Fast and Reliable Approximations for Interplanetary Low-Thrust Transfers

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Overall Problem and Approach

- Optimize a transfer from a fixed initial heliocentric state & time to a final one
- Inverse square power available from solar arrays
- Inverse square gravity available from the sun
- Constant specific Impulse

Solution Phase	Initial guess for state	Initial guess for control	Accounts for phase	# of state variables	Thrust constraint	Partials wrt state
First Approximation	no	no	no	4	no	no
Suboptimal Refinement	yes	no	yes	4	no	no
Optimal Bounded Control	yes	yes	yes	7	yes	yes

Perform fast global searches to enable real-time feedback during concurrent design sessions

Axial elements

$$\left\{\hat{f} \quad \hat{g} \quad \hat{h}\right\} = \left\{\hat{x} \quad \hat{y} \quad \hat{z}\right\} \begin{bmatrix} \cos h_X & 0 & \sin h_X \\ \sin h_X \sin h_Y & \cos h_Y & -\cos h_X \sin h_Y \\ -\sin h_X \cos h_Y & \sin h_Y & \cos h_X \cos h_Y \end{bmatrix}$$

- Defined by $\hat{g} \propto \hat{h} \times \hat{x}$
- Cf. equinoctial defined by single rotation along $\hat{h} \times \hat{z}$
- Angular momentum orthogonal to $\hat{x} \rightarrow h_X = 0$
- Rotate inertial frame so that $\hat{x} \propto \hat{h}_0 \times \hat{h}_f$

$$da/dt = \frac{2a^2}{h}\left[\left(e_X \sin L - e_Y \cos L\right)a_r + \left(1 + e_X \cos L + e_Y \sin L\right)a_\theta\right]$$

$$de_X/dt = \frac{h}{\mu}\sin L\ a_r + \left[\frac{h}{\mu}\cos L + \frac{r}{h}\left(e_X + \cos L\right)\right]a_\theta + e_Y \frac{r}{h}\tan h_X \cos L\ a_h$$

$$de_Y/dt = -\frac{h}{\mu}\cos L\ a_r + \left[\frac{h}{\mu}\sin L + \frac{r}{h}\left(e_Y + \sin L\right)\right]a_\theta - e_X \frac{r}{h}\tan h_X \cos L\ a_h$$

$$dh_X/dt = \frac{r \sin L}{h} a_h$$

$$dh_Y/dt = \frac{r \cos L}{h \cos h_X} a_h$$

$$dL/dt = \frac{h}{r^2} - \frac{r}{h}\tan h_X \cos L\ a_h$$

Trade singular pole in equinoctial elements for singular axis to decouple variation of axial elements

First Approximation

- Time/phase free transfer
  - math assumes many revolutions
- Analytic cost to minimize quadratic control on a single orbit
- Then minimize square root of that cost per orbit over entire transfer path
  - Approximately min ΔV
- Semi-analytic expressions for how any single element optimally varies given a variation in the other elements
- Assume straight path through eccentricity space
- Defines optimal path of orbit orientation
- Defines optimal path of *a*

$$a(s) = a_0 \left[\sin k \big/ \sin\left(\psi^*(s) + k\right) \right]^{2/3}, \; k = -\tan^{-1}\left[\sin\psi_f^* \big/ \left(\cos\psi_f^* - a_0^3 \big/ a_f^3\right) \right]$$

Approximate dynamics

- Analytically integrate dynamics over a segment (10–20 per rev)
- Assume constant elements and control during segment
- No need for *fancy* integrals

$$\Delta\alpha = -\frac{2\left(e\cos\nu^{+} - e\cos\nu^{-}\right)}{\left(1-e^{2}\right)}u_r + \frac{2\left(\Delta\nu + e\sin\nu^{+} - e\sin\nu^{-}\right)}{\left(1-e^{2}\right)}u_\theta$$

$$\Delta e_x = -\Delta\cos L\ u_r + \left(\ \Delta\sin L + \frac{e_X}{e^2}I_1 - \frac{e_Y}{e^2}I_2\right)u_\theta$$

$$\Delta e_y = -\Delta\sin L\ u_r + \left(-\Delta\cos L + \frac{e_X}{e^2}I_2 + \frac{e_Y}{e^2}I_1\right)u_\theta$$

$$\Delta h_x = \left(\frac{e_X}{e^2}I_2 - \frac{e_Y}{e^2}I_3\right)u_h$$

$$\Delta h_y = \left(\frac{e_X}{e^2}I_3 + \frac{e_Y}{e^2}I_2\right)u_h$$

$$\Delta t = \sqrt{\frac{a^3}{\mu\left(1-e^2\right)}}\left[\Delta\chi - \left(1-e^2\right)\left(\frac{e\sin\nu^{+}}{1+e\cos\nu^{+}} - \frac{e\sin\nu^{-}}{1+e\cos\nu^{-}}\right)\right]$$

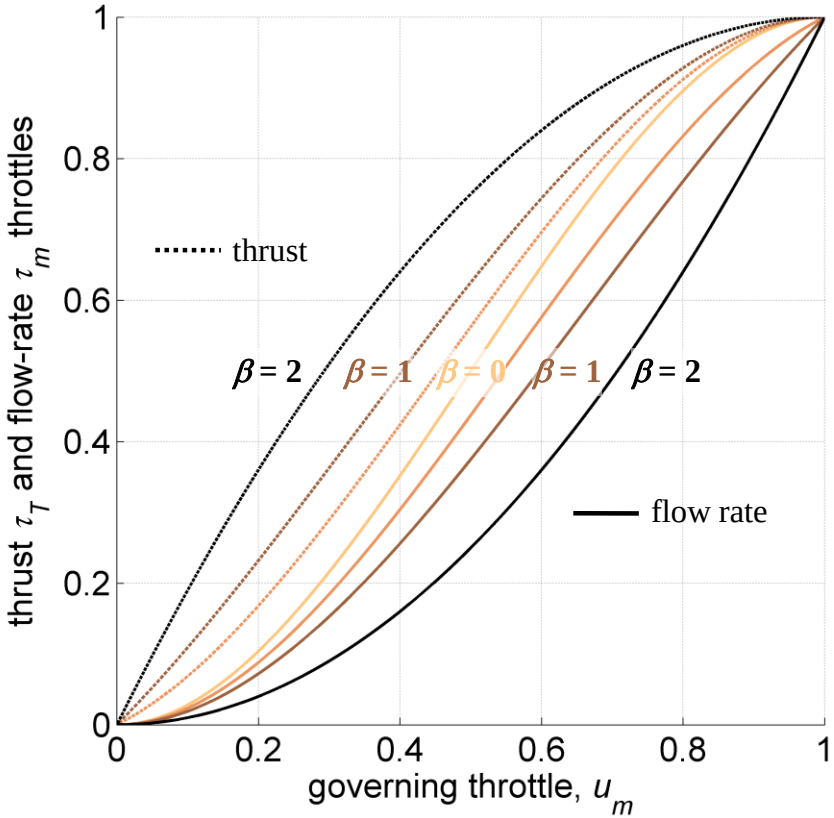
$$\Delta\chi = 2\sqrt{1-e^2}\ \tan^{-1}\left(\sqrt{1-e^2}\Big/\left[\left(1+e\cos\nu^{-}\right)\frac{\cos L^{+} + \cos L^{-}}{\sin L^{+} - \sin L^{-}} - e\sin\nu^{-}\right]\right)$$

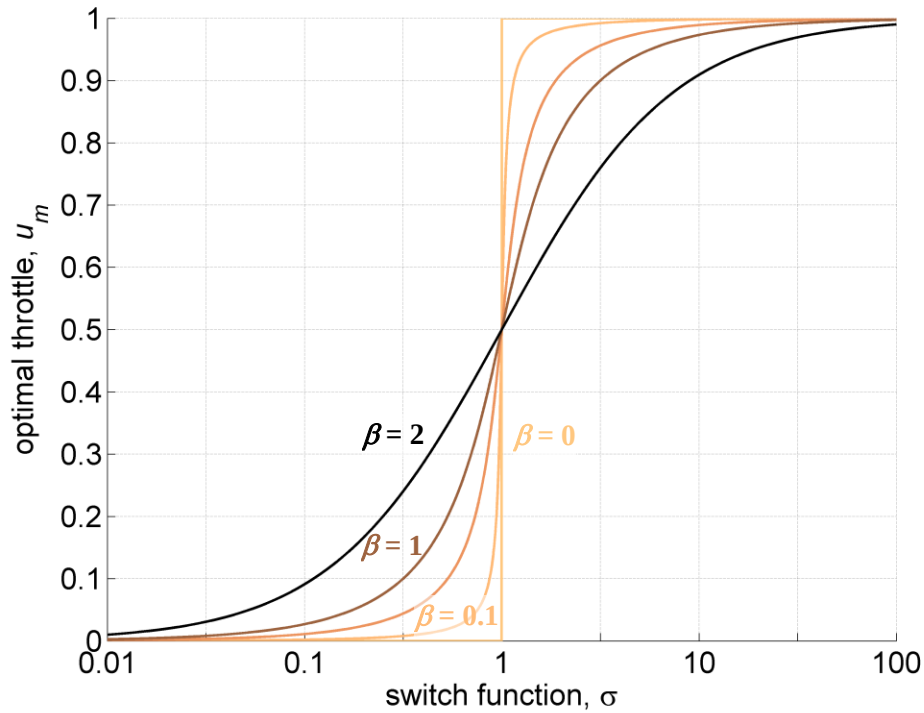
$$I_1 = \Delta\nu - \Delta\chi,\ I_2 = \log\left(1 - e\frac{\cos\nu^{+} - \cos\nu^{-}}{1+e\cos\nu^{+}}\right),\ I_3 = \frac{\Delta\chi}{\left(1-e^2\right)} - \Delta\nu$$

Suboptimal refinement

- Takes advantage of decoupled in-plane vs out-of-plane dynamics
- No need for initial guess of control
- Minimize quadratic cost
  - Min propellant with variable specific impulse
- Optimize control based on simplified suboptimal propagation of Lagrange multipliers
- Fast way to converge majority of transfers

Optimal Bounded Control

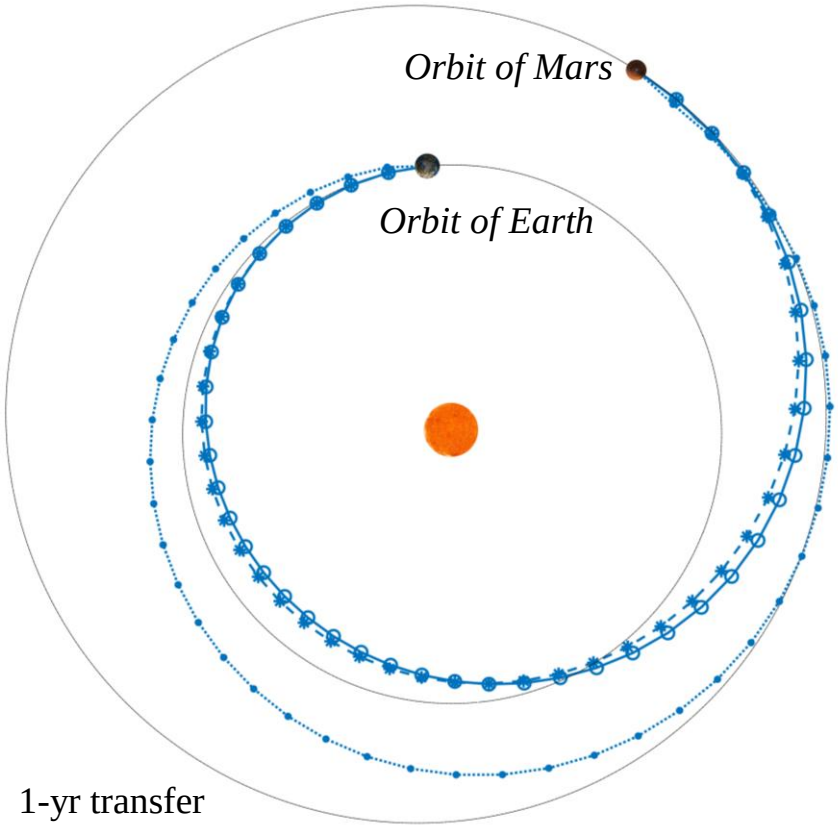




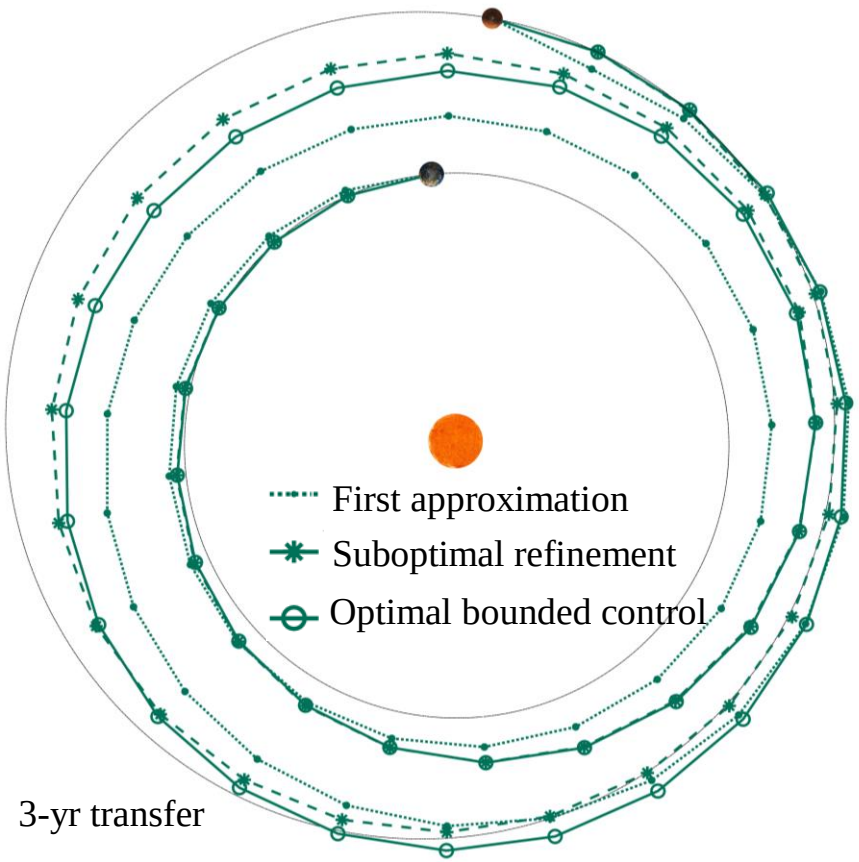
Optimal Bounded Control

- Initial guess of path, control, and Lagrange multipliers from suboptimal refinement
- Fix path and determine optimal control to target final state
- Propagate new path with fixed control
- More cases converge with decoupled state and co-state dynamics
  - Would take longer to drive to exact optimal conditions
- All the while the throttle remains optimally bounded

Example transfers to Mars

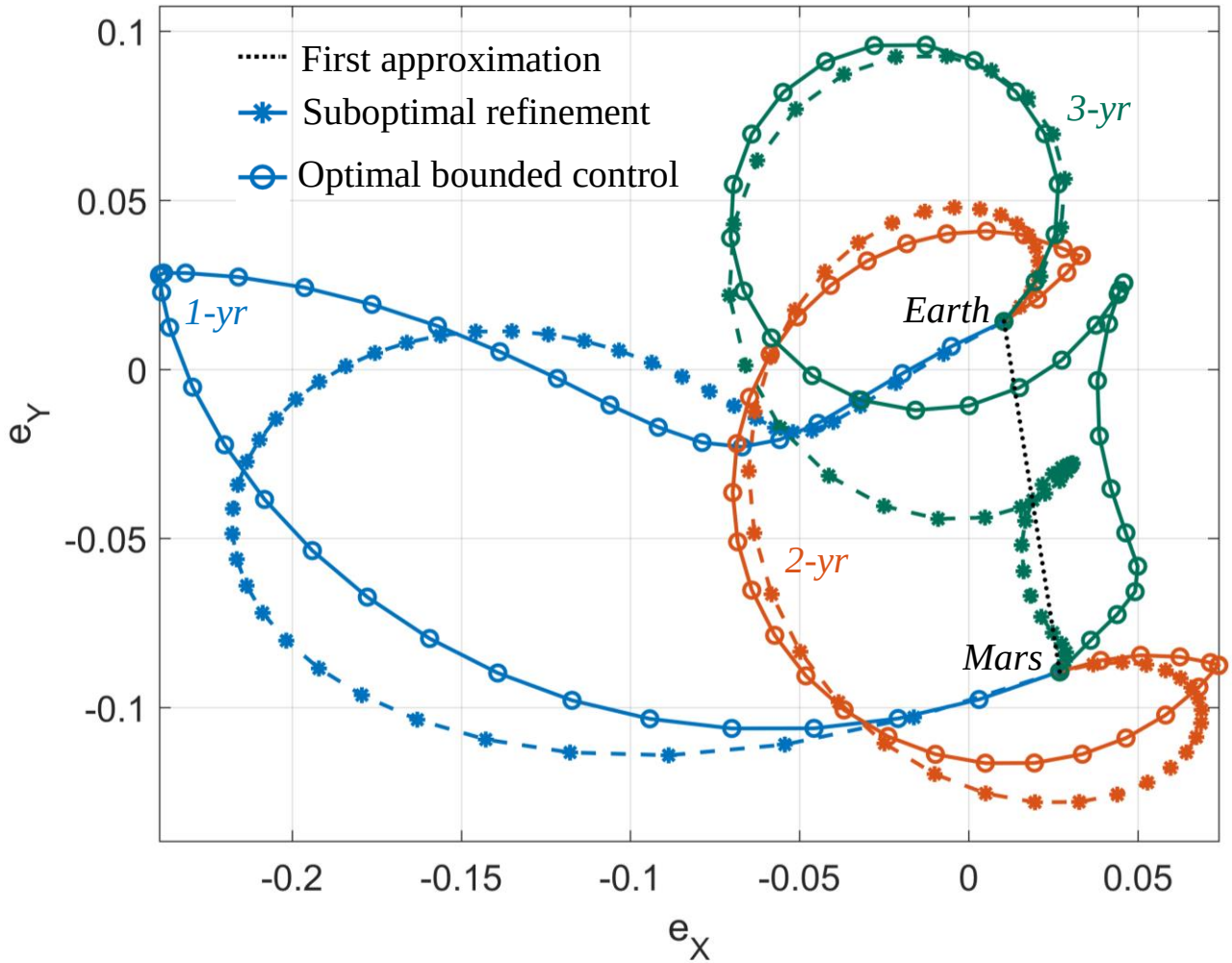


1-yr transfer

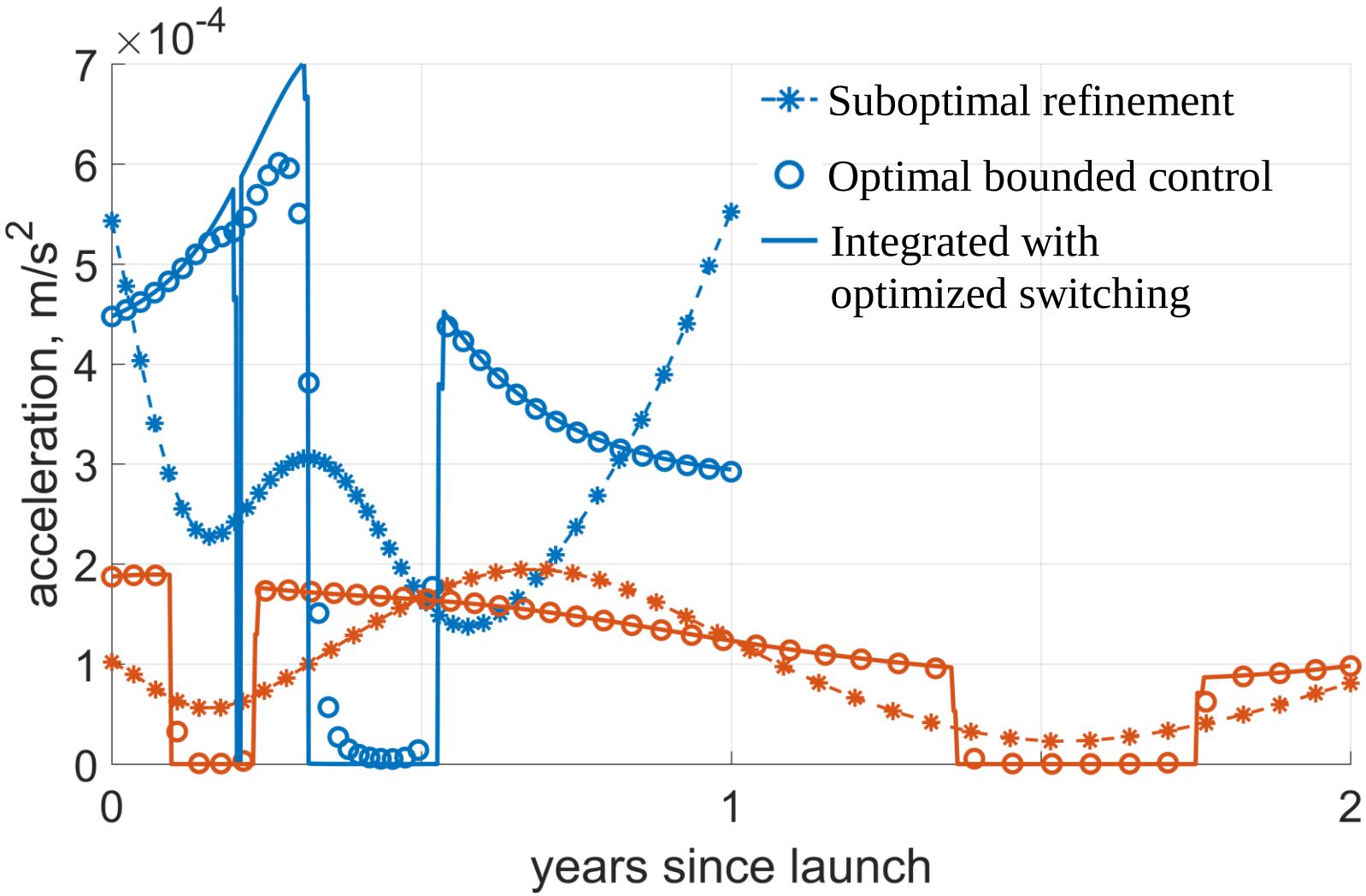


3-yr transfer

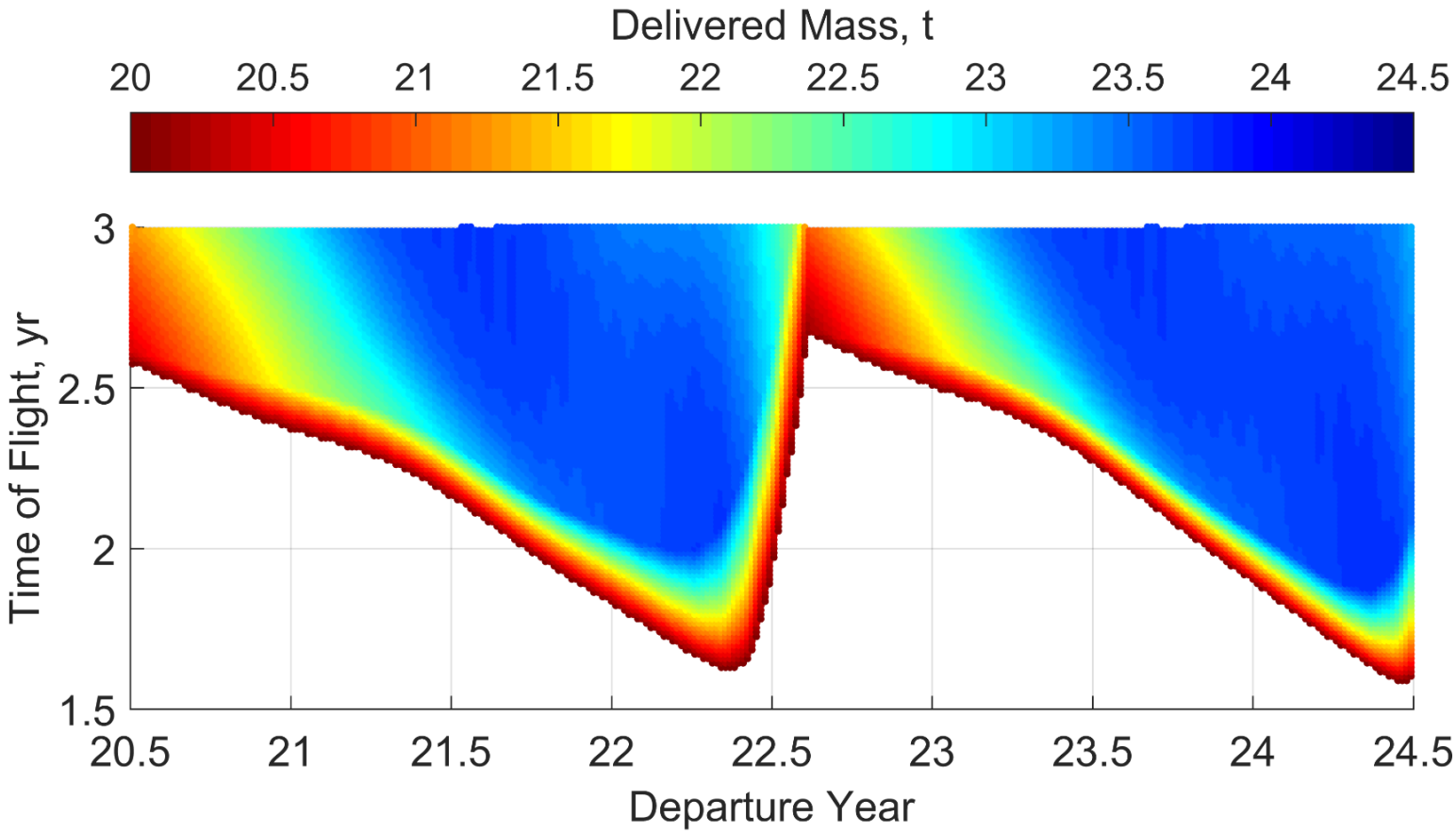
Evolution of eccentricity



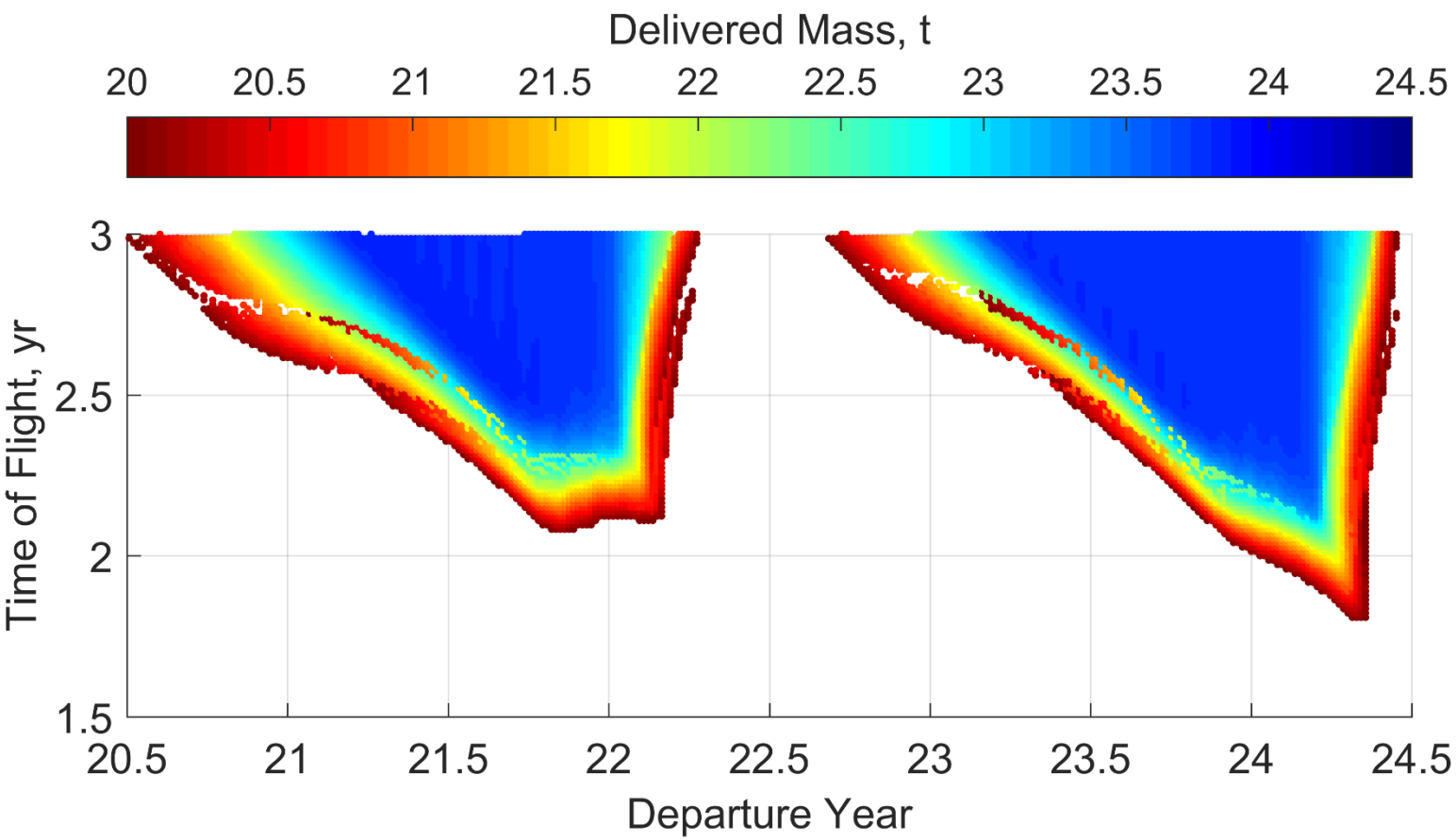
Adjust the thrust



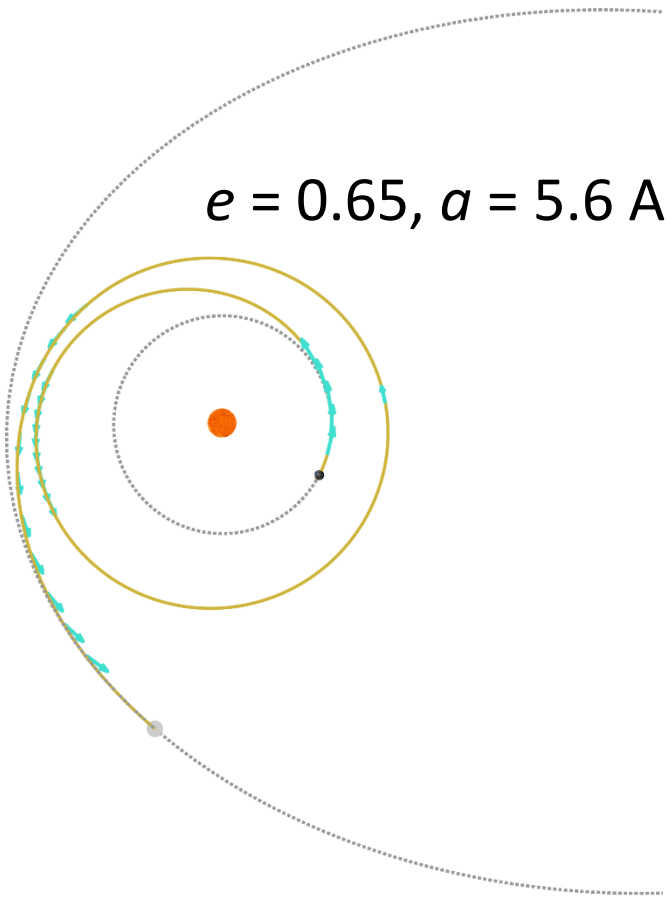
Mars: suboptimal refinement

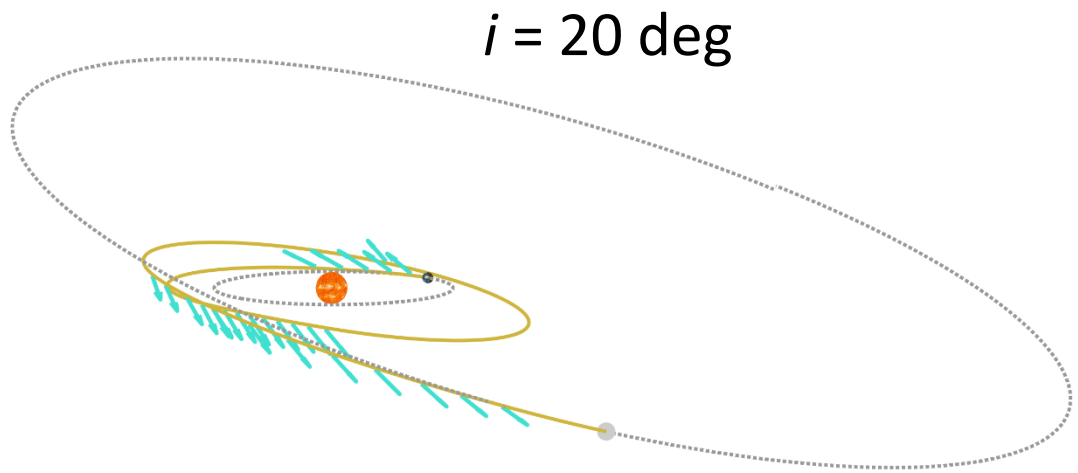


Mars: optimal bounded control

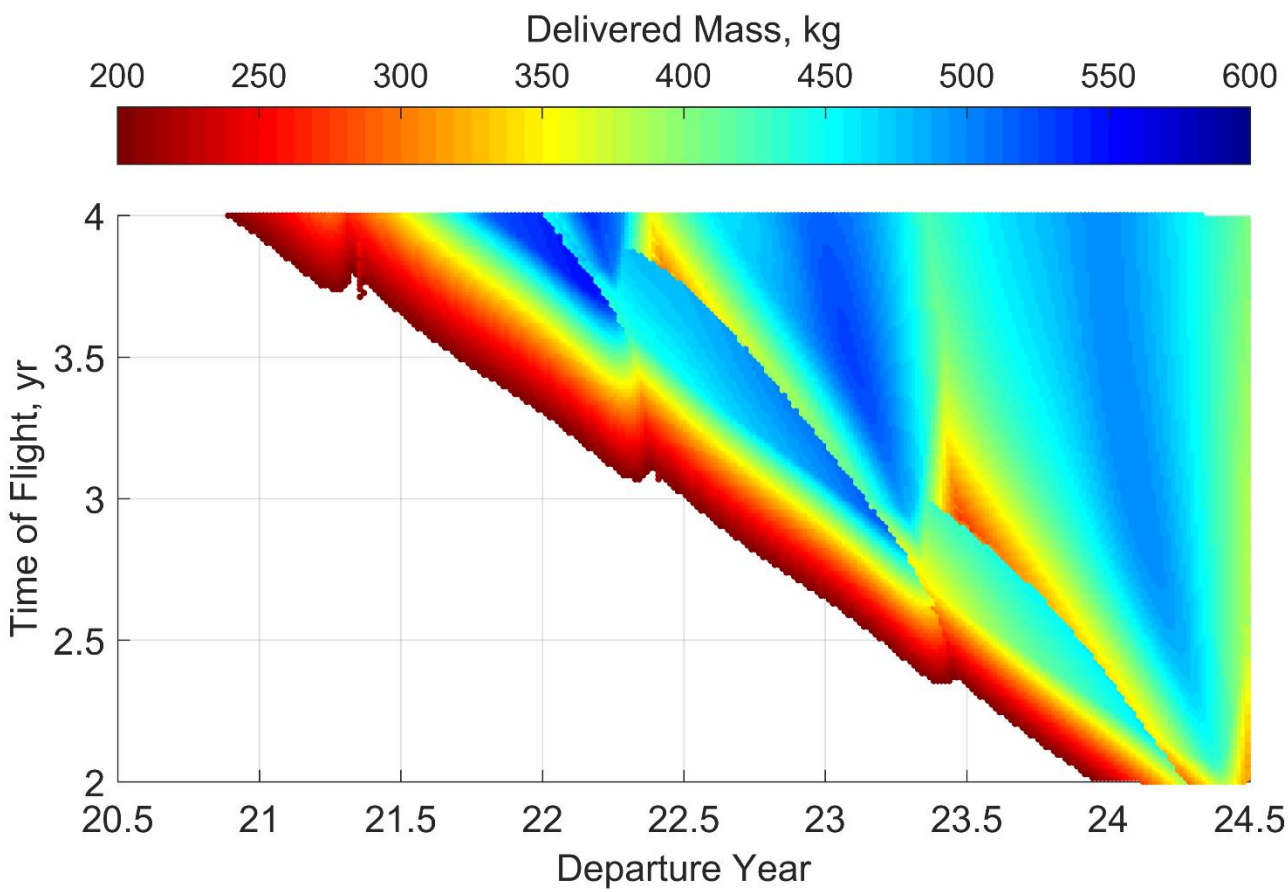


63P aka Wild 1

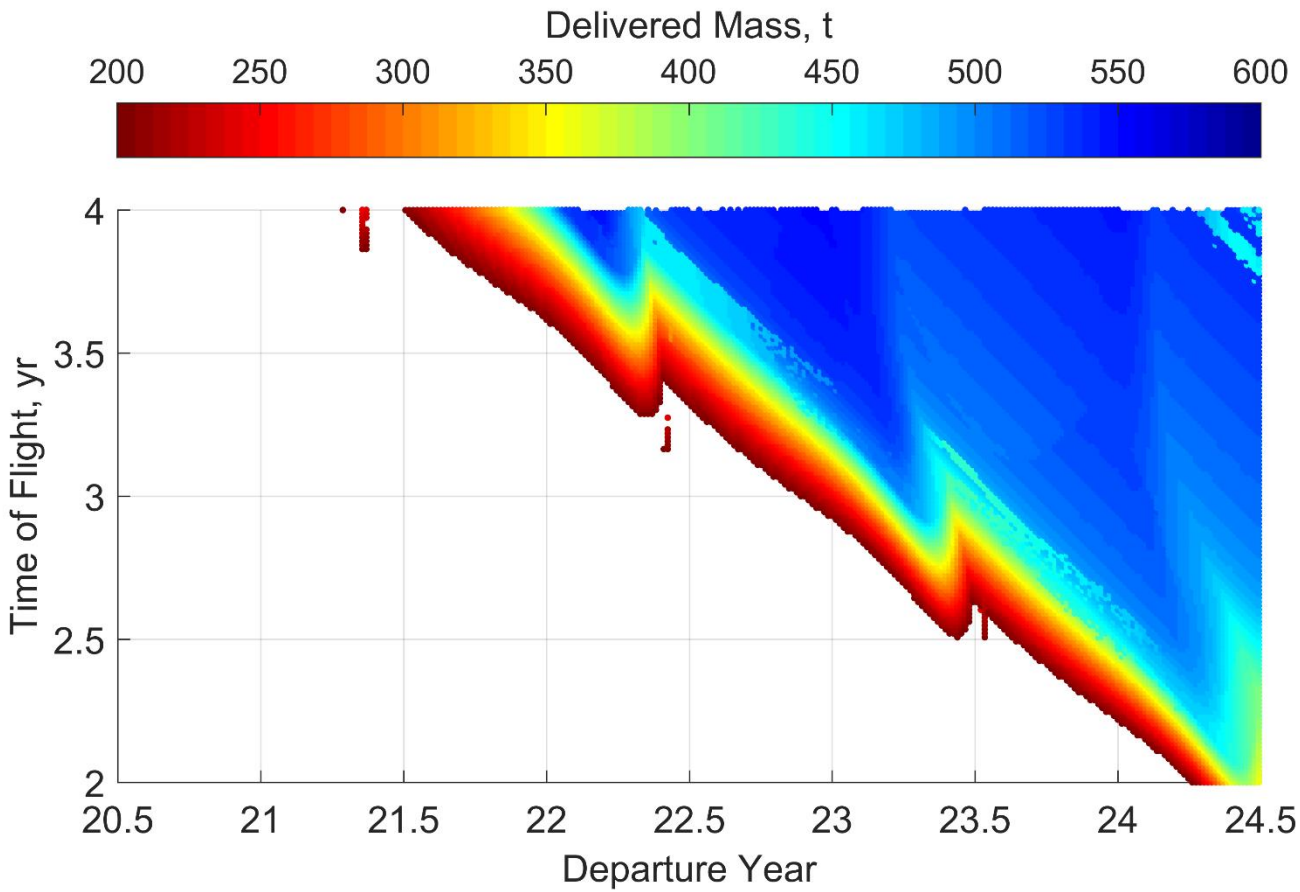




Wild 1: suboptimal refinement



Wild 1: optimal bounded control



Performance

Target	Mars		Wild 1	
Solution phase	Average error in final mass	Approx. CPU time per 100,000 cases	Average error in final mass	Approx. CPU time per 100,000 cases
First Approximation	10%	10 seconds	15%	10 seconds
Suboptimal Refinement	7%	3 minutes	10%	3 minutes
Optimal bounded control	**0.2%**	**½ hour**	**1%**	**1 hour**
MALTO (same # of segments)	0.2%	2 days	0.5%	4 days
Integrated with optimized switching	—	1 week	—	2 weeks

Conclusions

- Choice of coordinate frame and axial elements partially decouples equations
  - Permits semi-analytic initial guess for trajectory path
- Suboptimal refinement with simplified co-state dynamics bridges gap from no idea to optimal control
- Region of convergence increases when controls are optimized on fixed path, and path is propagated with fixed controls
- Produce global trade-space maps of low-thrust interplanetary transfers in real time during concurrent design sessions