Control by Coin Flips: Mixed Strategy for Finite-Horizon Stochastic Optimal Control

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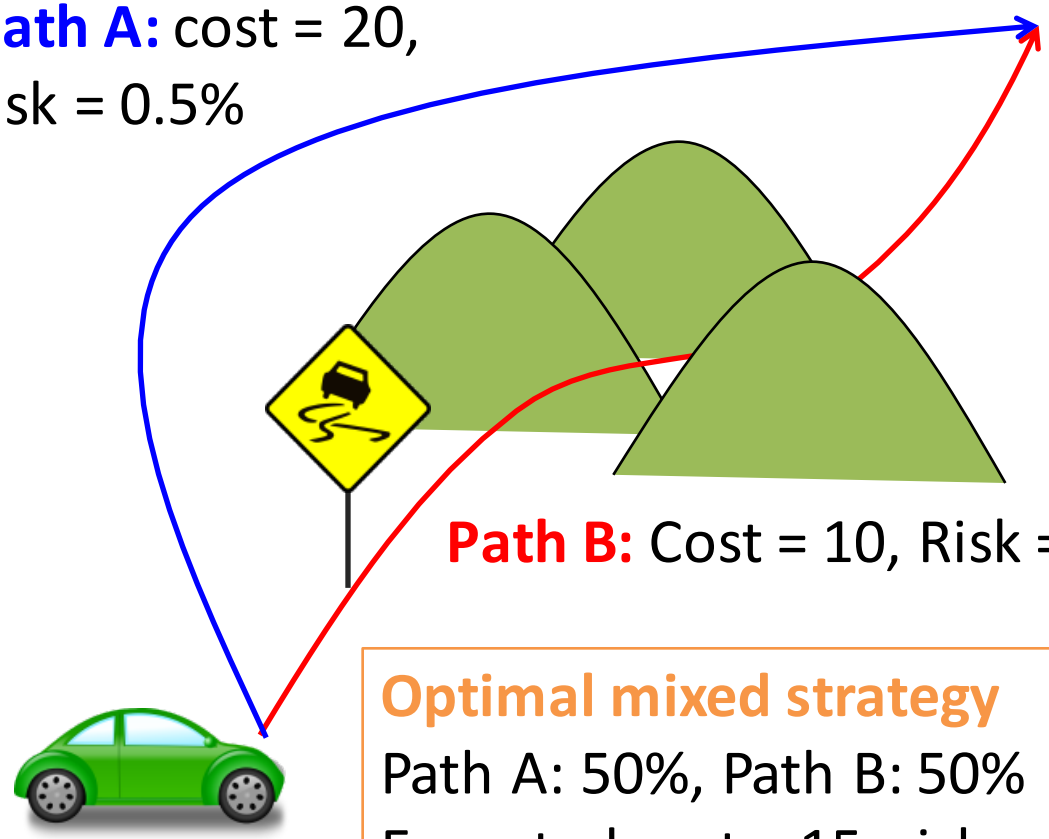
Intuition: Toy Example

Chance constraint: Risk $\leq$ 1%

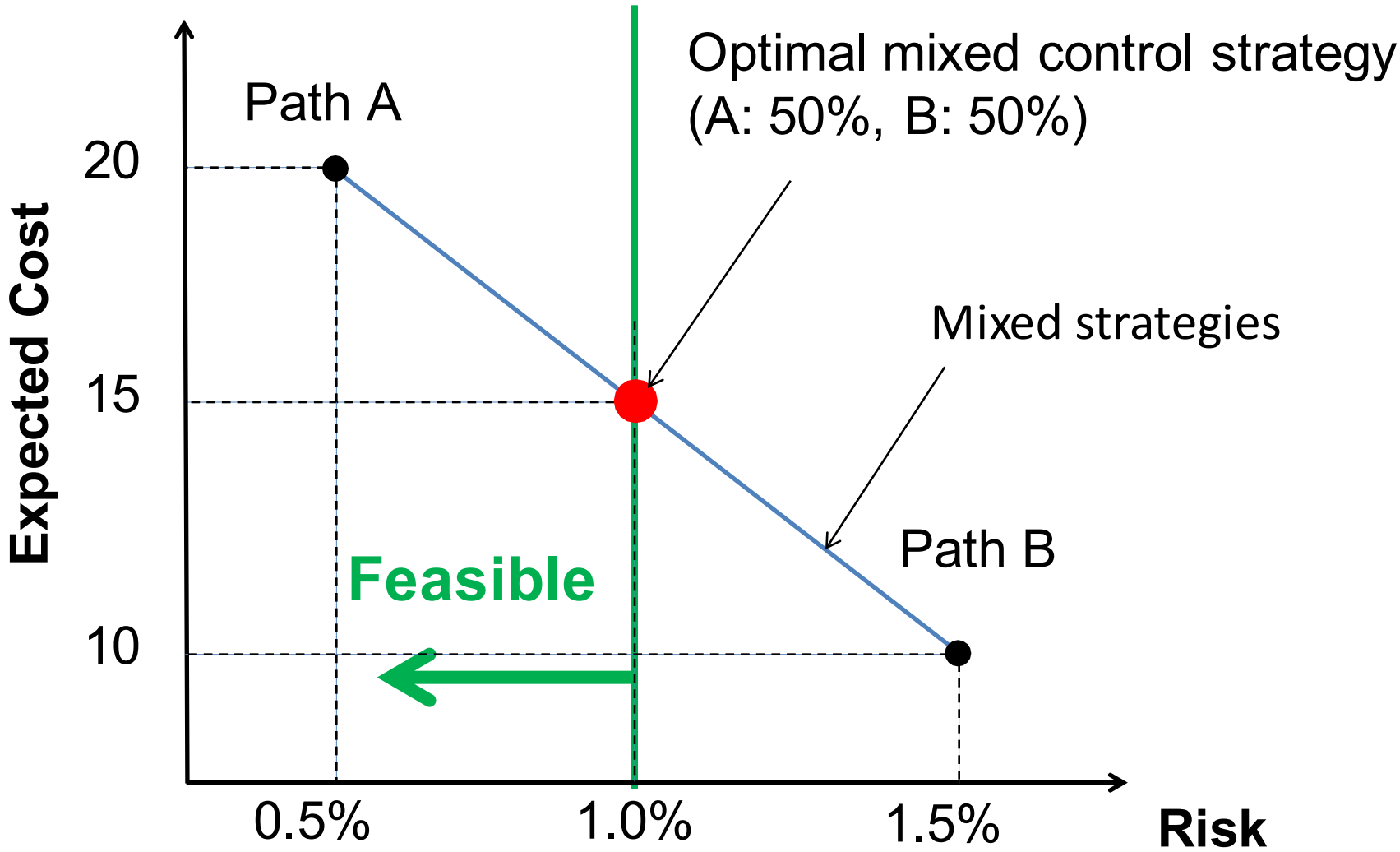
Path A: cost = 20,
risk = 0.5%



Path B: Cost = 10, Risk = 1.5%

Optimal mixed strategy
Path A: 50%, Path B: 50%
Expected cost = 15, risk = 1%

Intuition


Optimal mixed control strategy
(A: 50%, B: 50%)
Path A
Mixed strategies
Path B
Feasible
Expected Cost
20
15
10
0.5%
1.0%
1.5%
Risk

Problem Formulation (Pure SOC)

PSOC (Pure-strategy Stochastic Optimal Control

$$c_{\mathrm{P}}^{\star} = \min_{\substack{\mu \in \mathcal{M}^T \\ g(x,\mu,w)=0}} \mathbb{E}\left[f_0(x,\mu)\right]$$

$$s.t.\ \mathbb{E}\left[f_i(x,\mu)\right] \leq V_i, \quad i = 1 \cdots K.$$

Problem Formulation (Mixed SOC)

MSOC^N (Mixed-strategy Stochastic Optimal Control)

$$c_{\mathrm{M}}^{\star N} = \min_{\substack{\mu^1 \cdots \mu^{N+1} \in \mathcal{M}^T \\ \sum_{j=1}^{N} p^j = 1,\ p^j \geq 0 \\ g(x,\mu,w)=0}} \sum_{j=1}^{N+1} p^j \mathbb{E}\left[f_0(x,\mu^j)\right]$$

$$s.t. \sum_{j=1}^{N+1} p^j \mathbb{E}\left[f_i(x,\mu^j)\right] \leq V_i$$

$$i = 1 \cdots K$$

Question 1: What *N* is sufficient in order to achieve optimality?

Theorem 1: Sufficient Degree of Ranzomization

What N is sufficient in order to achieve optimality?

- If there are K stochastic constraints, $N=K$ (having K+1 options) is sufficient for optimality (called *K-randomization*)
  - Existing result for finite-state constrained MDP but not for continuous state/action space

Alternative Representation of the Problems

Let $c = (c_0, c_1, \cdots c_K)$ where c_i is the i-th cost value:

$$c_i = \mathbb{E}\left[f_i(x, \mu)\right].$$

We denote by $\mathcal{F} \subset \mathbb{R}^{K+1}$ the feasible set of the costs of the original problem, that is,

$$\mathcal{F} := \{c \mid \mu \in \mathcal{M} \wedge g(x, \mu, w) = 0\}. \tag{6}$$

PSOC':

$$\begin{aligned} c_{\mathrm{P}}^{\star} = \min_{c \in \mathcal{F}} \quad & c_0 \\ s.t. \quad & c_i \leq V_i, \quad i = 1 \cdots K. \end{aligned}$$

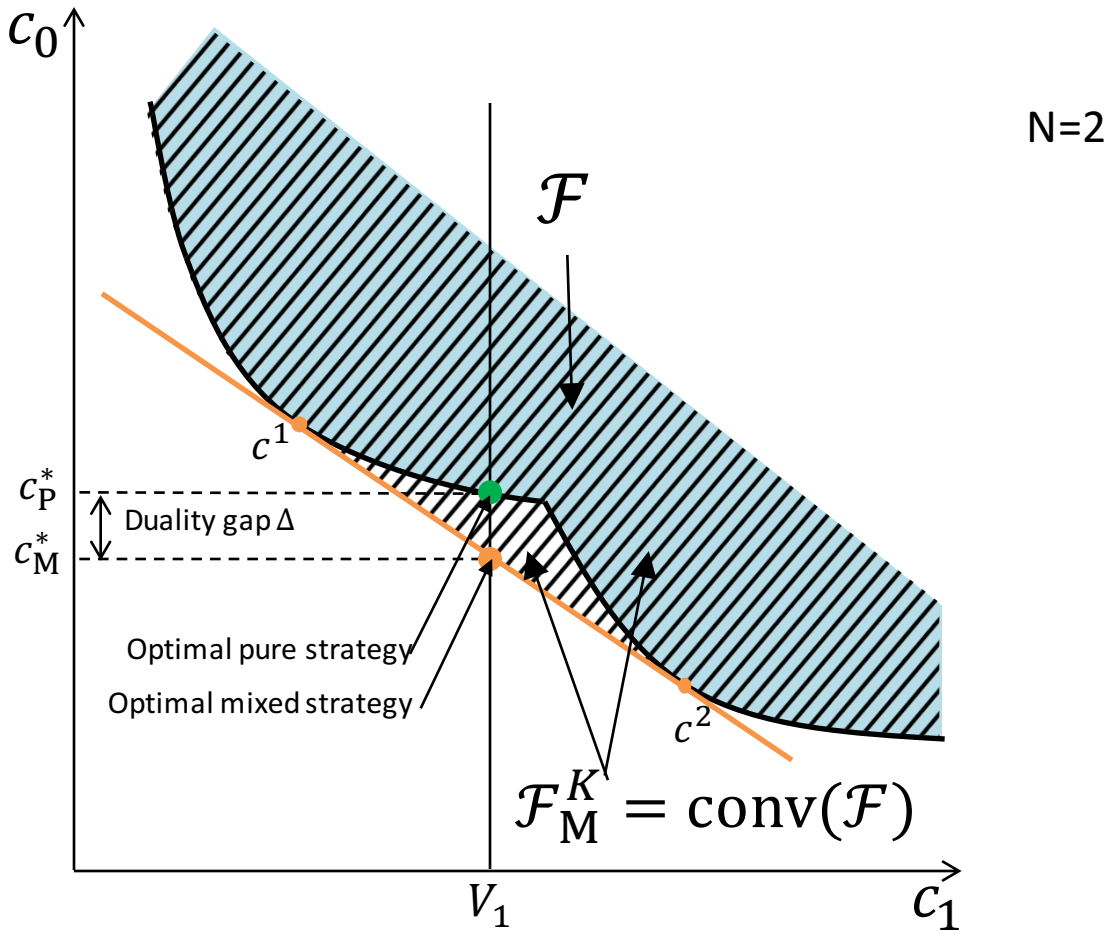
Alternative Representation of the Problems

MSOC^N':

$$\min_{\substack{c^1 \cdots c^{N+1} \in \mathcal{F} \\ \sum_{j=1}^{N+1} p^j = 1,\ p^j \geq 0}} \sum_{j=1}^{N+1} p^j c_0^j$$

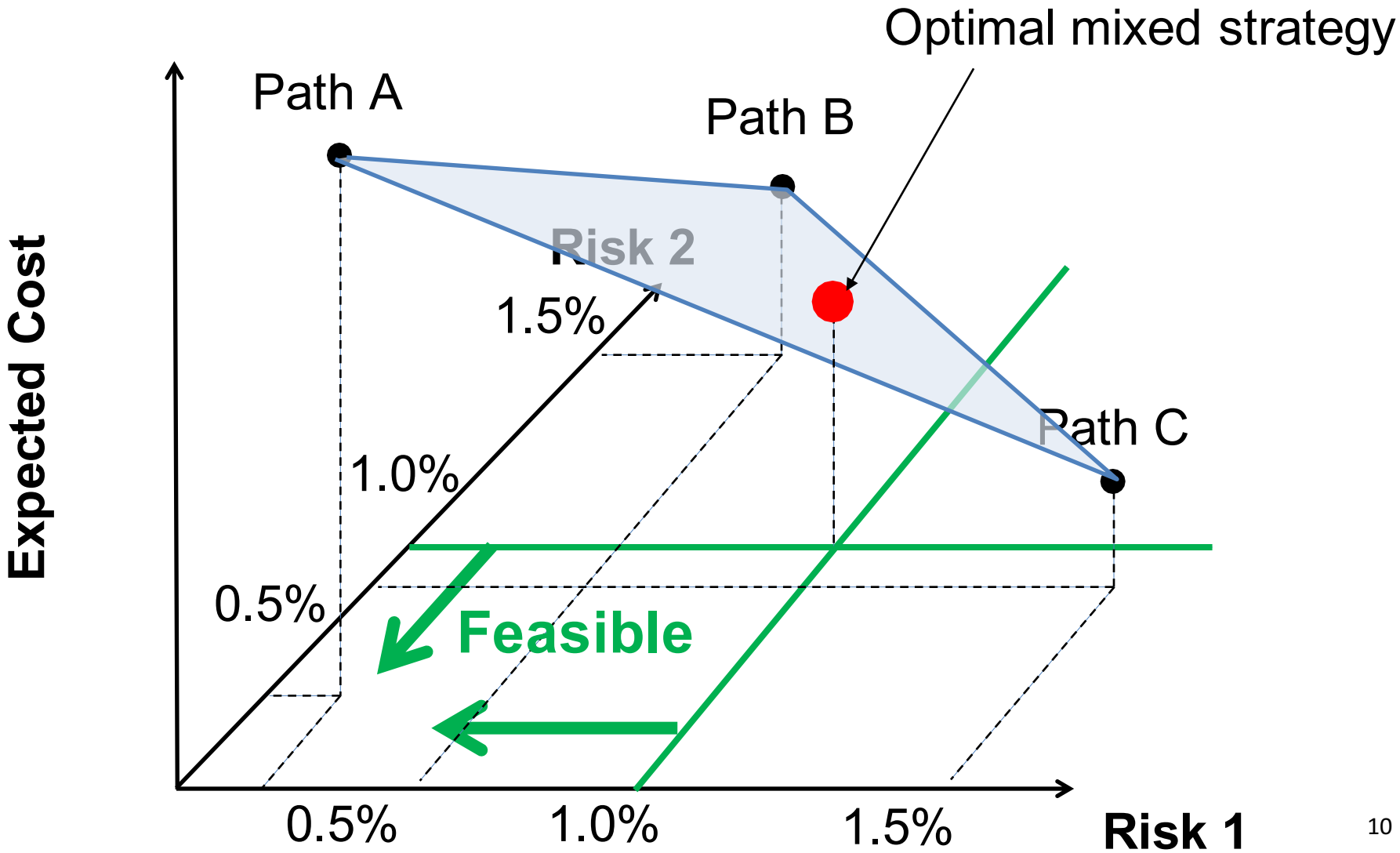
$$s.t. \quad \sum_{j=1}^{N+1} p^j c_i^j \leq V_i,$$

Graphical Interpretation



Theorem 1: Intuition

2 constraints => 3D cost-risk space => need 3 points to cover the convex hull



Question 2: How much we can gain by mixed strategy?

Theorem 2

$$c_{\mathrm{M}}^{\star} = c_{\mathrm{P}}^{\star} - \Delta.$$

- Where Δ is the duality gap of PSOC

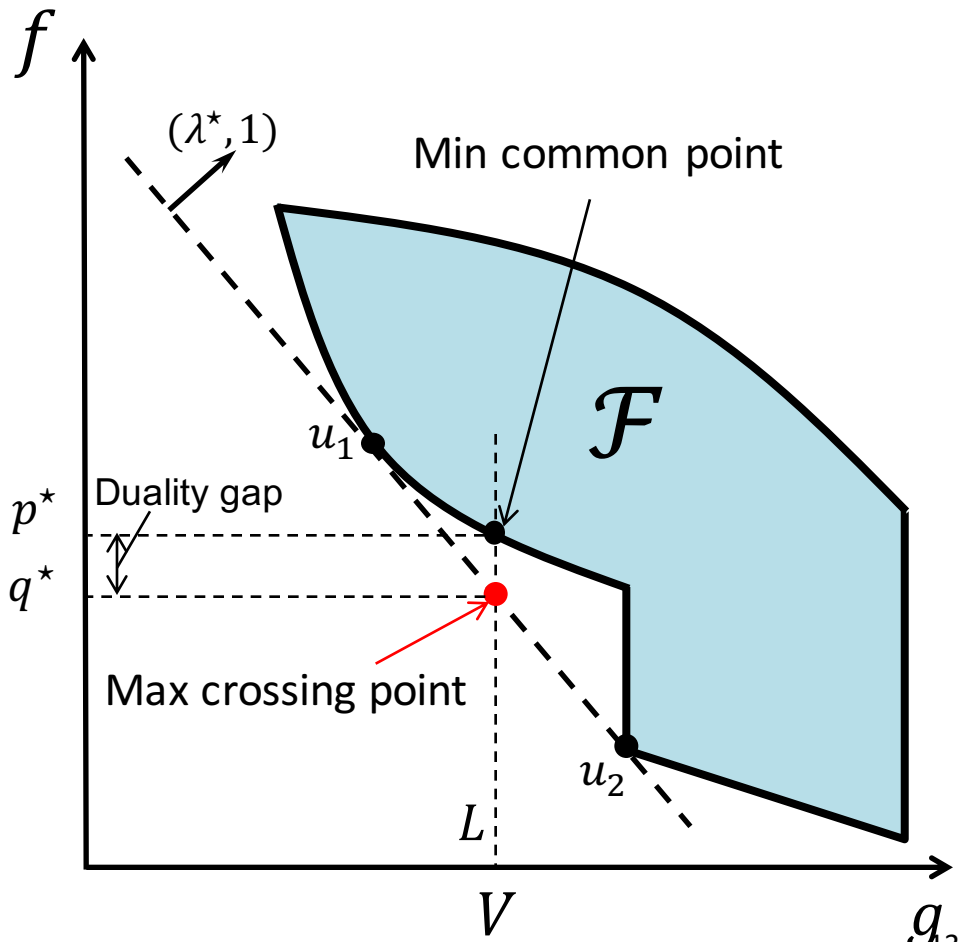
Min Common/Max Crossing (MCMC) Framework

Bertsekas 2009

- **Min common point =** primal optimal solution: the point with the least f among $\mathcal{F} \cap L$
- **Max crossing point** = dual optimal solution: the point with the greatest f among the intersection of L and hyperplanes that contain $\mathcal{F}$ in their upper closed halfspace

$$p^\star = \min_{x \in X} f(x) \quad \text{s.t.} \quad g(x) \leq c$$

$$q^\star = \max_{\lambda \geq 0} \min_{x \in X} f(x) + \lambda(g(x) - c)$$



$$\mathcal{F} = \{(f, g) | x \in X\}$$
$$L = \{(f, g) | g = c\}$$

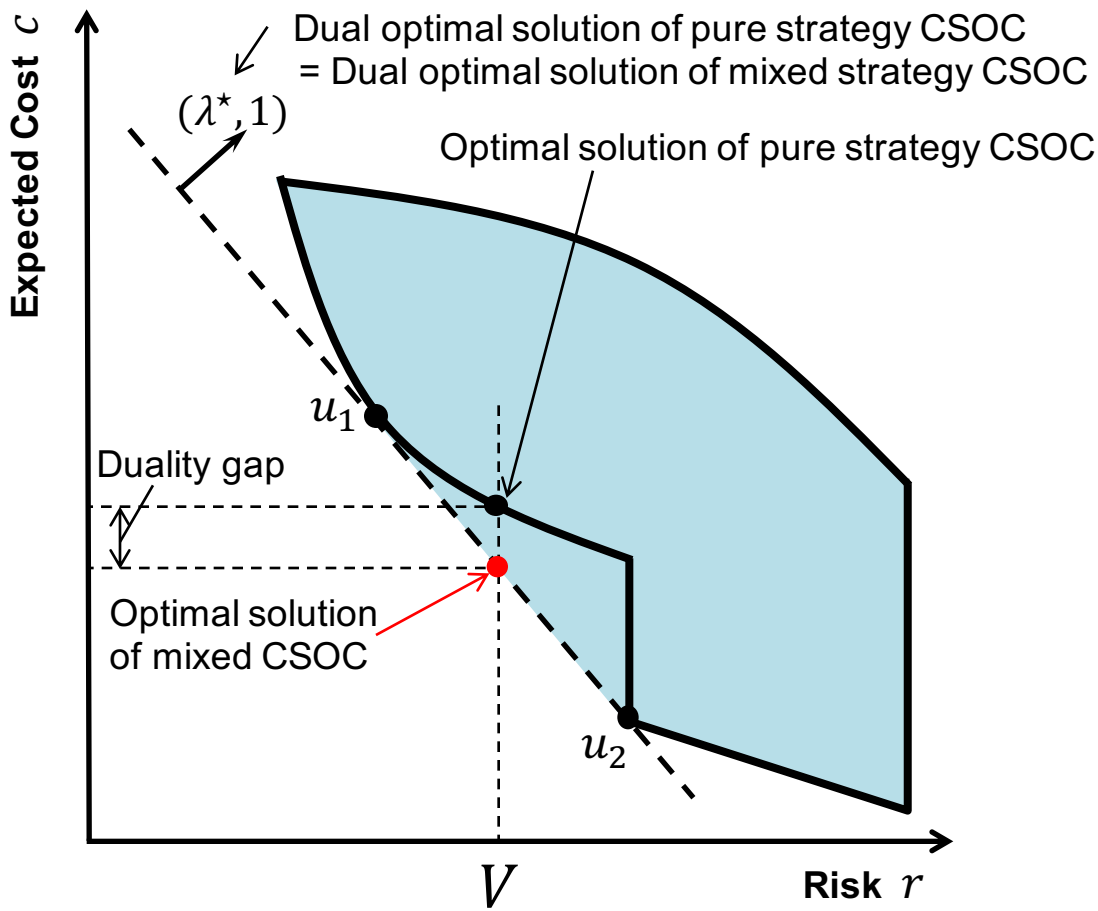
Strong Duality of Mixed Strategy CSOC

Remark 3

Mixed-strategy CSOC problem does not have a duality gap

Remark 4

The optimal mixed control strategy can consist of up to two pure control strategies



Solution Approach

- A naïve approach (for N-randomization): co-optimize $u^1 \cdots u^{N+1}$ and $p^1 \cdots p^{N+1}$
  - Difficult to solve, if not impossible…

$$c_{\mathrm{M}}^{\star N} = \min_{\substack{u^1 \cdots u^{N+1} \in \mathcal{U}^T \\ \sum_{j=1}^{N} p^j = 1,\ p^j \geq 0 \\ g(x,u,w)=0}} \sum_{j=1}^{N+1} p^j \mathbb{E}\left[f_0(x, u^j)\right]$$

$$s.t. \sum_{j=1}^{N+1} p^j \mathbb{E}\left[f_i(x, u^j)\right] \leq V_i$$

$$i = 1 \cdots K$$

Theorem 3: Condition for Optimality

- Necessary and Sufficient Condition

$$
\left.
\begin{aligned}
&a)\ c^i \in C(\lambda) \ \vee \ p^i = 0,\ \forall i = 1 \cdots K+1 \\
&b)\ \lambda \left(\sum_{i=1}^{K+1} p^i c^i_{1:K} - V \right) = 0, \\
&c)\ \sum_{i=1}^{K+1} p^i = 1, \\
&d)\ p^i \geq 0\ \forall i = 1 \cdots K+1, \\
&e)\ \sum_{i=1}^{K+1} p^i c^i_{1:K} \leq V, and \\
&f)\ c^i \in \mathcal{F},\ \forall i = 1 \cdots K+1.
\end{aligned}
\right\}
$$

KKT conditions

Solution Approach

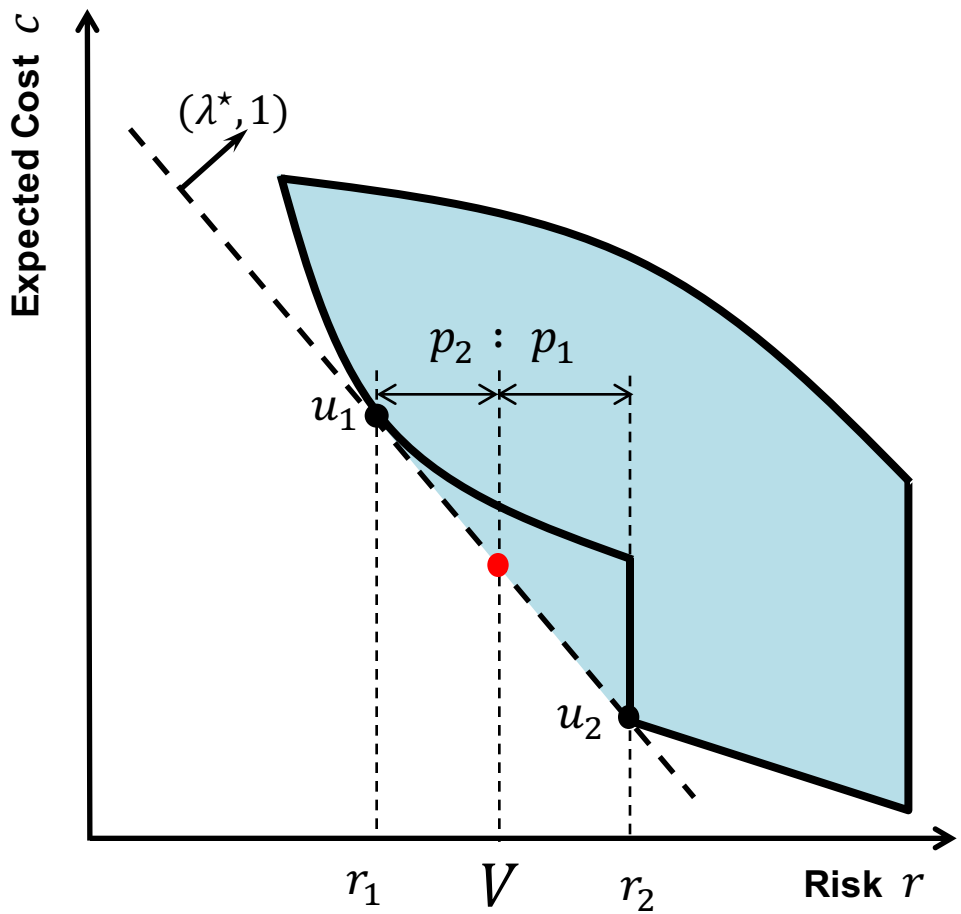
1. Solve the dual of the *original* (non-randomized) problem
   - Because the dual of the original and mixed problems are the same!
2. If $\lambda^* = 0$ (stochastic constraints are inactive)
   - The optimal solution to the original problem is also optimal for the mixed problem
3. If $\lambda^* > 0$
   - Get the primal solutions $u_1 \cdots u_{K+1}$ that minimizes the Lagrangian with λ^*
   - Get $p^1 \cdots p^{N+1}$ that makes the equality in the stochastic constraints hold:

$$\sum_{j=1}^{N+1} p^j \mathbb{E}\left[f_i(x, u^j)\right] = V_i \qquad \text{and} \qquad \sum_{j=1}^{N+1} p^j = 1$$

$$i = 1 \cdots K$$

Computing probability distribution

Get $p^1 \cdots p^{N+1}$ that makes the equality in the stochastic constraints hold



Special Case: K=1

- If there is only one stochastic constraint, the dual optimization can be solved efficiently by **root finding**
  - E.g., Joint chance-constrained optimal control

Special case with K=1

Dual Problem

$$\max_{\lambda} \min_{u \in \mathcal{U}} \mathbb{E}[f_0(x,u)] + \lambda(\mathbb{E}[f_1(x,u)] - \mathrm{V})$$

$$\|$$

$$q(\lambda)$$

Primal Problem

$$\min_{u \in \mathcal{U}} \mathbb{E}[f_0(x,u)]$$

$$\text{s.t. } \mathbb{E}[f_1(x,u)] \leq V$$

Special case with K=1

Dual Problem

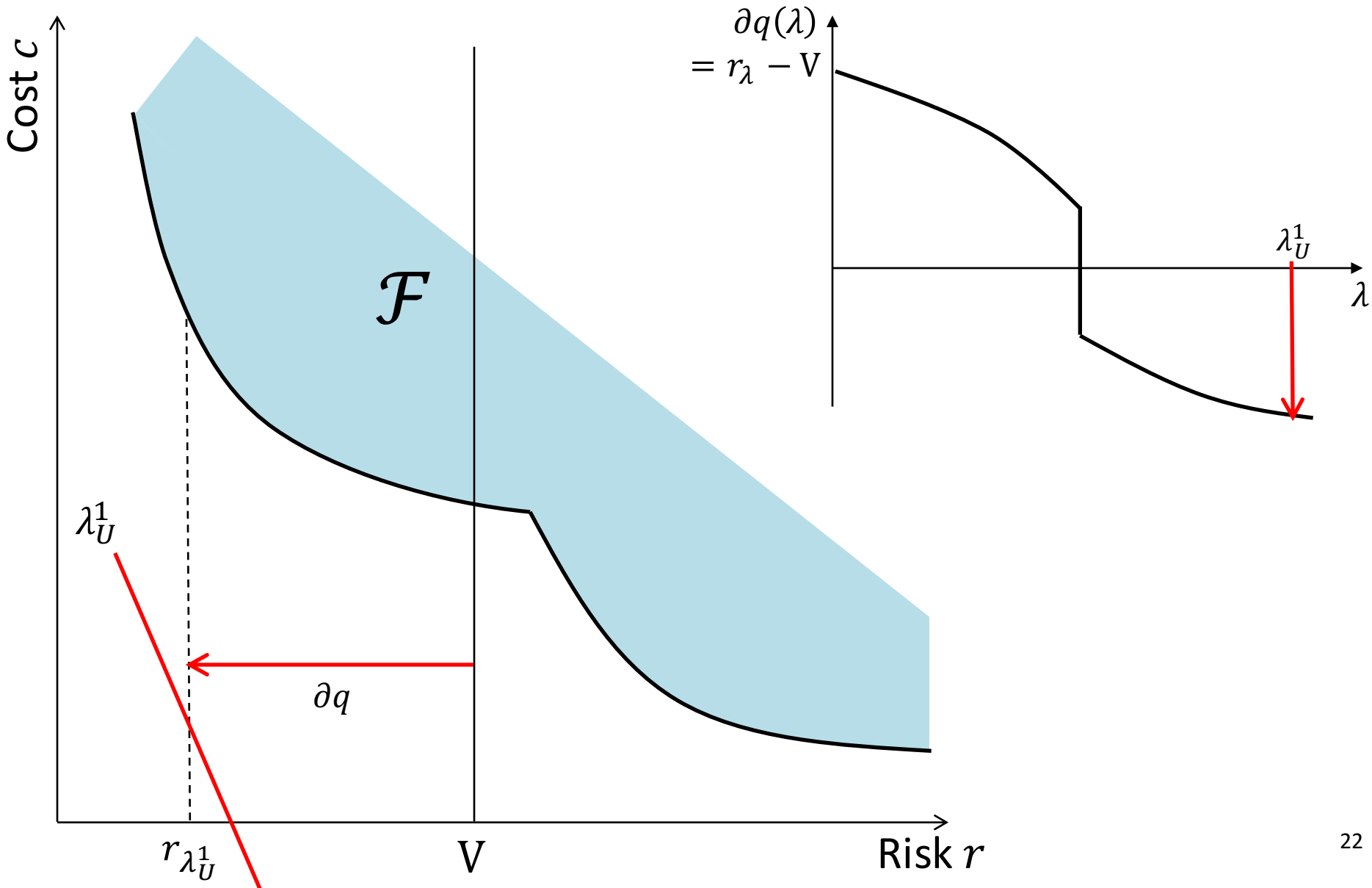
$$\max_{\lambda} \min_{u \in \mathcal{U}} \mathbb{E}[f_0(x,u)] + \lambda(\mathbb{E}[f_1(x,u)] - \mathrm{V})$$

$$\|$$

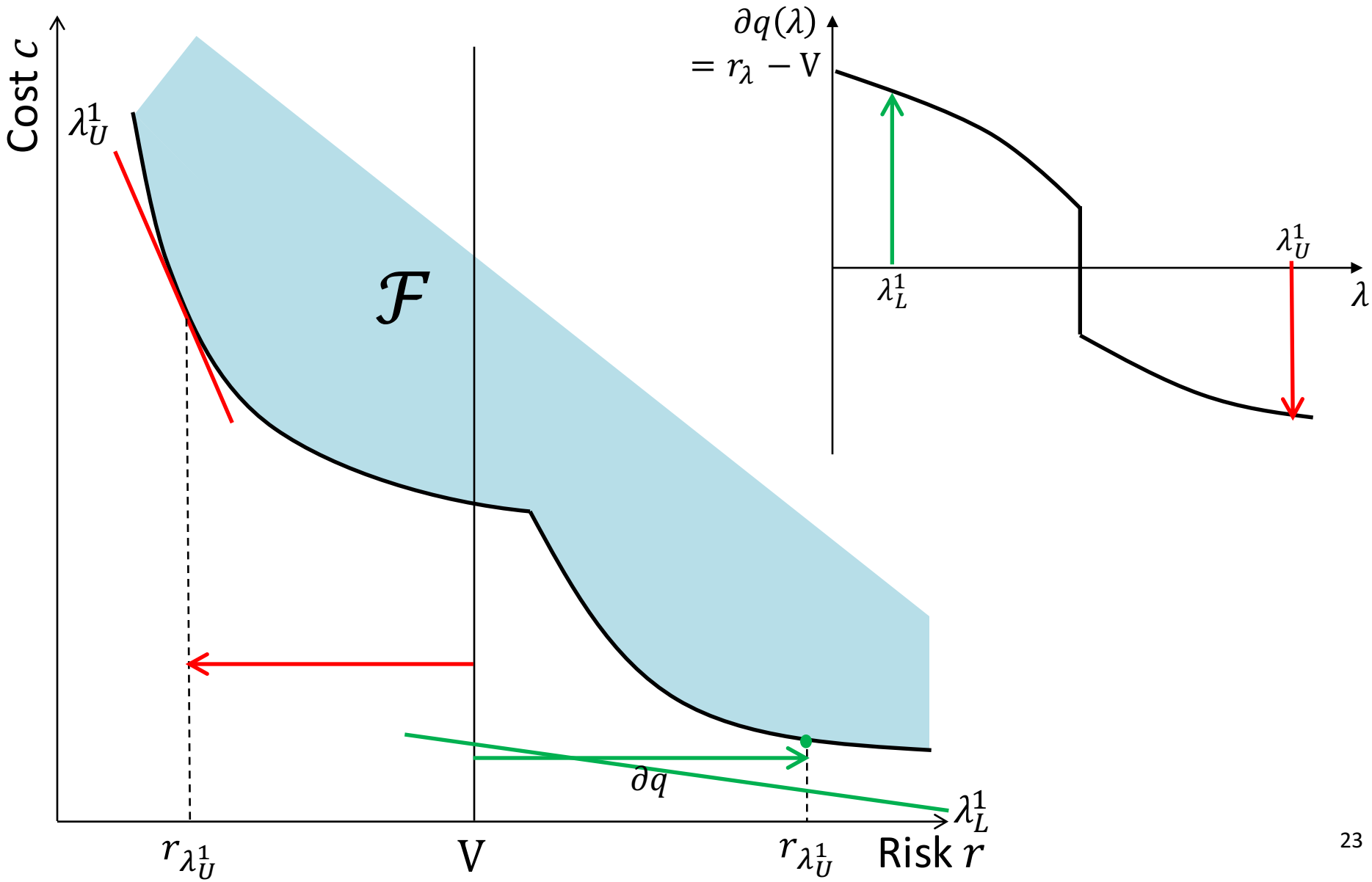
$$q(\lambda)$$

- $q(\lambda)$ is always concave
- Optimality condition: $0 \in \partial q(\lambda)$
  - $\partial q(\lambda)$: subgradient
- Therefore, dual optimization is reduce to a zero-finding problem over $\partial q(\lambda)$

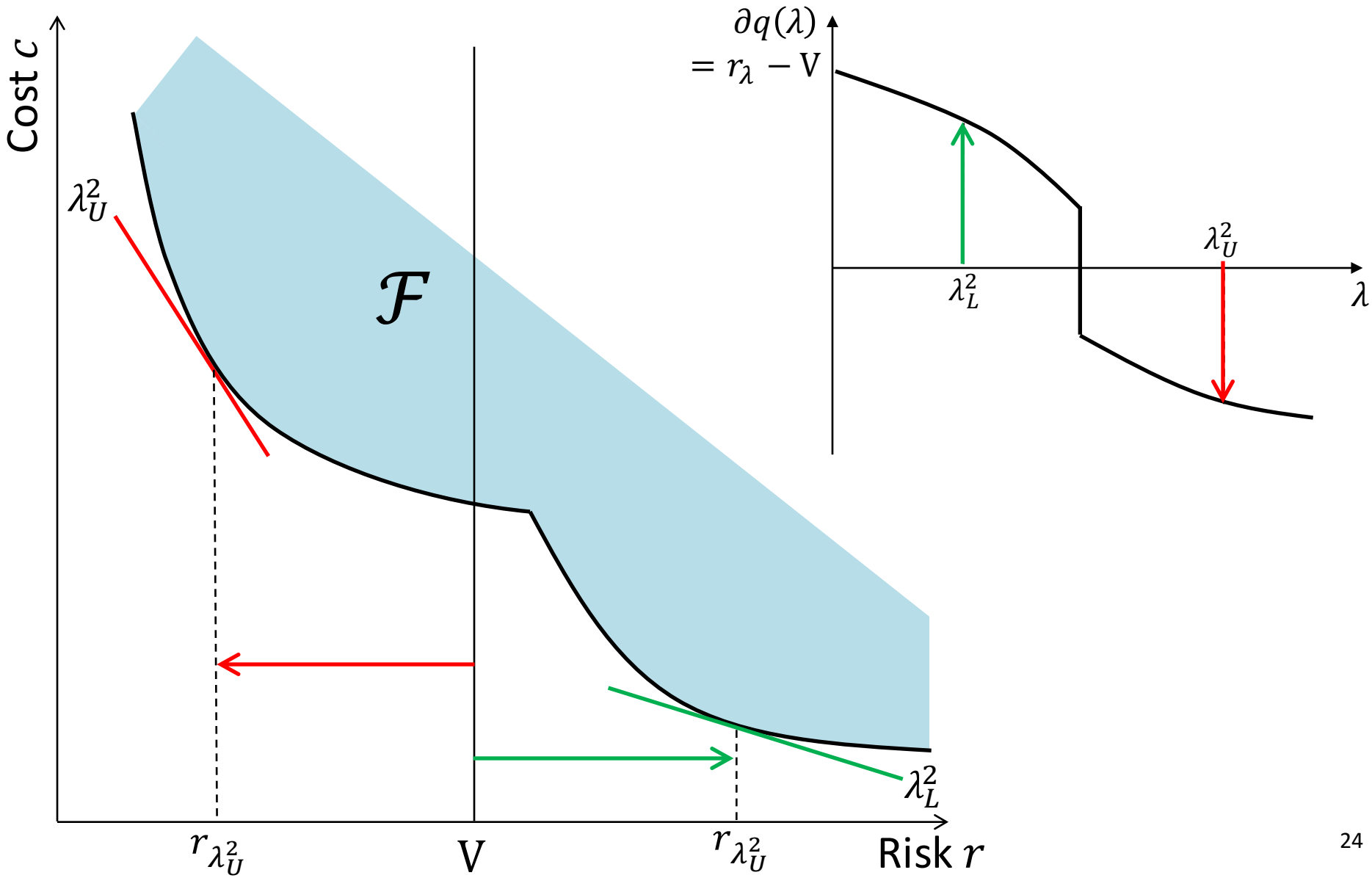
Dual Solution through Root Finding



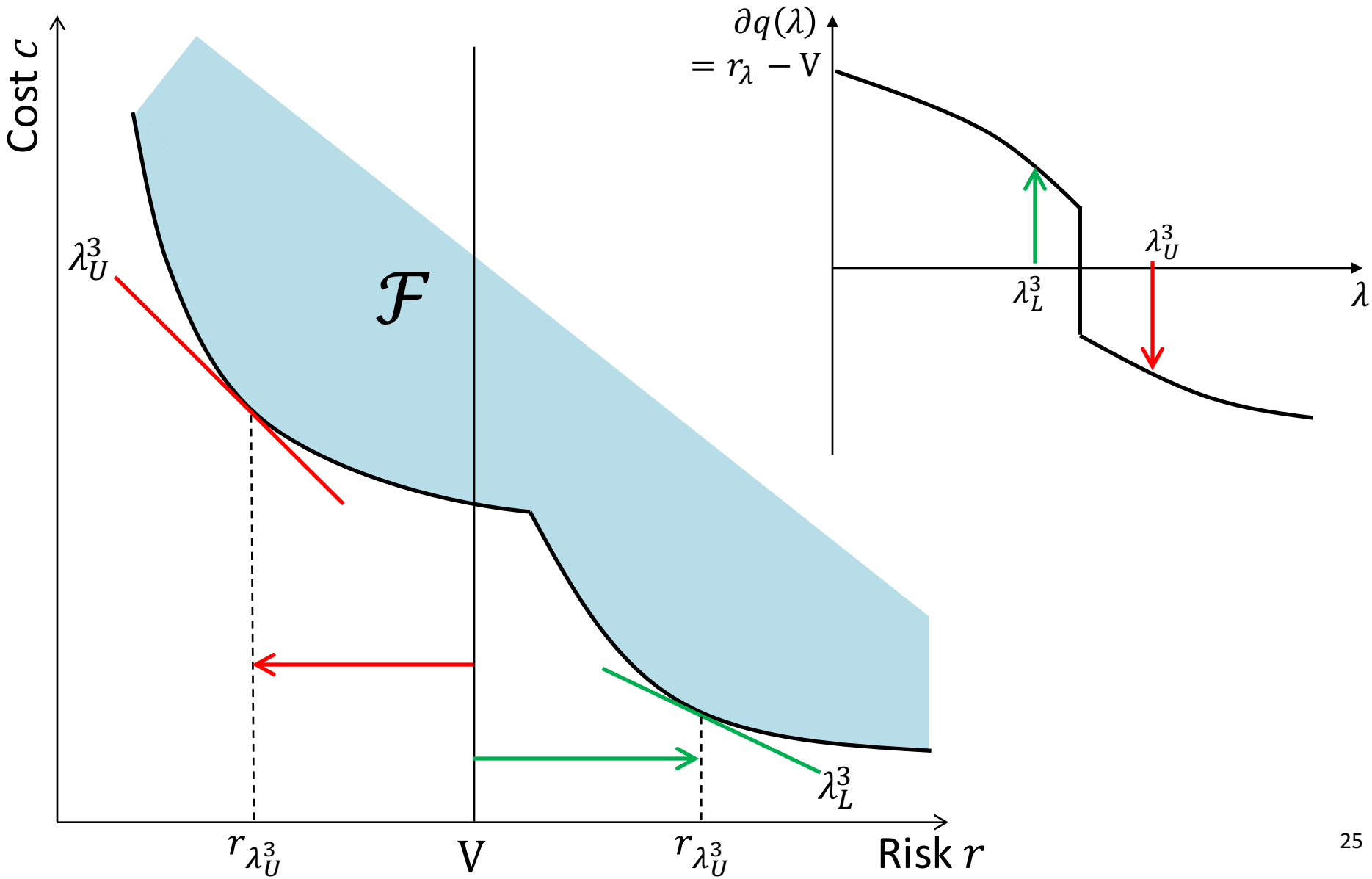
Dual Solution through Root Finding



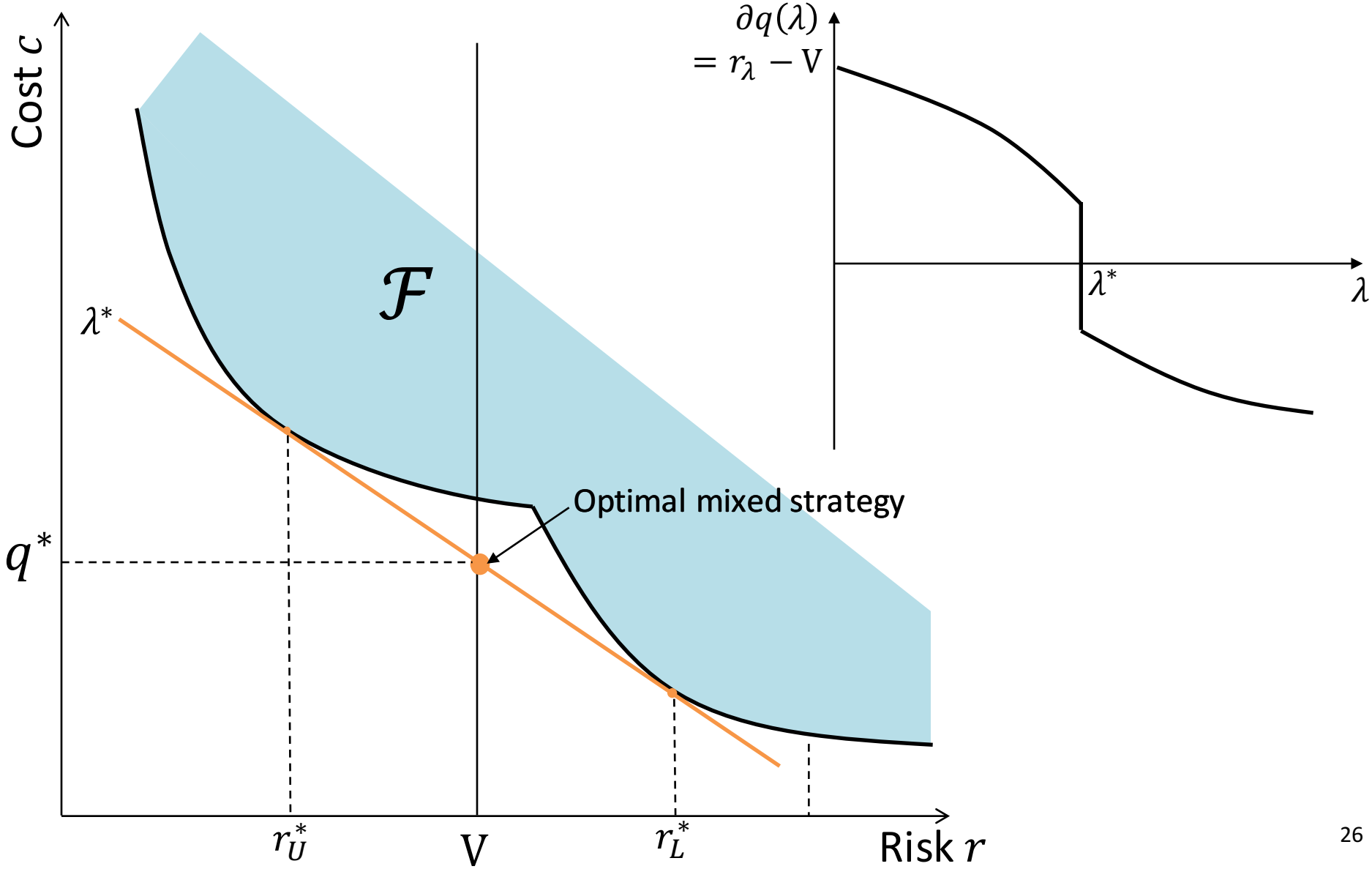
Dual Solution through Root Finding



Dual Solution through Root Finding



Dual Solution through Root Finding



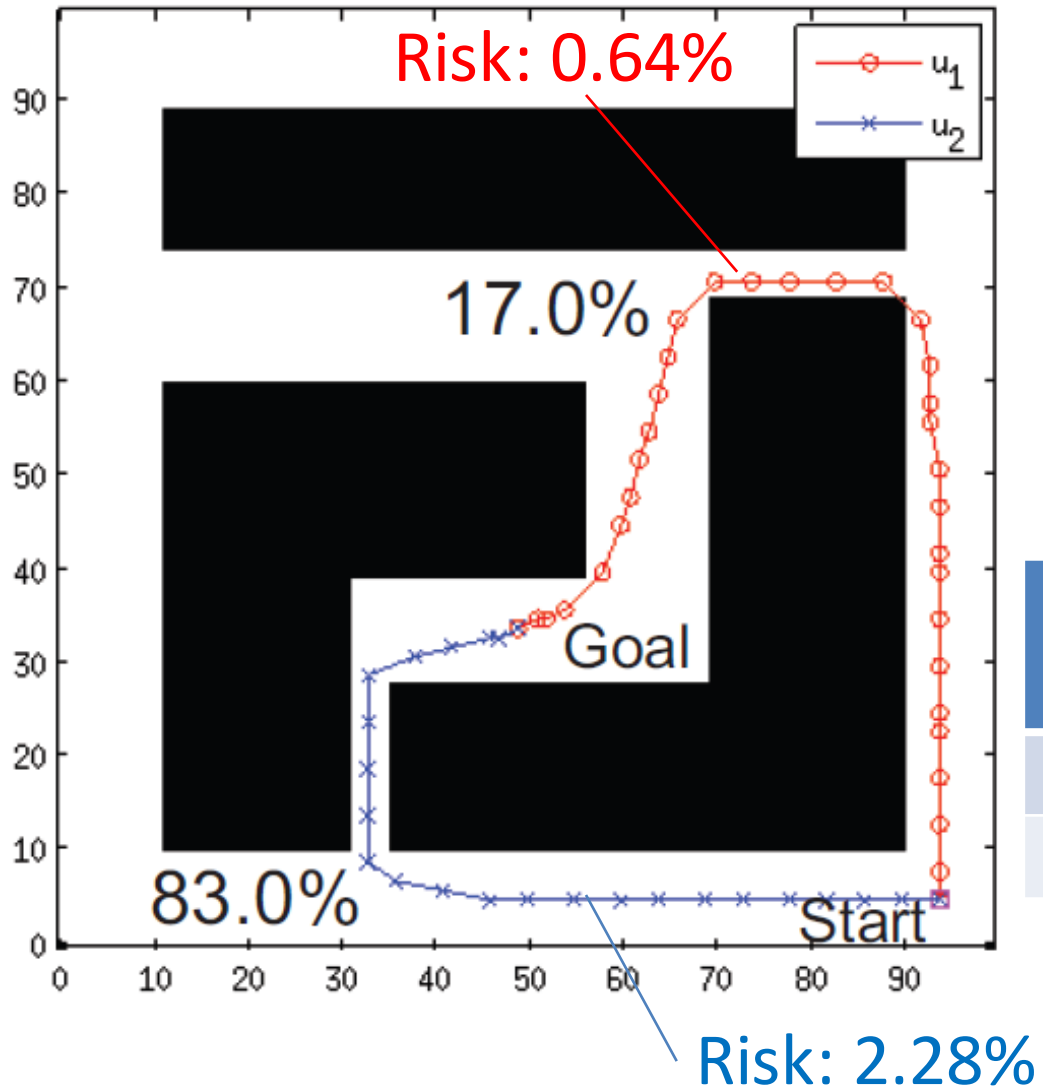
Implementation

- Can be used with any chance-constrained optimal control problem

- Implemented on chance-constrained dynamic programming* and MPC**

*Ono, Kuwata, Baralam, "Joint chance-constrained dynamic programming," CDC-12
**Ono, "Control by Coin Flips: Mixed Strategy for Stochastic MPC," Submitted to ACC-16

Application to Path Planning



Problem:

- Minimize path length
- Risk $\leq$ 2%

- $x_{k+1} = x_k + u_k + w_k$
- 100x100 discrete state space

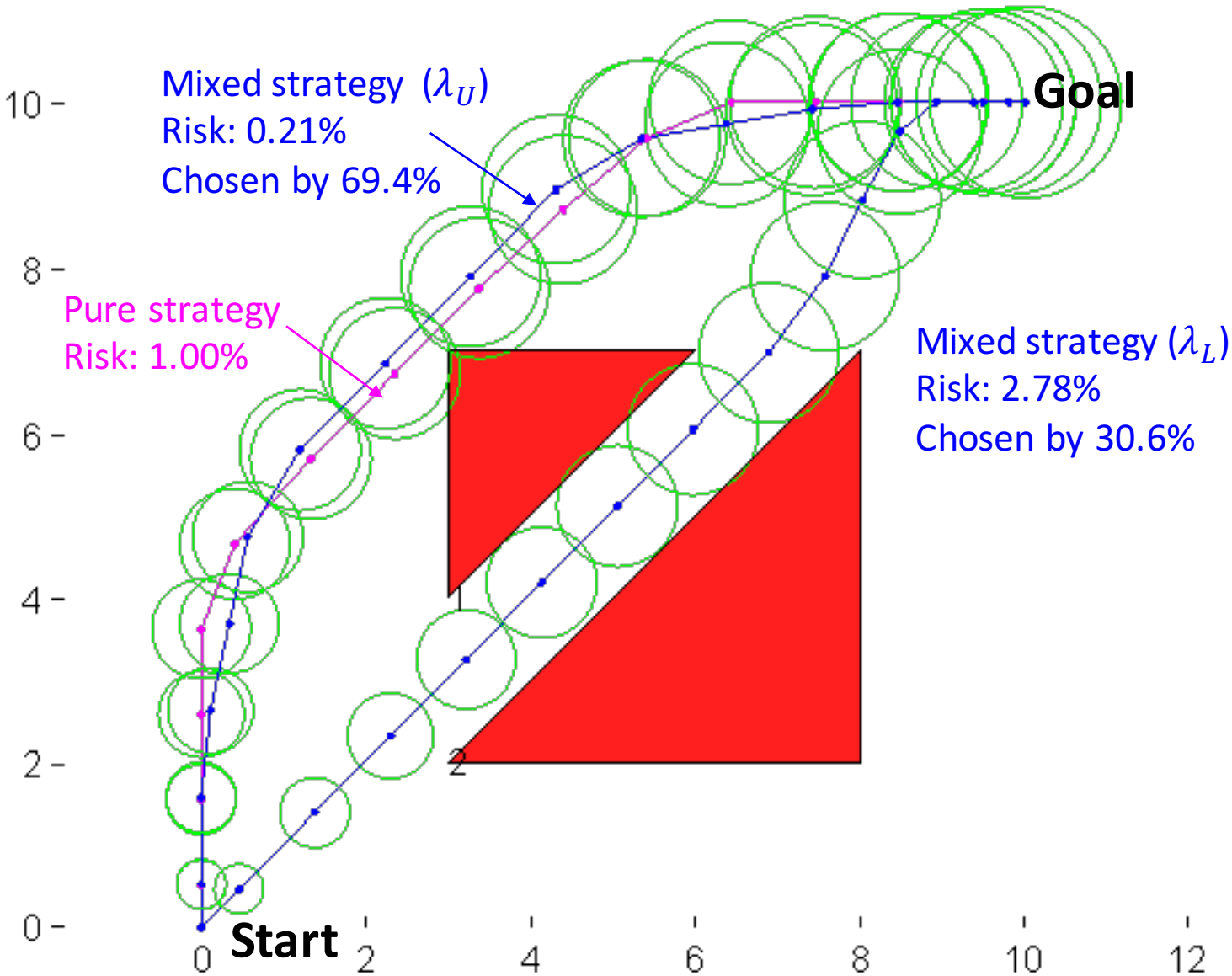
Control strategy	Expected path length	Risk
Pure	130.8	0.64%
Mixed	104.2	2.0%

Piecewise Linear Approximation of SMPC

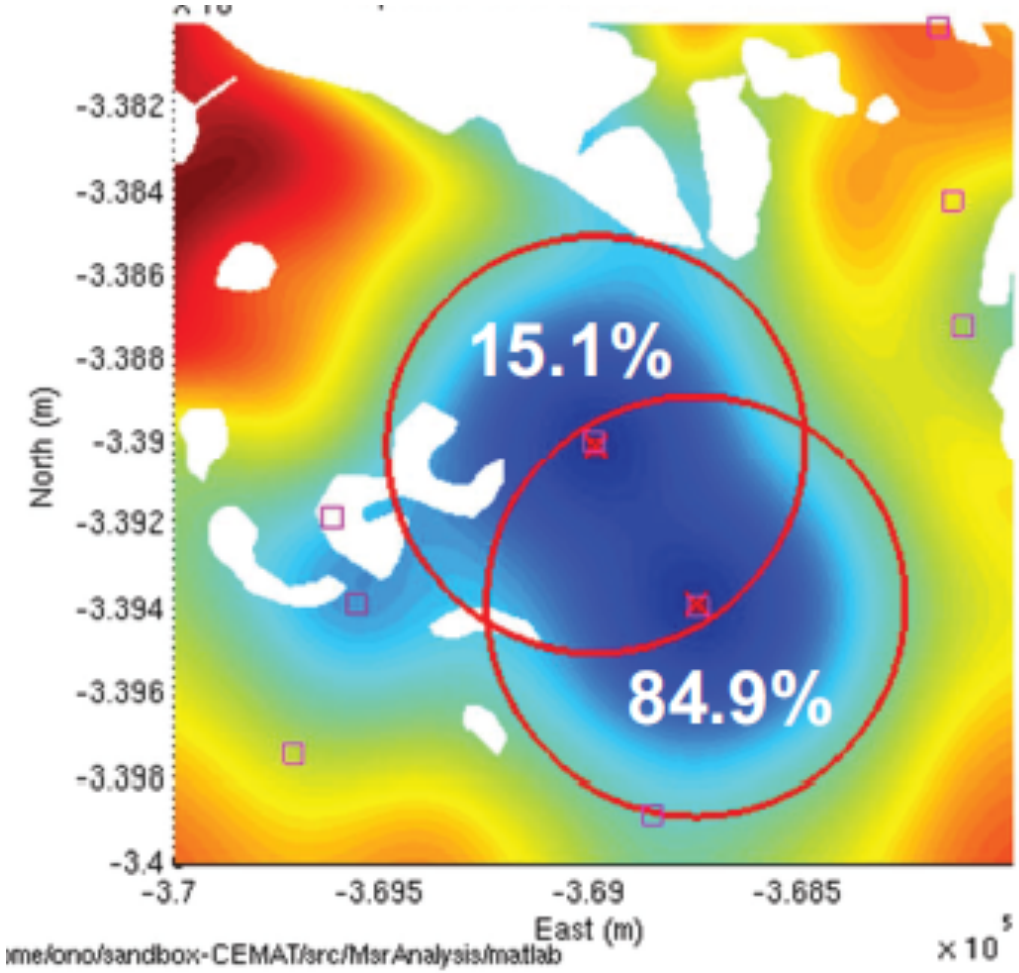
MILP

$$
\begin{aligned}
\min \ & \sum_{k=1}^{N}\sum_{i=1}^{m} v_{ki} + \lambda\left(\sum_{i=1}^{M}\sum_{k=1}^{N}\delta_{ik} - V\right) \\
\text{s.t. } & v_{kd} \geq u_{kd}, \ v_{kd} \geq -u_{kd}, Pu_k \preceq q \\
& \bar{x}_{k+1} = A\bar{x}_k + Bu_k + w_k \\
& a_l\left(\frac{h_{ij}\bar{x}_k - g_{ij}}{h_{ij}\Sigma_{x_k}h_{ij}^T}\right) + b_l \leq \delta_{ik} + \mathcal{M}z_{ij} \\
& \sum_{j=1}^{R_i} z_{ij} \leq R_i - 1, \quad z_{ij} \in \{0,1\} \\
& k = 1\cdots N, d = 1\cdots m, \ i = 1\cdots M, \\
& j = 1\cdots R_i, \ l = 1\cdots L.
\end{aligned}
$$

Path Planning with CCMPC



Mars Landing Scenario



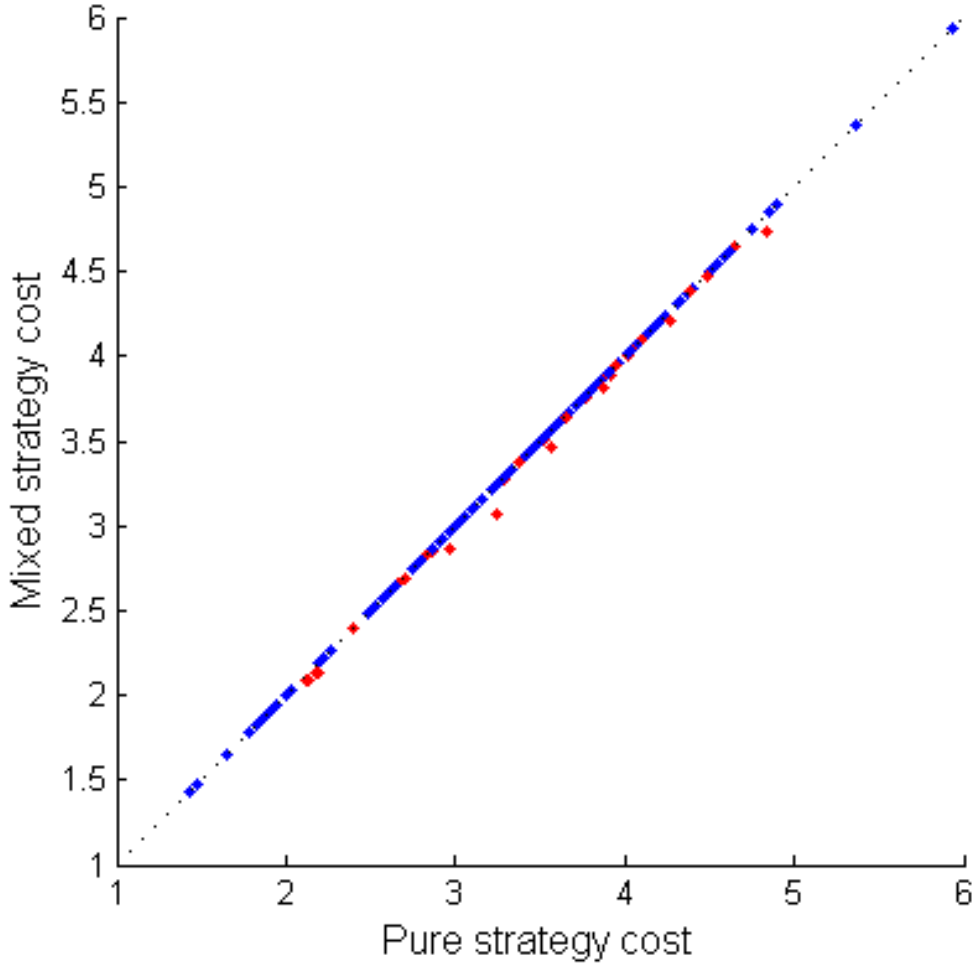
Problem:

- Minimize driving distance to visit two science targets after landing
- Risk $\leq$ 0.1%

- 2000x2000 discrete state space
- Used terrain data at E. Margaritifer on Mars

Control strategy	Expected cost	Risk
Pure	645.49	0.016%
Mixed	644.81	0.1%

Validation with Monte Carlo Simulation

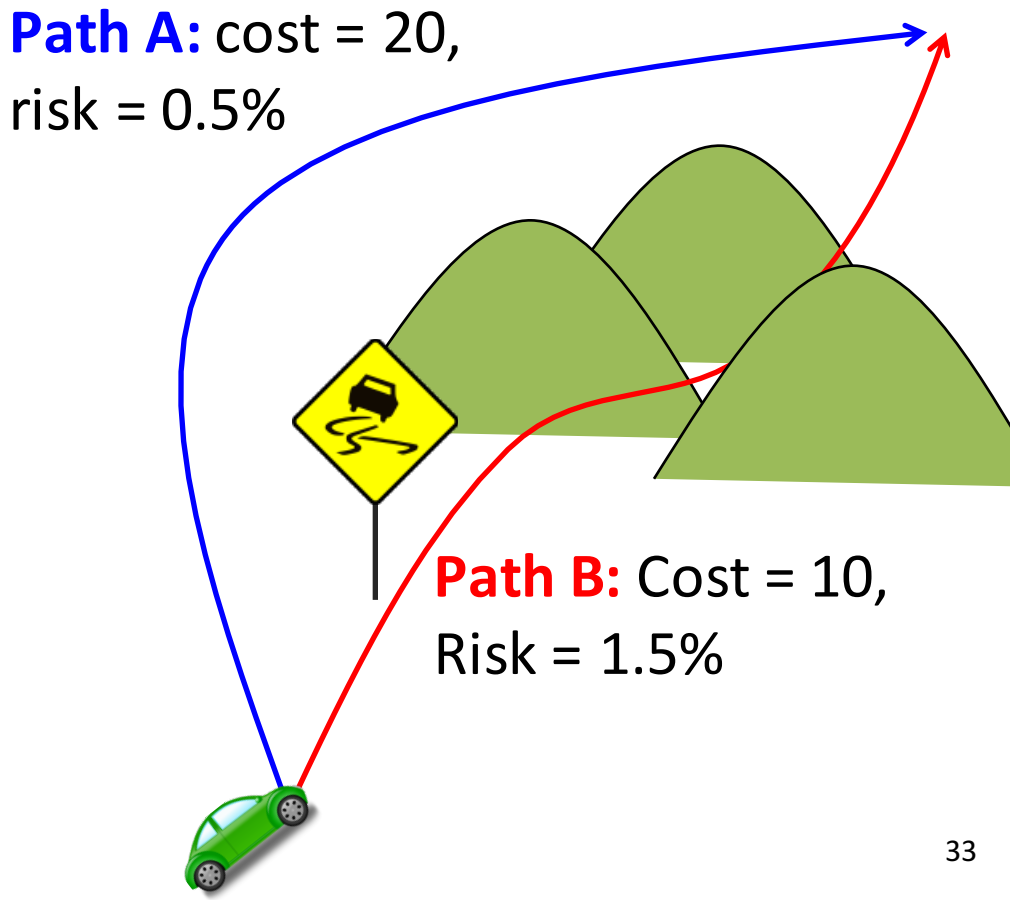


- 200 CCMPC problems with random obstacle locations
- Mix solutions outperforms in red dots
- No pure solutions outperforms the mixed ones

A Paradox

- Before the coin flip, risk is 1%
- But after seeing the result of the coin flip, the risk is NOT 1% any more!
- If you *know* that Path B is chosen, the chance constraint is violated!

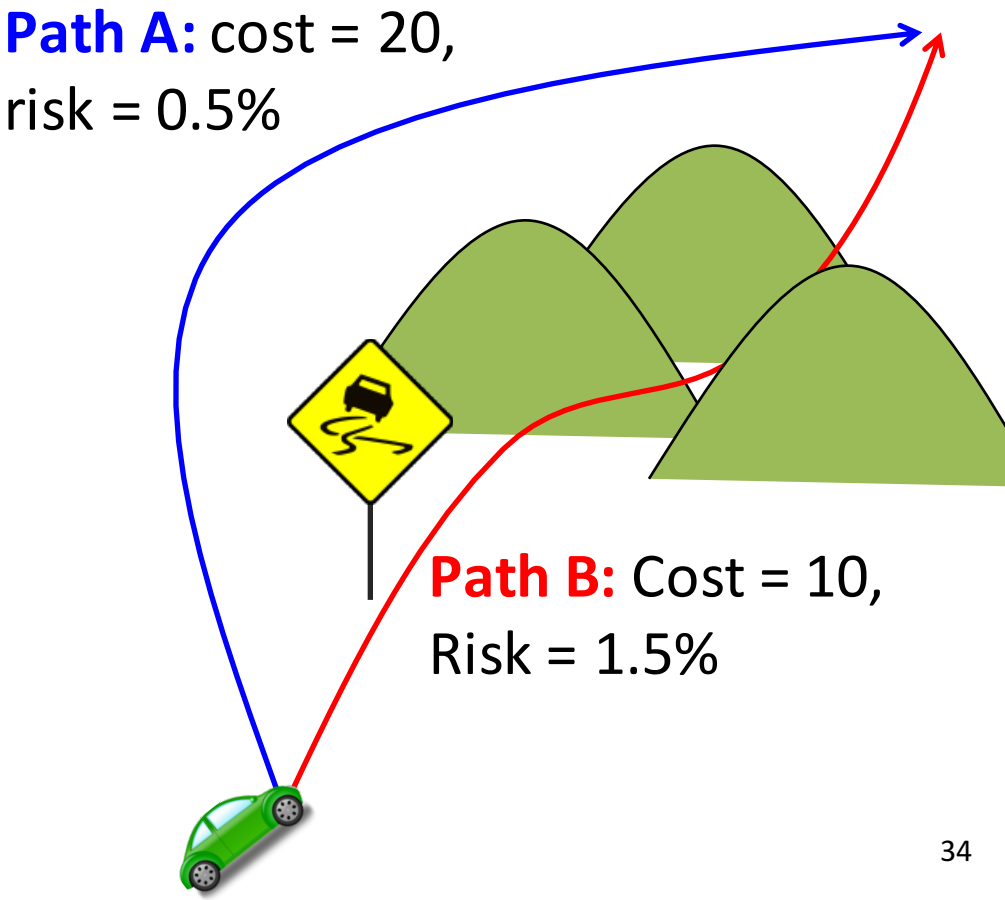
Optimal mixed strategy
Path A: 50%, Path B: 50%
Expected cost = 15, risk = 1%



Solution to the Paradox

- Don't let the operator know the result of the coin flip
  - Then the probability of failure stays at 1%
- Like Schrödinger's cat, the vehicle's state is a superimposition of Path A and Path B

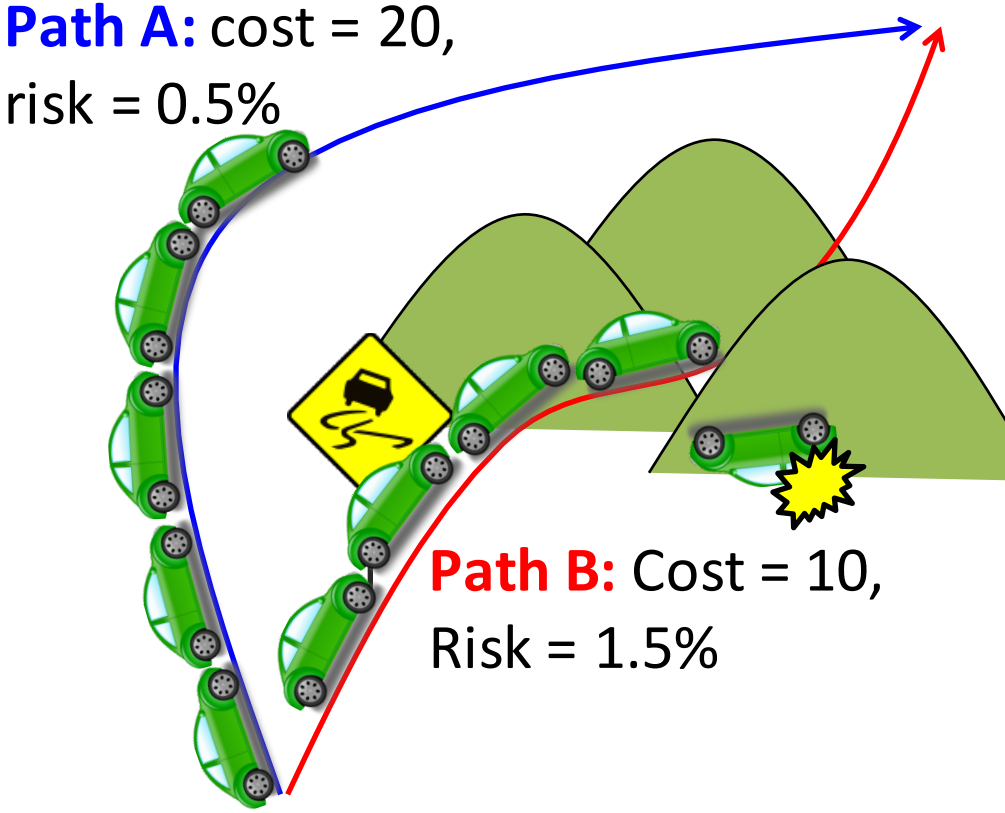
Optimal mixed strategy
Path A: 50%, Path B: 50%
Expected cost = 15, risk = 1%



Multi-agent Interpretation

- If there are 100 homogeneous agents
  - Minimize ***total*** cost while ensuring that 99 of them are expected to be safe
- Optimal strategy: send 50 to Path A, send 50 to Path B.

Optimal mixed strategy
Path A: 50%, Path B: 50%
Expected cost = 15, risk = 1%



Conclusions

- In a nonconvex constrained stochastic optimal control problem, randomizing control may result in a reduced expected cost
- If there are *K* stochastic constraints, *K*-randomization is sufficient
- Developed a solution approach to the mixed strategy control through dual optimization
- Developed an efficient solution approach to a problem with K=1 using root finding

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  - Office of Naval Research Grant N00014-15-IP-00052