identity A.15. Provided $\mathscr{I} \subseteq \mathbb{N}_0$ and $s \in \mathbb{N}_0$, then:

$${}^k a_n^{\bullet(s)} = 0 \quad \text{if } n < ks \tag{A.43}$$

Proof. The definition of a dot-power, from (A.27), is: $a_n^{\bullet(s)} = \beta_n^{\langle \rho_m \rangle} a_{\rho_s}$. Each $a_n^{\bullet(s)}$ term is comprised of a sum of β-products of the a_{ρ_s} coefficients such that s of the ρ_s indices sum to n. If the smallest allowed value for the ρ_s is k, then the smallest value for this sum is ks. Therefore we must have $n \geq ks$ for $a_n^{\bullet(s)}$ to be nonzero. □

Identity A.16. Provided $\mathscr{I} \subseteq \mathbb{N}_0$ and $s \in \mathbb{N}_0$, then:

$${}^k a_n^{\bullet(s)} = \delta_k^r \beta_n^{\langle r\rangle^s} (a_k)^s \quad \text{if } n = ks \tag{A.44}$$

Proof. Since there are no subscripts of ${}^k a_n$ less than k, the only way to sum s subscripts to ks is to have s factors of $a_k[x]_k$ multiplied together: $(a_k[x]_k)^s = \frac{(ks)!}{(k!)^s}(a_k)^s[x]_{(ks)} = {}^k a_{(ks)}^{\bullet(s)}[x]_{(ks)}$. The δ_k^r in (A.44) is used to prevent summation over k. □

A.4.2 Rational and Complex Powers

We can also use binomial expansions for rational and complex number powers using Newton's generalized binomial coefficient:

Definition A.17. Given $n \in \mathbb{N}_0$ and $z \in \mathbb{C}$, then *Newton's Binomial Coefficient* is given by:

$$\binom{z}{n} = \frac{(z)_n}{n!} \tag{A.45}$$

where $(z)_n$ is the descending Pochhammer symbol, representing:

$$(z)_n = z(z-1)(z-2)\ldots(z-n+1) \tag{A.46}$$

This then allows binomial expansions for any power $z \in \mathbb{C}$:

Theorem A.18. *Given $\mathscr{I} \subseteq \mathbb{N}_0$ and the a_n,$z \in \mathbb{C}$, if $a_0 \neq 0$ then:*

$$a_n^{\bullet(z)} = \Sigma_s \binom{z}{s} (a_0)^{(z-s)} \ {}^1 a_n^{\bullet(s)} \tag{A.47}$$

Proof. We start by writing $(a_n[x]^n)^z$ in binomial form:

$$(a_n[x]^n)^z = (a_0 + {}^1 a_n[x]^n)^z \tag{A.48}$$

Let $y(x) = {}^1 a_n[x]^n$. If we take the s-th derivative of the left hand side with respect to y, we get:

$$\frac{d^s}{dy^s}(a_0 + {}^1 a_n[x]^n)^z = (z)_s (a_0 + {}^1 a_n[x]^n)^{(z-s)} \tag{A.49}$$

The Taylor series expansion of $(a_n[x]^n)^z$ in y with $\tilde{y} = 0$ is:

$$\begin{aligned}(a_n[x]^n)^z &= \Sigma_s \frac{(z)_s}{s!}(a_0)^{(z-s)}(y)^s \\ &= \Sigma_s \binom{z}{s}(a_0)^{(z-s)}({}^1 a_n[x]^n)^s \\ &= \Sigma_s \binom{z}{s}(a_0)^{(z-s)} \ {}^1 a_n^{\bullet(s)}[x]^n \end{aligned} \tag{A.50}$$

where $\binom{z}{s}$ is Newton's Binomial Coefficient from Definition A.17. Note that we need $a_0 \neq 0$ when $z \notin \mathbb{N}_0$ because the exponent of $(a_0)^{(z-s)}$ in (A.47) can go negative if $z \notin \mathbb{N}_0$. □

In Theorem A.18, (A.47) is a finite sum in s because ${}^{1}a_n^{\bullet(s)} = 0$ when $s > n$. We can now take a series, $a_n[x]^n$ to any rational, real, or complex power when $a_0 \neq 0$. And even when $a_0 = 0$, we could shift the center point (provided a(x) is not identically 0), and then apply Theorem A.18 to the shifted series where $a_0 \neq 0$.

For cases where there are multiple possible values, the $(a_0)^{(z-s)}$ factor will determine which branch is generated by the series. I.e., if $(a_0)^{(z-s)}$ is multi-valued there will be a different series for each branch of $(a_0)^{(z-s)}$.