Quaternary Triangular Mesh: A Hierarchical coördinate System for Geoprocessing and Cartography

Van Snyder

van.snyder@jpl.nasa.gov

Jet Propulsion Laboratory   California Institute of Technology

10 August 2016

Geoffrey Dutton

This presentation is partly based upon the work of Geoffrey Dutton, from a monograph of the same title as this seminar, based upon his Ph.D. dissertation at the University of Zürich-Irchel in 1998.

The monograph was published by Springer Verlag as part of the series “Lecture Notes in Earth Sciences” in 1997.

ISBN 3-540-64980-8.

This seminar focuses on its applications to geoprocessing, leaving aside its applications to cartography.

Prior Work

An Earth-orbiting instrument was described in a seminar previously presented in this venue. It consists of a spectroradiometer that passively observes thermal microwave emission from the limb of the atmosphere. Its antenna scans from 100 km to the surface in the orbit plane. A coördinate system to represent the results of these observations can be a two-dimensional coördinate system that represents an annulus within the orbit plane, with the horizontal coördinate being orbit angle, and the vertical coördinate being the logarithm of pressure, which is roughly consistent with geometric height, by way of hydrostatic equilibrium.

Proposed New Work

A successor instrument has been proposed that will scan vertically at some angle left or right from the orbit plane, then step closer to the orbit plane and scan again, continue to the same distance on the other side of the orbit plane, and then repeat. A coördinate system to represent the results of these observations is necessarily a three-dimensional coördinate system that represents a shell of the Earth figure. The horizontal basis is a mesh on the Earth's surface, and the vertical coördinate is the logarithm of pressure.

Global Surface Mesh

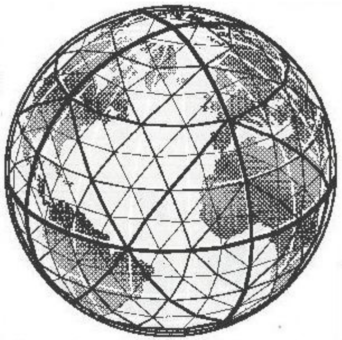
A simple global surface mesh could use parallels of latitude and meridians of longitude as coördinates. This has the defect that as the resolution of the mesh is refined, facets of the mesh at the poles become very small. The ratio of the area of a facet near the pole to the area of a facet near the equator approaches zero. Also, facets adjacent to the poles are triangles, while other facets are quadrilaterals.

A more uniform mesh is desired.

The Quaternary Triangular Mesh described by Geoffrey Dutton, denoted hereafter by QTM, is such a mesh. As the resolution is refined the ratio of the area of the smallest facet to the largest facet approaches 6/11.

Starting and Refining the QTM

A QTM begins by dividing the Earth into octants, with meridians at multiples of 90°. Each facet is a triangle. If a facet is too large, it is divided into four sub-facets by connecting the midpoints of the edges of the facet; each of the sub-facets is also a triangle, hence the term "Quaternary Triangular Mesh." A figure from Dutton's monograph shows uniform refinement to the fourth level:



Size of the QTM

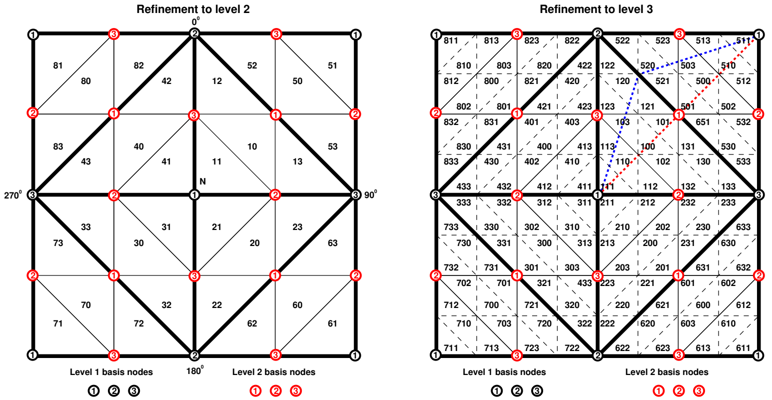
By construction, the number of facets f_ℓ at the ℓ^{th} refinement, where the QTM that consists only of the original octants is level 1, is $f_\ell = 8 \times 4^{\ell-1} = 2^{2\ell+1}$. Because smaller facets are formed by bisecting each edge, the number of vertices in the ℓ^{th} refinement is $v_\ell = v_{\ell-1} + e_{\ell-1}$, where e_ℓ is the number of edges. From Euler's formula for genus-zero polyhedra, $f_\ell = e_\ell - v_\ell + 2$. Solving these equations, we have $v_\ell = 2^{2\ell} + 2$ and $e_\ell = 3 \times 2^{2\ell}$.

Counting the poles and equator, the vertices are on $2^\ell + 1$ equally-spaced parallels of latitude. For example, after 7 levels of refinement, the meridional extent of a facet is about $20000/2^7$ km ≈ 156 km.

Along the k^{th} parallel from each pole, vertices are at $4k$ equally-spaced longitudes.

ZOT projection

Geoffrey Dutton developed a simple way, called the *Zenithial Ortho Triangular* projection, or ZOT projection, to identify facets within a QTM, and to compute the facet within which a position given by latitude and longitude is contained. A figure from his monograph shows ZOT projections of refinements to levels 2 and 3:



ZOT projection (cont.)

The heavy lines are edges of the octants. The outer boundary consists of southern hemisphere meridians of the octants, and the halves of each outer boundary are aliased. For example, the points labeled 3 on the top edge are the same point. The north pole is the center. The south pole is the outer corners. The equator is the inscribed square with diagonal edges. Diagonal lines are parallels of latitude. Meridians are straight lines from each pole to the equator. For example, the 22.5° meridian is the blue line, and the 45° meridian is the red line. The only other meridians shown are edges of the octants. The narrow lines are boundaries of the level-2 refinement. Dashed lines are boundaries of the level-3 refinement.

Every triangle in the mesh is an isosceles right triangle. The right angle is called the *polar angle* because it is always nearest one of the poles of the sphere. One edge incident on the pole vertex is horizontal in the ZOT projection, and the other is vertical.

Vertex and facet numbering in the ZOT projection

Vertices of the mesh are not uniquely identified by the QTM identifier (QID) of a facet. At each level of refinement, they are assigned a *basis number* in the range $1 \cdots 3$. Initially, the poles are assigned 1, the 0° and 180° equatorial vertices of the octants are assigned 2, and the 90° and 270° vertices are assigned 3. Thereafter, as each edge is bisected, the basis number of the midpoint is $B_n = 6 - (B_a + B_b)$, where B_a and B_b are the basis numbers of the end points. The QID of each facet at level 1 is $8 +$ its octant number. When a facet is subdivided, the QID of the central facet is the QID of the parent facet, with zero appended. The QID of each other facet is the QID of the parent facet, with the basis number of the vertex of the larger facet that is also a vertex of the smaller facet appended.

ZOT coordinates

Given longitude ϕ and latitude θ, ZOT coordinates x and y are initially computed in the ℓ_1 metric using

$$\delta x = 1 - \frac{|\theta|}{90}$$
$$x = \delta x \frac{\lfloor \phi \rfloor \mod 90 + \phi - \lfloor \phi \rfloor}{90}$$
$$y = \delta x - x$$

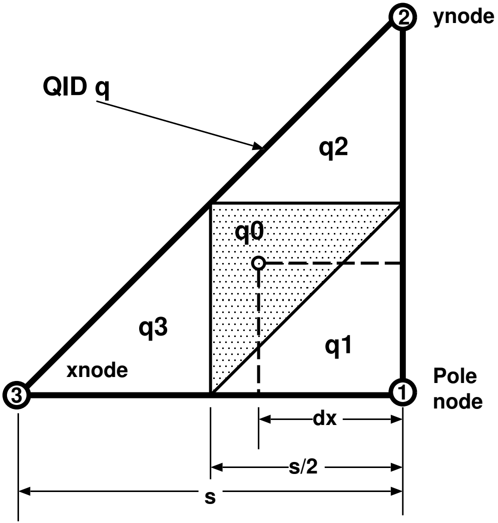
Then in the southern hemisphere, replace $x = 1 - y$ and $y = 1 - x$. Then in even-numbered octants, exchange x and y. Then in the top half of ZOT space, negate y, and in the left half of ZOT space, negate x.

ZOT coördinates are in the range $-1 \cdots 1$, with the north pole at $(0, 0)$, and the south pole at $(\pm 1, \pm 1)$.

Finding a Facet

To find a facet, compute the x and y ZOT coördinates of a point, and use 8 + the octant number as the initial QID. If the level of facet identified does not have sufficient resolution, the QID of a sub-facet is determined by computing the distances dx and dy of the x and y ZOT coördinates from the pole vertex. Let the length of a horizontal or vertical edge of a facet (which is incident on the pole vertex) be s. If $|dx| + |dy| < s/2$, append the basis number of the pole vertex to the QID of the parent facet. Otherwise if either $|dx| > s/2$ or $|dy| > s/2$ append the basis number of the x or y vertex respectively. Otherwise, append zero. If $|dx| > s/2$ and $|dy| > s/2$, you made a mistake along the way and you're working in the wrong facet. No square roots or trigonometric functions are needed. The process is repeated if the size of the facet is not sufficiently small. The maximum number of repetitions is equal to the degree of refinement of the mesh.

Calculation of Sub-Facet and QID



Construct an Integer from the ZOT Coördinates

At level ℓ there are $2^\ell + 1$ possible values of each ZOT coördinate, with spacing $2^{1-\ell}$. Therefore, $2^{\ell-1}(1+x)$ and $2^{\ell-1}(1+y)$, where x and y are ZOT coördinates in a level ℓ refinement, are each integers in the range $0 \cdots 2^l$, and $2^{\ell-1}(1+x+(1+2^\ell)(1+y))$ is an identifier in the range $0 \cdots 2^{2\ell} + 2^{\ell+1}$ for each ZOT coördinate. This is too many identifiers because there are only $2^{2\ell}+2$ vertices. The outer edges of the ZOT projection are meridians of southern-hemisphere octants, and the negative half of each outer edge represents the same points as the positive half. ZOT coördinates are disambiguated by using $|y|$ if $|x| = 1$ and $|x|$ if $|y| = 1$. This reduces the number of identifiers to $2^{2\ell}+2$, and reduces the range to $2^{\ell-1} \cdots 2^{2\ell} + 2^{\ell+1}$. If ℓ is restricted so that a QID fits in an integer, and a QID identifies a facet, and there are $2^{2\ell+1}$ facets, this vertex identifier also fits in an integer.

Data Structure to Find Facets and Vertices

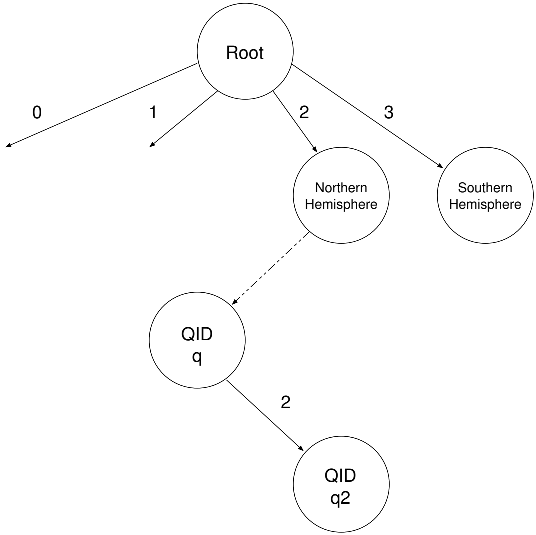
A quadtree, that is, a tree in which each vertex has four sons, is used to find the representations of facets and vertices. The value of a nonzero `son` field of each vertex is the index of another vertex in the tree.

Because the highest order nonzero bit of a QID is the high-order bit of an octant index, which is $8 +$ the octant number, and therefore in the range $8 \cdots 15$, one can always isolate the four-bit octant number unambiguously. The high-order two bits are the hemisphere number, in the range $2 \cdots 3$, and the low-order two bits are the octant number within the hemisphere, in the range $0 \cdots 3$.

The first three vertices of the quadtree then have `son` fields with values $[0, 0, 2, 3]$, $[4, 5, 6, 7]$, and $[8, 9, 10, 11]$, and the `son` fields of other vertices are all zero.

A recursive process is used to construct the remainder of the quadtree, and refine the QTM within a specified polygon.

Data structure to find a facet



Refining a QTM Within a Polygon

The concept "inside a polygon" is ambiguous on a sphere, so an additional point defined to be inside a specified polygon is also required. A vertex is considered to be inside the polygon if an "inside a polygon" algorithm reports that the designated point and the vertex are either both within or both outwith the polygon.

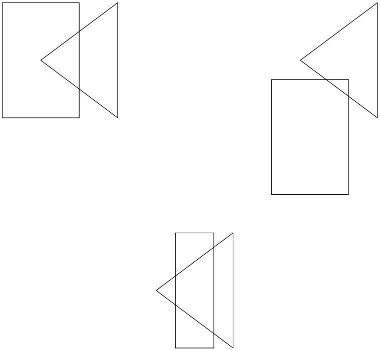
Start from the initial octants. Refine a facet and add its sub-facets to the quadtree if it is not sufficiently small and if

- a vertex of the facet is within the polygon,
- a vertex of the polygon is within the facet, or
- an edge of the facet intersects an edge of the polygon.

If any vertex of a sufficiently small facet is within the polygon, assign serial numbers to all its vertices that do not have one, put the serial numbers into the quadtree, and (using a hash table) associate the quadtree vertex number to the vertices' serial numbers, to avoid giving a vertex more than one serial number.

Refining a QTM Within a Polygon (cont.)

Here are the three cases needing refinement:



Refining a QTM Within a Polygon (cont.)

An "inside a polygon" algorithm for a plane is simpler and faster than one for a sphere, so you might prefer to test in the ZOT projection.

If your polygon has an edge that crosses a southern-hemisphere meridian of one of the octants, the polygon will appear in two (or more) pieces in the ZOT projection, because these meridians are aliased on the ZOT projection boundary.

There are two possibilities:

- Break the polygon into pieces. This is nontrivial because there is not necessarily another edge that crosses the same meridian, even in the northern hemisphere. Then search in all the pieces.
- Break the edge into two edges, each ending at the meridian. That point is two points on a boundary of the ZOT projection. Introduce two antiparallel edges at that point, between the aliased points of the meridian, along a boundary of the ZOT projection. Now it's just one polygon in the ZOT projection.

Atmospheric Remote Sensing

For cartography, or a geophysical phenomenon on the Earth's surface, a two-dimensional mesh is sufficient, and a QTM as so far described is sufficient.

For atmospheric remote sensing, a three-dimensional mesh is needed.

We construct a third dimension by extending a line orthogonally from the Earth's surface at each vertex of the QTM, and put points of the three-dimensional mesh only on these lines. We call such a mesh "stacked." If, furthermore, the n^{th} point above the Earth's surface on every line has the same vertical coördinate value, we call such a mesh "coherent."

A stacked mesh, whether it is coherent or not, can be thought of as a collection of prisms.

If the vertical dimension has a finite extent, it represents a shell about the Earth's surface.

Ray Tracing

Many problems in atmospheric remote sensing require to trace a ray through the Earth's atmosphere.

If the radiative-transfer equation is being integrated along the ray, one wishes to know the points at which the ray intersects a vertical or horizontal face of a prism. One then integrates the radiative-transfer equation from one such point to the next.

If geophysical values, such as temperature or composition, are represented at points on the mesh, and interpolated to points within prisms using multilinear interpolation, there are discontinuities in the first derivatives of the interpolated values along the ray at points where it crosses a prism boundary. The error term in a quadrature or differential-equation formula depends upon the maximum value of a high-order derivative within the range. If there is a discontinuity in the first derivative, and the range spans the point where that discontinuity, the algebraic order of the formula is only 1, no matter how many abscissae are used.

Intersections of the Ray with Prism Boundaries

There are four kinds of surfaces the ray might intersect:

- A vertical surface of constant latitude.
- A vertical surface not of constant latitude.
- A horizontal surface within the QTM.
- A horizontal surface outwith the QTM.

Intersection With a Surface of Constant Latitude

A vertical surface of constant latitude is a sector of a cone. For atmospheric remote sensing, geodetic latitude and geodetic height are more useful than geocentric measures. Therefore, the apex of the cone is not at the Earth's center. The intersection is determined by inserting the equation for a line, *viz.*

$$\mathbf{X} = \mathbf{C} + t\mathbf{U}$$

where **C** is a point on the line and **U** is a vector along it, into the equation for a cone, *viz.*

$$(\mathbf{X} - \mathbf{V})^T \mathbf{M} (\mathbf{X} - \mathbf{V}) = 0\,,$$

where

$$\mathbf{V} = [0, 0, v]^T\,,\ \mathbf{M} = \mathbf{D}\mathbf{D}^T - \sin^2\theta\,\mathbf{I}\,,\ \mathbf{D} = [0, 0, 1]^T\,,$$

v is the distance along the polar axis of the apex from the Earth's center, θ is the geodetic latitude, and **I** is the identity matrix.

Intersection With a Surface of Constant Latitude (cont.)

This results in the equation

$$(\mathbf{C} + t\mathbf{U} - \mathbf{V})^T \mathbf{M} (\mathbf{C} + t\mathbf{U} - \mathbf{V}) = 0\,.$$

The points where the line intersects the cone are the roots of the polynomial $a_2 t^2 + a_1 t + a_0 = 0$, where

$$a_2 = u_3^2 - \mathbf{U} \cdot \mathbf{U} \sin^2\theta = u_3^2 - |\mathbf{U}|^2 \sin^2$$
$$a_1 = (c_3 - v)u_3 - (\mathbf{C} - \mathbf{V}) \cdot \mathbf{U} \sin^2\theta$$
$$a_0 = (c_3 - v)^2 - (\mathbf{C} - \mathbf{V}) \cdot (\mathbf{C} - \mathbf{V}) \sin^2\theta = (c_3 - v)^2 - |\mathbf{C} - \mathbf{V}|^2 \sin^2\theta$$

If the latitude θ is geodetic then

$$v = \frac{ae^2 \sin\theta}{\sqrt{1 - e^2 \sin^2\theta}}$$

where $e^2 = 1 - \frac{b^2}{a^2}$ is the square of meriodonal eccentricity, and a and b are equatorial and polar radii of the Earth, respectively.

Intersection With a Vertical Plane

A plane is the set of points $\mathbf{X}$ such that

$$(\mathbf{X} - \mathbf{P}_0) \cdot \mathbf{n} = 0$$

where $\mathbf{P}_0$ is a point on the plane and $\mathbf{n}$ is a normal vector to it. Substituting the equation of a line

$$\mathbf{X} = \mathbf{C} + t\mathbf{U}$$

into the equation for a plane gives

$$(t\mathbf{C} + \mathbf{U} - \mathbf{P}_0) \cdot \mathbf{n} = t\mathbf{C} \cdot \mathbf{n} + (\mathbf{U} - \mathbf{P}_0) \cdot \mathbf{n} = 0\,.$$

Solving for t gives

$$t = \frac{(\mathbf{U} - \mathbf{P}_0) \cdot \mathbf{n}}{\mathbf{C} \cdot \mathbf{n}}\,.$$

Intersection With a Horizontal Surface Within the QTM

A horizontal surface within the QTM is approximated by a sector of a sphere that has the same radius of curvature as the Earth has at the circumcenter of the QTM facet at the Earth's surface, plus the average geodetic altitude of the points on the lines incident on QTM vertices, with a center on the normal to the plane containing those points.

Define the plane by three points $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. Arbitrarily choose $\mathbf{C}$ as a reference point and define two vectors

$$\mathbf{a} = \mathbf{A} - \mathbf{C} \text{ and } \mathbf{b} = \mathbf{B} - \mathbf{C}\,.$$

A normal vector to that plane is $\mathbf{n} = \mathbf{a} \times \mathbf{b}$. The vector from $\mathbf{C}$ to the circumcenter $\mathbf{P}_0$ is given by

$$\mathbf{v} = \frac{(|\mathbf{a}|^2\mathbf{b} - |\mathbf{b}|^2\mathbf{a}) \times \mathbf{n}}{2|\mathbf{n}|^2}\,.$$

The circumcenter of the three points is therefore $\mathbf{P}_0 = \mathbf{C} + \mathbf{v}$.

Intersection With a Horizontal Surface (cont.)

The centers of the spheres defined by the three points and the radius are on the line

$$\mathbf{p}(t) = \mathbf{P}_0 + t\mathbf{n} = \mathbf{V} + \mathbf{C} + t\mathbf{n}\,.$$

The distance from any of the three points, say $\mathbf{C}$ to the center of the sphere is its radius r. Therefore

$$|\mathbf{p}(t) - \mathbf{C}|^2 = |\mathbf{V} + t\mathbf{n}|^2 = |\mathbf{n}|^2 t^2 + t\mathbf{V} \cdot \mathbf{n} t + |\mathbf{V}|^2 = r^2\,.$$

Since $\mathbf{V}$ is in the plane defined by the three points, $\mathbf{V}$ is orthogonal to $\mathbf{n}$, i.e., $\mathbf{V} \cdot \mathbf{n} = 0$. Therefore $|\mathbf{n}|^2 t^2 + |\mathbf{V}|^2 = r^2$, and

$$t = \pm\sqrt{\frac{r^2 - |\mathbf{V}|^2}{|\mathbf{n}|^2}}\,,$$

with the sign of t chosen to minimize $|\mathbf{p}(t)|$.

Intersection With a Horizontal Surface (cont.)

A sphere is the set of points $\mathbf{X}$ such that

$$|\mathbf{X} - \mathbf{P}_0|^2 = r^2 .$$

By substituting the equation of a line, *viz.* $\mathbf{X} = \mathbf{C} + t\mathbf{U}$, and letting $\mathbf{V} = \mathbf{C} - \mathbf{P}_0$, the points of intersection of the line with the sphere are the roots of the polynomial

$$\mathbf{U} \cdot \mathbf{U}t^2 + 2\mathbf{V} \cdot \mathbf{U}t + \mathbf{V} \cdot \mathbf{V} = r^2 .$$

The radius of curvature r that is used is the mean curvature

$$R_A = \frac{2}{\frac{1}{R_M(\theta)} + \frac{1}{R_N(\theta)}}$$

where $R_M(\theta) = \frac{b^2}{a(1-e^2 \sin^2 \theta)^{3/2}}$ is the meridional radius of curvature, and $R_N = \frac{a}{\sqrt{1-\sin^2 \theta}}$ is the normal radius of curvature.

Intersection With a Horizontal Surface Outwith the QTM

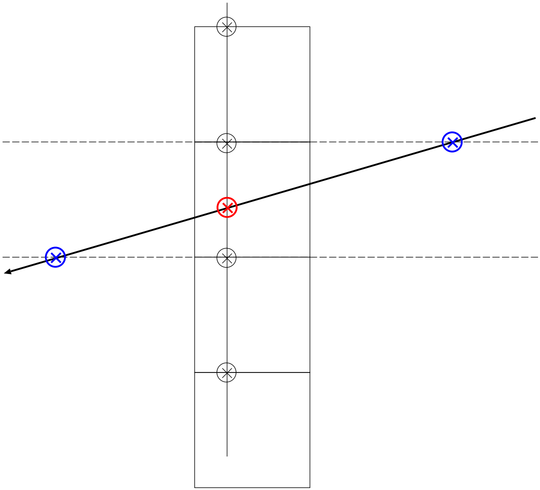
We have no temperature information outside the polygon. Therefore the temperature outside the polygon is assumed not to have any horizontal variation, and therefore the relationship between pressure and height (according to hydrostatic equilibrium) is assumed to be constant outside the polygon.

Therefore, a horizontal surface outwith the QTM is assumed to be a surface at a constant geodetic height above the Earth reference ellipsoid. The heights of the surfaces considered are the heights at the horizontal position where the ray first intersects a vertical surface of a prism of the QTM within the polygon, interpolated from the two adjacent vertices at each height.

This assumes that the numbers of positions at which height (or pressure) are tabulated on every vertical line from a vertex of the QTM are the same on every line. The sets of heights (or pressures) are not necessarily assumed to be the same on every line.

Interesting Vertical Surfaces Outwith the QTM

This is how the levels of vertical surfaces outwith the QTM are chosen:



A Surface a Constant Distance from an Ellipsoid

A surface at a constant distance h from an ellipsoid is not an ellipsoid. Let $\mathbf{V} = [x, y, z]^T$ and

$$\mathbf{M} = \begin{bmatrix} \frac{1}{a+h} & 0 & 0 \\ 0 & \frac{1}{b+h} & 0 \\ 0 & 0 & \frac{1}{c+h} \end{bmatrix} .$$

where a, b, and c are the ellipsoid's semiminor axes. If h is small relative to those axes, an ellipsoid that is near to a surface a constant distance h from the given ellipsoid is given by

$$(\mathbf{M}\mathbf{V}^T)(\mathbf{M}\mathbf{V}) = \frac{x}{(a+h)^2} + \frac{y}{(b+h)^2} + \frac{z}{(c+h)^2} = 1 .$$

As a first approximation, compute an intersection of the ray with this ellipsoid.

Surface a Constant Distance from an Ellipsoid (cont.)

Intersections of a line with an ellipsoid can be calculated by substituting the equation of a line, *viz.* $\mathbf{V} = \mathbf{C} + t\mathbf{U}$ into the equation of the ellipsoid. This leads to the quadratic polynomial

$$(\mathbf{MU})^T(\mathbf{MU})t^2 + 2(\mathbf{MC})^T(\mathbf{MU})t + (\mathbf{MC})^T(\mathbf{MC}) = 1$$

If there is no intersection of the ray with the ellipsoid that approximates the desired surface, compute the point closest to it. Start by defining the ellipsoid in homogeneous coordinates using the matrix

$$\mathbf{M} = \begin{bmatrix} 1/a^2 & 0 & 0 & 0 \\ 0 & 1/b^2 & 0 & 0 \\ 0 & 0 & 1/c^2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

If a point $\mathbf{P}$ on its surface is given in homogeneous coordinates as $\mathbf{P} = [\alpha x, \alpha y, \alpha z \alpha]^T$ such that $\mathbf{P}^T\mathbf{MP} = 0$, we have the familiar equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$.

Nearest Points on Line and Ellipsoid

The point on the ellipsoid that is closest to the line $\mathcal{L}(t) = \mathbf{C} + t\mathbf{U}$ lies on the plane $\mathbf{P}_1$ defined by $\mathcal{L}$ and the origin.

The plane $\mathbf{P}_2$ that is tangent to the ellipsoid at the point closest to $\mathcal{L}$ is perpendicular to $\mathbf{P}_1$. A normal to this plane is the vector $\mathbf{r}$ to the point on $\mathcal{L}$ that is nearest to the origin. The vector $\mathbf{r}$ is necessarily orthogonal to $\mathcal{L}$. The point on the ellipsoid that is nearest to $\mathcal{L}$ is on any plane normal $\mathbf{r}$, but not necessarily on a line collinear with $\mathbf{r}$. The normal from $\mathbf{r}$ to the origin is given by

$$\mathbf{r} = \mathbf{C} - (\mathbf{C} \cdot \mathbf{U})\mathbf{U} = [i, j, k]^T$$

The homogeneous coordinates of $\mathbf{P}_2$ are $\mathbf{W} = [i, j, k, -\ell]^T$ with normal direction $\mathbf{r}$ and unknown distance d from the origin, where

$$d = \frac{\ell}{|\mathbf{r}|} = \frac{\ell}{\sqrt{i^2 + j^2 + k^2}}$$

Nearest Points on Line and Ellipsoid (cont.)

To make sure $\mathbf{P}_2$ is tangent to the ellipsoid we set $\mathbf{W}^T\mathbf{C}^{-1}\mathbf{W} = 0$ and solve for

$$\ell = \pm\sqrt{a^2 i^2 + b^2 j^2 + c^2 k^2}\,,$$

where the sign of ℓ is chosen to put the plane on the side of the ellipsoid nearest to $\mathcal{L}$.

The point on the ellipsoid where the tangent plane touches (i.e., the point closest to $\mathcal{L}$) is defined in homogeneous coordinates by $\mathbf{P} = \mathbf{C}^{-1}\mathbf{W}$, or

$$\mathbf{P} = [\alpha x, \alpha y, \alpha z \alpha]^T = [a^2 i, b^2 j, c^2 k, -\ell]^T\,,$$

or

$$\mathbf{P} = [x, y, z]^T = \frac{1}{\ell}[a^2 i, b^2 j, c^2 k]^T\,.$$

Nearest Points on Line and Ellipsoid (cont.)

$\mathcal{L}$ is parallel $\mathbf{P}_2$, so the distance from $\mathbf{P}_2$ to $\mathcal{L}$, i.e., the tangent height $h = |\mathbf{r}| - d$.

The point on $\mathcal{L}$ that is nearest to $\mathbf{P}$ can be found by solving for the intersection of $\mathcal{L}(t)$ with $\mathbf{P} + h\nabla F(\mathbf{P})$, i.e., by solving $\mathbf{C} + t\mathbf{U} = \mathbf{P} + h\nabla F(\mathbf{P})$ for t. Alternatively, $t = |\mathbf{P} + h\nabla F(\mathbf{P}) - \mathbf{C}|/|\mathbf{U}| = |\mathbf{P} + h\nabla F(\mathbf{P}) - \mathbf{C}|$ (assuming $|\mathbf{U}| = 1$) $= (\mathbf{P} + h\nabla F(\mathbf{P}) - \mathbf{C}) \cdot \mathbf{U}$.

The last expression arises because $\mathbf{P} + h\nabla F(\mathbf{P})$ and $\mathbf{C}$ are on $\mathcal{L}$, and the distance between them, t, is the difference of their projections onto $\mathbf{U}$.

Surface a Constant Distance from an Ellipsoid (cont.)

Let $\mathcal{L}(t_0)$ be the point (given in Earth Centered Rotating Cartesian coördinates) on $\mathcal{L}(t)$ that intersects the ellipsoid, or that is closest to it. Calculate the geodetic height h_0 at t_0, e.g., by using Bowring's or Fukushima's algorithm. If $h_0 > h$ there is no intersection, and there is nothing more to do.

Fukushima's algorithm is faster then Bowring's because it is a Halley iteration on the tangent of the geodetic latitude. On the first iteration, it usually produces θ_i that is accurate to within a few micro arcseconds if $h < 30,000$ km.

If there is an intersection . . . .

Surface a Constant Distance from an Ellipsoid (cont.)

for i = 0, ...

1. Evaluate $\mathcal{L}(t_i)$ in Earth Centered Rotating Cartesian coördinates.
2. Using Fukushima's algorithm, compute ϕ_i, θ_i and h_i at $\mathcal{L}(t_i)$.
3. If $|\delta h| = |h - h_i| \leq h_{\text{tol}}$ **exit**.
4. Compute

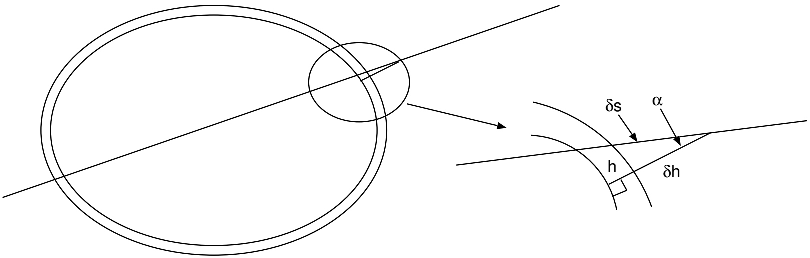
$$t_{i+1} = t_i + \frac{\delta h}{\text{sign}(\max(|\cos\alpha_i|, \frac{1}{2}), \cos\alpha_i)}$$

   where $\cos\alpha_i = \mathbf{N}_i \cdot \mathcal{L}(t_i)$ and $\mathbf{N}_i = [\cos\theta_i \cos\phi_i, \cos\theta_i \sin\phi_i, \sin\theta_i]^T$ is the unit normal to the ellipsoid at the point nearest to $\mathcal{L}(t_i)$.

end for

Surface a Constant Distance from an Ellipsoid (cont.)

Here is an illustration of that iteration:



Interpolation Coefficients and the Jacobian Matrix

After the points where the ray along which the radiative transfer equation is being integrated intersect surfaces of prisms in the QTM are computed, interpolation coefficients from the vertices bounding those faces are computed.

Those interpolation coefficients are then used to compute the Jacobian matrix used in the Newton iteration to solve for temperature and the atmospheric constituents, as described in a previous seminar in this venue.

Thanks for your attention

van.snyder@jpl.nasa.gov