(Preprint) AAS 15-730

EFFICIENT TRAJECTORY PROPAGATION FOR ORBIT DETERMINATION PROBLEMS

Javier Roa* **and Jesús Peláez**†

Regularized formulations of orbital motion apply a series of techniques to improve the numerical integration of the orbit. Despite their advantages and potential applications little attention has been paid to the propagation of the partial derivatives of the corresponding set of elements or coordinates, required in many orbit-determination scenarios and optimization problems. This paper fills this gap by presenting the general procedure for integrating the state-transition matrix of the system together with the nominal trajectory using regularized formulations and different sets of elements. The main difficulty comes from introducing an independent variable different from time, because the solution needs to be synchronized. The correction of the time delay is treated from a generic perspective not focused on any particular formulation. The synchronization using time-elements is also discussed. Numerical examples include strongly-perturbed orbits in the Pluto system, motivated by the recent flyby of the New Horizons spacecraft, together with a geocentric flyby of the NEAR spacecraft.

INTRODUCTION

The propagation of the variational equations of orbital motion is required in most orbit determination problems, situational awareness scenarios, numerical searching methods, computation of periodic orbits, *etc.* The solution to the variational equations determines how sensitive the solution is to initialization errors or to uncertainties in the physical parameters.[1] The linear propagation of the covariance matrix is performed by means of the state-transition matrix,[2] which is solved from the variational equations. Introducing a different set of variables to describe the motion requires additional transformations for computing the state-transition matrix and for propagating the covariance. Vallado[3] analyzes the performance of the transformation involving the equinoctial and spherical variables, as well as the accuracy of the covariance estimation.[4]

Regularized formulations of orbital motion take advantage of different techniques for removing the $r \to 0$ singularity in the equations of motion in Cartesian coordinates and to improve the stability of the integration. Different approaches to regularizing the equations of orbital motion can be found in the literature. Levi-Civita introduced a transformation in the complex plane to formulate the planar problem, avoiding the singularity associated to a direct impact with the attractive center. The Levi-Civita variables were extended to the three dimensional space by Kustaanheimo and

*PhD Candidate, Space Dynamics Group, Technical University of Madrid, Pza. Cardenal Cisneros 3, 28040, Madrid. Student Member AIAA. javier.roa@upm.es
Present address: Jet Propulsion Laboratory, California Institute of Technology, 4800 Oak Grove Drive, Pasadena, CA 91109-8099, USA
†Professor and Head, Space Dynamics Group, Technical University of Madrid, Pza. Cardenal Cisneros 3, 28040, Madrid. Member AIAA.

Stiefel.[5] Burdet, based on previous work by Sperling,[6] embedded the Laplace-Runge-Lenz vector and the energy in the equations of motion arriving to a fully regularized formulation. Both the Kustaanheimo-Stiefel (KS) and the Sperling-Burdet (SB) regularizations succeed in transforming the nonlinear and strongly coupled system of equations defining Keplerian motion into a linear and decoupled system, which greatly improves the numerical stability. The advantages hold when introducing perturbations. Deprit *et al.*[7] wrote an exhaustive essay on linearization. The practical use of integrals of motion was explored by Milankovitch,[8,9] who reduced the problem to propagating the Laplace-Runge-Lenz and the angular momentum vectors, together with an angular measurement. Baumgarte[10] and Janin[11] stabilized Cowell's method by introducing a fictitious time and embedding the energy into the equations of motion. Based on the concept of Hansen ideal frames Peáez *et al.*[12] formulated the special perturbations method called Dromo. A modified version for propagating hyperbolic orbits has recently been published.[13,14]

The physical time is replaced by a fictitious time by means of the Sundman transformation. The fictitious time behaves as an analytic step-size adaption that improves the discretization of the orbit. A root-finding algorithm is required to detect when the physical time reaches its target value. Despite the many advantages of these more sophisticated propagators very few applications to real problems can be found. For example, little attention has been paid to how the partial derivatives of the state vector are propagated adopting this formalism. The goal of this paper is to derive the general theory that makes it possible to propagate the partial derivatives together with the reference trajectory. It is formulated from a generic perspective and not focused on any particular formulation. The main difficulty comes from the synchronism of the solutions. Roa and Peláez have studied this phenomenon in the context of spacecraft relative motion.[15,16,17,18]

The Paper is organized as follows. The first section is devoted to the governing equations of motion, with an emphasis on the variational formulation. The concept of synchronism is presented and its relevance when propagating the state-transition matrix is discussed. The section closes with the required transformations for propagating the covariance matrix attached to a certain set of elements or coordinates. The steps for integrating the variational equations and propagating the covariance matrix are presented in the following section. Next, two practical considerations are discussed: the importance of the initialization of the independent variable and the computation of the time delay from a time-element. The theory is validated by integrating two elliptic orbits in the Pluto system, and a geocentric flyby.

DYNAMICS

Let $\mathbf{r}, \mathbf{v} \in \mathbb{R}^3$ denote the position and velocity vectors of a particle immerse in a central gravitational field, respectively. Its dynamics is defined by the initial value problem (IVP)

$$\begin{cases} \dfrac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}\,t^2} = -\dfrac{\mu}{r^3}\,\mathbf{r} + \mathbf{a}_\mathrm{p} \\ \mathbf{r}(t_0) = \mathbf{r}_0, \quad \mathbf{v}(t_0) = \mathbf{v}_0 \end{cases} \tag{1}$$

where $r = \|\mathbf{r}\|$, μ is the gravitational parameter of the attracting body, t is the physical time and $\mathbf{a}_\mathrm{p}$ denotes the perturbing acceleration. The components of the position and velocity vectors of the particle form the state vector in Cartesian coordinates $\mathbf{x}(t; \mathbf{x}_0) \in \mathbb{R}^6$. When introducing the state

vector $\mathbf{x}(t;\mathbf{x}_0)$ the second order system given in Eq. (1) reduces to the first order system

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{x}}{\mathrm{d}\,t} = \mathbf{f}(t;\mathbf{x},\mathbf{a}_\mathrm{p}) \\ \mathbf{x}(t_0) = \mathbf{x}_0 \end{cases} \tag{2}$$

Although there are integration schemes that deal with second order systems, such as the Störmer-Cowell or the Runge-Kutta-Nyström, the representation of the system given in Eq. (2) is preferred for many applications.

Consider a set of elements* or generalized coordinates $\boldsymbol{\text{œ}} \in \mathbb{R}^n$ that alternatively defines the state of the particle, together with a generic, monotonically increasing independent variable ϑ. When the independent variable is different from the physical time then the time is defined as a function $t = t(\vartheta)$, and $t \in \boldsymbol{\text{œ}}$. In what follows we assume that $\boldsymbol{\text{œ}} = (\text{œ}_1, \text{œ}_2, \ldots, \text{œ}_n)^\top$ and $t = \text{œ}_1$. The IVP defined in Eq. (2) transforms into

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{\text{œ}}}{\mathrm{d}\vartheta} = \mathbf{g}(\vartheta;\boldsymbol{\text{œ}},\mathbf{a}_\mathrm{p}) \\ \boldsymbol{\text{œ}}(\vartheta_0) = \boldsymbol{\text{œ}}_0 \end{cases} \tag{3}$$

This new representation of the motion might be better suited for numerical integration, be regular, or enjoy interesting properties. Regularization is a broad line of research that looks for nonsingular sets of variables. Examples of such transformations of the dynamical system are the Gauss planetary equations, the Kustaanheimo-Stiefel (KS) transformation, the Sperling-Burdet regularization (SB), the Burdet-Ferrándiz approach, Dromo, Deprit's elements, and many others.

Consider the problem of propagating an orbit from an initial epoch t_0 to a final epoch t_1. If the independent variable is different from the physical time, the integrator requires a root-finding algorithm to solve for ϑ_1 in the equation

$$t_1 = t(\vartheta_1;\boldsymbol{\text{œ}}_0)$$

The solution to this equation determines when the integration stops. The final value of the independent variable relates to the rest of variables in the problem by means of

$$\vartheta_1 = \vartheta(t_1,\boldsymbol{\text{œ}}_0) \tag{4}$$

It is clear that a neighbor trajectory departing from $\boldsymbol{\text{œ}}_0 + \Delta\boldsymbol{\text{œ}}_0$ will reach t_1 with a different value of the independent variable,

$$\vartheta_2 = \vartheta(t_1,\boldsymbol{\text{œ}}_0 + \Delta\boldsymbol{\text{œ}}_0)$$

and the difference between both values is a function of the time t_1 and the initial conditions:

$$\Delta\vartheta = \Delta\vartheta(t_1,\boldsymbol{\text{œ}}_0,\Delta\boldsymbol{\text{œ}}_0) \tag{5}$$

For any given value of the independent variable ϑ, the function $\mathbf{q} : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^6$ transforms the set $\boldsymbol{\text{œ}}$ to the state vector, $\mathbf{x}$. That is,

$$\mathbf{x} = \mathbf{q}(\vartheta;\boldsymbol{\text{œ}}) \tag{6}$$

*According to Stiefel and Scheifele[19] an element either remains constant or grows linearly with the independent variable in the unperturbed problem.

Such transformation is typically nonlinear, and ideally the mapping is injective, although exceptions such as the KS transformation exist. The transformation must be invertible so that the initial values of the set $\boldsymbol{\text{œ}}_0$ can be obtained from the state vector:

$$\boldsymbol{\text{œ}}_0 = \mathbf{q}^{-1}(0; \mathbf{x}_0)$$

Some formulations require a particular treatment of the inverse map $\mathbf{q}^{-1}$. The KS transformation is a good example: a single point in $\mathbb{R}^3$ is transformed into a entire Hopf fibration in the KS space. More details on this particular Hopf map have been discussed by different authors.[20,17]

The variational equations

Consider a nominal trajectory, $\mathbf{x}(t; \mathbf{x}_0)$, and a neighbor trajectory, $\mathbf{x}(t; \mathbf{x}_0 + \Delta\mathbf{x}_0)$, defined by a small offset in the initial conditions $\|\Delta\mathbf{x}_0\| \ll \|\mathbf{x}_0\|$. The relative dynamics is represented by the differential state

$$\Delta\mathbf{x}(t) = \mathbf{x}(t; \mathbf{x}_0 + \Delta\mathbf{x}_0) - \mathbf{x}(t; \mathbf{x}_0) \tag{7}$$

A Taylor expansion about the nominal trajectory provides

$$\Delta\mathbf{x}(t) = \nabla_{\mathbf{x}_0}\mathbf{x}(t, t_0)\,\Delta\mathbf{x}_0 + \mathcal{O}(\|\Delta\mathbf{x}_0\|^2)$$

Here, the Jacobian matrix $\nabla_{\mathbf{x}_0}\mathbf{x}(t, t_0)$ is referred to as the state-transition matrix, $\mathcal{A}(t, t_0)$. In the linearized model it maps the initial separation $\Delta\mathbf{x}_0$ to the separation at any given time, $\Delta\mathbf{x}(t)$. The components of the state-transition matrix are

$$\mathcal{A}_{i,j} = \frac{\partial\, x_i}{\partial x_{0j}}$$

Clearly the initial state-transition matrix reduces to the identity matrix, $\mathcal{A}(t_0, t_0) = \mathbb{I}$, and therefore $\Delta\mathbf{x}(t_0) \equiv \Delta\mathbf{x}_0$. Under these conditions Eq. (7) can be rewritten in terms of the state-transition matrix,

$$\Delta\mathbf{x}(t) = \mathcal{A}(t, t_0)\,\Delta\mathbf{x}_0$$

The solution $\mathbf{x}(t; \mathbf{x}_0)$ to Eq. (2) can be differentiated with respect to both the time and the initial conditions,[1] resulting in the differential equations governing the evolution of the state-transition matrix

$$\begin{cases} \dfrac{\partial}{\partial t}\mathcal{A}(t, t_0) = \nabla_{\mathbf{x}}\mathbf{f}(t; \mathbf{x}, \mathbf{a}_{\mathrm{p}}) \circ \mathcal{A}(t, t_0) \\ \mathcal{A}(t_0, t_0) = \mathbb{I} \end{cases}$$

In practice this IVP is integrated simultaneously with Eq. (2). To clarify the notation the symbol $\circ$ denotes the standard product of two matrices.

The same discussion applies to any set of variables different from the Cartesian coordinates. Consider the relative dynamics between two neighbor solutions to Eq. (3), $\boldsymbol{\text{œ}}(\vartheta; \boldsymbol{\text{œ}}_0)$ and $\boldsymbol{\text{œ}}(\vartheta; \boldsymbol{\text{œ}}_0 + \Delta\boldsymbol{\text{œ}}_0)$. Assuming that the initial separation is small the linearized solution reads

$$\Delta\boldsymbol{\text{œ}}(\vartheta) = \boldsymbol{\text{œ}}(\vartheta; \boldsymbol{\text{œ}}_0 + \Delta\boldsymbol{\text{œ}}_0) - \boldsymbol{\text{œ}}(\vartheta; \boldsymbol{\text{œ}}_0) \simeq \mathcal{B}(\vartheta, \vartheta_0)\,\Delta\boldsymbol{\text{œ}}_0 \tag{8}$$

where $\mathcal{B}(\vartheta, \vartheta_0) = \nabla_{\boldsymbol{\text{œ}}_0}\boldsymbol{\text{œ}}$ is the state-transition matrix associated to the set of variables $\boldsymbol{\text{œ}}$. Its components are

$$\mathcal{B}_{i,j} = \frac{\partial\, \text{œ}_i}{\partial \text{œ}_{0j}}$$

The state-transition matrix at each integration step is solved from the IVP

$$\begin{cases} \dfrac{\partial}{\partial \vartheta}\mathcal{B}(\vartheta,\vartheta_0) = \nabla_{\boldsymbol{\varpi}}\mathbf{g}(\vartheta;\boldsymbol{\varpi},\mathbf{a}_\mathrm{p}) \circ \mathcal{B}(\vartheta,\vartheta_0) \\ \mathcal{B}(\vartheta_0,\vartheta_0) = \mathbb{I} \end{cases} \tag{9}$$

The variational equations in Eq. (9) are integrated together with Eq. (3).

Synchronism of the solution

The main difficulty one encounters when integrating the variational equations with an independent variable different from time is the *synchronism* of the solution. The right-hand side of the variational equations is evaluated at every step in ϑ, and not in t. Hence, the partial derivatives solved from the variational equations define the state-transition matrix for constant ϑ,

$$\mathcal{B}(\vartheta,\vartheta_0) = \nabla_{\boldsymbol{\varpi}_0}\boldsymbol{\varpi}\Big|_\vartheta \tag{10}$$

that is different from the state-transition matrix computed for constant time. These results are called the ϑ-synchronous and t-synchronous solutions, respectively. To clarify the synchronism of the solution all ϑ-synchronous results will be denoted $|_\vartheta$, whereas t-synchronous variables correspond to $|_t$. A time delay Δt appears in the ϑ-synchronous case. Indeed, provided that $t \in \mathbf{oe}(\vartheta)$ Eq. (8) shows that

$$\Delta t\Big|_\vartheta = t(\vartheta;\boldsymbol{\varpi}_0 + \Delta\boldsymbol{\varpi}_0) - t(\vartheta;\boldsymbol{\varpi}_0) = t_2 - t_1$$

The perturbed time is $t_2 = t_1 + \Delta t\,|_\vartheta$, whereas t_1 is simply the reference time for which the solution is to be given. It is barely necessary to note that $\Delta t\,|_t = 0$, due to the definition of synchronism. This delay must be corrected to recover the physical sense of the solution. It is obtained from

$$\Delta t\Big|_\vartheta = \nabla_{\boldsymbol{\varpi}_0} t\Big|_\vartheta \cdot \Delta\boldsymbol{\varpi}_0 \tag{11}$$

Note that the gradient $\nabla_{\boldsymbol{\varpi}_0} t|_\vartheta$ is simply the first row of the state-transition matrix $\mathcal{B}|_\vartheta(\vartheta,\vartheta_0)$ given in Eq. (10). The symbol $|_\vartheta$ is redundant in this case and will be omitted.

Equation (4) readily suggests that the independent variable ϑ can be seen in practice as a function of the form $\vartheta = \vartheta(t;\boldsymbol{\varpi}_0)$. Equation (9) can then be rewritten in terms of the physical time:

$$\Delta\boldsymbol{\varpi}\Big|_\vartheta = \boldsymbol{\varpi}(t_2;\boldsymbol{\varpi}_0 + \Delta\boldsymbol{\varpi}_0) - \boldsymbol{\varpi}(t_1;\boldsymbol{\varpi}_0)$$

The effect of the time delay can be observed in this equation: the variational solution $\Delta\boldsymbol{\varpi}(\vartheta)$ relates the set of elements at two different times, $t_2 \neq t_1$. Assuming that the time delay is small compared to the time scale of the problem* the variational solution can be referred to the reference time

$$\Delta\boldsymbol{\varpi}\Big|_t = \boldsymbol{\varpi}(t_2 - \Delta t;\boldsymbol{\varpi}_0 + \Delta\boldsymbol{\varpi}_0) - \boldsymbol{\varpi}(t_1;\boldsymbol{\varpi}_0) = \boldsymbol{\varpi}(t_2;\boldsymbol{\varpi}_0 + \Delta\boldsymbol{\varpi}_0) - \boldsymbol{\varpi}(t_1;\boldsymbol{\varpi}_0) - \frac{\mathrm{d}\boldsymbol{\varpi}}{\mathrm{d}\,t}(t_2;\boldsymbol{\varpi}_0)\,\Delta t + O(\Delta t^2)$$

or simply

$$\Delta\boldsymbol{\varpi}\Big|_t = \Delta\boldsymbol{\varpi}\Big|_\vartheta - \mathbf{g}(\vartheta;\boldsymbol{\varpi}_0,\mathbf{a}_\mathrm{p})\,\frac{\mathrm{d}\vartheta}{\mathrm{d}\,t}\,\Delta t \tag{12}$$

*This hypothesis holds for $\|\Delta\boldsymbol{\varpi}_0\| \ll \|\boldsymbol{\varpi}\|$ and for sufficiently short propagations.

The function $\mathbf{g} \in \mathbb{R}^n$ defines the right-hand side of Eq. (3). In addition, since $\text{œ}_1 = t$ then $\mathrm{d}\vartheta/\mathrm{d}t$ is the inverse of the first component of $\mathbf{g}$, g_1. The ϑ-synchronous solution is given by the state-transition matrix $\mathcal{B}|_\vartheta(\vartheta, \vartheta_0)$. With this result and the expression for the time delay given in Eq. (11) it follows

$$\Delta\boldsymbol{\text{œ}}\Big|_t = \mathcal{B}\Big|_\vartheta(\vartheta, \vartheta_0)\,\Delta\boldsymbol{\text{œ}}_0 - \mathbf{g}(\vartheta; \boldsymbol{\text{œ}}_0, \mathbf{a}_\mathrm{p})\,\frac{\mathrm{d}\vartheta}{\mathrm{d}\,t}\,(\nabla_{\boldsymbol{\text{œ}}_0} t \cdot \Delta\boldsymbol{\text{œ}}_0)$$

and therefore

$$\Delta\boldsymbol{\text{œ}}\Big|_t = \left[\mathcal{B}\Big|_\vartheta(\vartheta, \vartheta_0) - \frac{1}{g_1}\mathbf{g}(\vartheta; \boldsymbol{\text{œ}}_0, \mathbf{a}_\mathrm{p}) \otimes \nabla_{\boldsymbol{\text{œ}}_0} t\right]\Delta\boldsymbol{\text{œ}}_0 = \mathcal{B}\Big|_t(\vartheta, \vartheta_0)\,\Delta\boldsymbol{\text{œ}}_0$$

Here $\otimes$ denotes the dyadic product*. The t-synchronous state-transition matrix, $\mathcal{B}|_t(\vartheta, \vartheta_0)$, is obtained when correcting the time delay in the ϑ-synchronous state-transition matrix:

$$\mathcal{B}\Big|_t(\vartheta, \vartheta_0) = \mathcal{B}\Big|_\vartheta(\vartheta, \vartheta_0) - \frac{1}{g_1}\mathbf{g}(\vartheta; \boldsymbol{\text{œ}}_0, \mathbf{a}_\mathrm{p}) \otimes \nabla_{\boldsymbol{\text{œ}}_0} t \tag{13}$$

Recall that the gradient $\nabla_{\boldsymbol{\text{œ}}_0} t$ corresponds to the first row of $\mathcal{B}\Big|_\vartheta(\vartheta, \vartheta_0)$, and g_1 is the first component of $\mathbf{g}(\vartheta; \boldsymbol{\text{œ}}_0, \mathbf{a}_\mathrm{p})$. The matrix $\mathcal{B}|_t(\vartheta, \vartheta_0)$ propagates the variational solution from the initial state up to a certain time t,

$$\Delta\boldsymbol{\text{œ}}\Big|_t(t) = \mathcal{B}\Big|_t(\vartheta(t), \vartheta_0)\,\Delta\boldsymbol{\text{œ}}_0 \tag{14}$$

Linear mapping $\Delta\mathbf{x} \leftrightarrow \Delta\boldsymbol{\text{œ}}$

The change on the state vector given an initial separation is denoted by $\Delta\mathbf{x} = \Delta\mathbf{x}(t)$, as shown in Eq. (7). The t-synchronous solution is the only physically admissible and therefore the subscripts defining the synchronism are omitted when referred to $\Delta\mathbf{x}$, *i.e.* $\Delta\mathbf{x} \equiv \Delta\mathbf{x}|_t$. The state vector of the particle relates to $\boldsymbol{\text{œ}}$ by means of the transformation $\mathbf{q} : \boldsymbol{\text{œ}} \mapsto \mathbf{x}$ defined in Eq. (6). The solution at a given time $t = t_1$ then reads

$$\Delta\mathbf{x}(t_1) = \mathbf{q}(\vartheta_2(t_1); \boldsymbol{\text{œ}}_2) - \mathbf{q}(\vartheta_1(t_1); \boldsymbol{\text{œ}}_1)$$

In this expression $\vartheta_2 = \vartheta_1 + \Delta\vartheta|_t$, according to Eq. (5). In addition, $\boldsymbol{\text{œ}}_2 = \boldsymbol{\text{œ}}_1 + \Delta\boldsymbol{\text{œ}}|_t$. Assuming that both $\Delta\vartheta|_t$ and $\Delta\boldsymbol{\text{œ}}|_t$ are small the previous equation can be expanded about the solution at ϑ_1 to provide

$$\Delta\mathbf{x}(t_1) = \nabla_{\boldsymbol{\text{œ}}}\mathbf{q}\Big|_\vartheta(\vartheta_1; \boldsymbol{\text{œ}}_1)\,\Delta\boldsymbol{\text{œ}}\Big|_t + \frac{\partial\mathbf{q}}{\partial\vartheta}\,\Delta\vartheta\Big|_t + \ldots \tag{15}$$

The second term is a correction applied to the Jacobian of $\mathbf{q}$ to recover the t-synchronism of the solution. In what remains of the paper the gradient $\nabla_{\boldsymbol{\text{œ}}}\mathbf{q}\Big|_\vartheta(\vartheta_1; \boldsymbol{\text{œ}}_1)$ is denoted by $\boldsymbol{Q} = \nabla_{\boldsymbol{\text{œ}}}\mathbf{q}\Big|_\vartheta(\vartheta_1; \boldsymbol{\text{œ}}_1)$. Roa *et al.*[18] deduced the following identity:

$$\frac{\partial\mathbf{q}}{\partial\vartheta}\,\Delta\vartheta\Big|_t = -\frac{\partial\mathbf{q}}{\partial\,t}\,\Delta t\Big|_\vartheta$$

*The dyadic (or outer) product of two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)^\top$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)^\top$ defines the matrix

$$\mathbf{u} \otimes \mathbf{v} = \mathbf{u}\,\mathbf{v}^\top = \begin{bmatrix} u_1v_1 & u_1v_2 & \ldots & u_1v_n \\ u_2v_1 & u_2v_2 & & u_2v_n \\ \vdots & & \ddots & \\ u_nv_1 & u_nv_2 & & u_nv_n \end{bmatrix}$$

that combined sequentially with Eqs. (6) and (1) provides

$$\Delta\mathbf{x}(t_1) = \boldsymbol{Q}\,\Delta\boldsymbol{\text{œ}}\Big|_t - \frac{\partial\mathbf{x}}{\partial t}\,\Delta t\Big|_\vartheta = \boldsymbol{Q}\,\Delta\boldsymbol{\text{œ}}\Big|_t - \mathbf{f}(t_1;\mathbf{x}_0,\mathbf{a}_\mathrm{p})\,\Delta t\Big|_\vartheta \tag{16}$$

For transformations not depending explicitly on the independent variable ϑ, $\mathbf{q} = \mathbf{q}(\boldsymbol{\text{œ}})$, the t-synchronous and ϑ-synchronous Jacobians are the same, the Jacobian needs no additional corrections and the relative state vector reduces to

$$\mathbf{q} = \mathbf{q}(\boldsymbol{\text{œ}}) \implies \Delta\mathbf{x}(t_1) = \boldsymbol{Q}\,\Delta\boldsymbol{\text{œ}}\Big|_t \tag{17}$$

This result follows naturally from Eq. (15) provided that $\partial\mathbf{q}/\partial\vartheta = 0$.

Given an initial separation $\Delta\mathbf{x}_0$ the relative dynamics can be propagated using the state-transition matrix $\mathcal{A}(t,t_0)$:

$$\Delta\mathbf{x}(t) = \mathcal{A}(t,t_0)\,\Delta\mathbf{x}_0$$

Alternatively, the relative state vector can be propagated using the set $\boldsymbol{\text{œ}}$ thanks to Eq. (16). This equation can be rewritten as

$$\Delta\mathbf{x}(t) = \boldsymbol{Q}\left[\mathcal{B}\Big|_t(\vartheta(t),\vartheta_0)\,\Delta\boldsymbol{\text{œ}}_0\right] - \mathbf{f}(t;\mathbf{x}_0,\mathbf{a}_\mathrm{p})\,(\nabla_{\boldsymbol{\text{œ}}_0}t\cdot\Delta\boldsymbol{\text{œ}}_0)$$

thanks to Eqs. (11) and (14). This equation is expanded to provide

$$\Delta\mathbf{x}(t) = \left[\boldsymbol{Q}\circ\mathcal{B}\Big|_t(\vartheta(t),\vartheta_0) - \mathbf{f}(t;\mathbf{x}_0,\mathbf{a}_\mathrm{p})\otimes\nabla_{\boldsymbol{\text{œ}}_0}t\right]\Delta\boldsymbol{\text{œ}}_0 \tag{18}$$

The vector $\Delta\boldsymbol{\text{œ}}_0$ can be obtained from the initial separation $\Delta\mathbf{x}_0$ under the small-displacements assumption:

$$\Delta\boldsymbol{\text{œ}}_0 = \mathbf{q}^{-1}(t_0;\mathbf{x}_0+\Delta\mathbf{x}_0) - \mathbf{q}^{-1}(t_0;\mathbf{x}_0) = \nabla_{\mathbf{x}_0}\mathbf{q}^{-1}(t_0;\mathbf{x}_0)\,\Delta\mathbf{x}_0 + \ldots \tag{19}$$

Writing $\boldsymbol{Q}_0^\dagger = \nabla_{\mathbf{x}_0}\mathbf{q}^{-1}(t_0;\mathbf{x}_0)$, this result transforms Eq. (18) into

$$\Delta\mathbf{x}(t) = \left[\boldsymbol{Q}\circ\mathcal{B}\Big|_t(\vartheta(t),\vartheta_0) - \mathbf{f}(t;\mathbf{x}_0,\mathbf{a}_\mathrm{p})\otimes\nabla_{\boldsymbol{\text{œ}}_0}t\right]\boldsymbol{Q}_0^\dagger\,\Delta\mathbf{x}_0$$

meaning that the state-transition matrix $\mathcal{A}(t,t_0)$ can be constructed from the set $\boldsymbol{\text{œ}}$,

$$\mathcal{A}(t,t_0) = \left[\boldsymbol{Q}\circ\mathcal{B}\Big|_t(\vartheta(t),\vartheta_0) - \mathbf{f}(t;\mathbf{x}_0,\mathbf{a}_\mathrm{p})\otimes\nabla_{\boldsymbol{\text{œ}}_0}t\right]\boldsymbol{Q}_0^\dagger$$

Finally, it can be referred to the ϑ-synchronous solution by virtue of Eq. (13):

$$\mathcal{A}(t,t_0) = \left\{\boldsymbol{Q}\,\mathcal{B}\Big|_\vartheta(\vartheta(t);\boldsymbol{\text{œ}}) - \frac{1}{g_1}\boldsymbol{Q}\circ\left[\mathbf{g}(\vartheta;\boldsymbol{\text{œ}}_0,\mathbf{a}_\mathrm{p})\otimes\nabla_{\boldsymbol{\text{œ}}_0}t\right] - \mathbf{f}(t;\mathbf{x}_0,\mathbf{a}_\mathrm{p})\otimes\nabla_{\boldsymbol{\text{œ}}_0}t\right\}\boldsymbol{Q}_0^\dagger \tag{20}$$

This expression proves that the state-transition matrix that propagates the relative state vector $\Delta\mathbf{x}(t)$ can be constructed from any generic formulation. The correction including the vector function $\mathbf{f}$ is only included if the transformation $\mathbf{q}$ depends explicitly on the independent variable, ϑ.

The covariance matrix

Let $\mathcal{P}(t)$ denote the covariance matrix of the system. The uncertainty in the definition of the initial conditions is $\mathcal{P}(t_0)$. The covariance matrix associated to the state vector $\mathbf{x}(t)$ is denoted by $\mathcal{P}_{\mathbf{x}}(t)$, and defined by the expectation

$$\mathcal{P}_{\mathbf{x}}(t) = \mathrm{E}[\Delta\mathbf{x} \otimes \Delta\mathbf{x}]$$

Provided that $\Delta\mathbf{x}(t)$ relates to the initial conditions through the state-transition matrix $\mathcal{A}(t, t_0)$, the covariance matrix reads

$$\mathcal{P}_{\mathbf{x}}(t) = \mathrm{E}[(\mathcal{A}(t, t_0)\,\Delta\mathbf{x}_0) \otimes (\mathcal{A}(t, t_0)\,\Delta\mathbf{x}_0)]$$

By virtue of the linearity of expectation it follows

$$\mathcal{P}_{\mathbf{x}}(t) = \mathcal{A}(t, t_0) \circ \mathrm{E}[\Delta\mathbf{x}_0 \otimes \Delta\mathbf{x}_0] \circ \mathcal{A}^{\top}(t; \mathbf{x}) = \mathcal{A}(t, t_0) \circ \mathcal{P}_{\mathbf{x}}(t_0) \circ \mathcal{A}^{\top}(t; \mathbf{x}) \tag{21}$$

This is the standard procedure for propagating the covariance matrix.

Alternatively, let $\mathcal{P}_{\boldsymbol{\varpi}}(\vartheta)$ denote the covariance referred to the set $\boldsymbol{\varpi}$. Considering the transformation given in Eq. (19) and the linearity of expectation, the covariance matrix $\mathcal{P}_{\boldsymbol{\varpi}}$ is solved initially from

$$\mathcal{P}_{\boldsymbol{\varpi}}(\vartheta_0) = \mathrm{E}[\Delta\boldsymbol{\varpi}_0 \otimes \Delta\boldsymbol{\varpi}_0] = \mathrm{E}[(\boldsymbol{Q}_0^{\dagger}\,\Delta\mathbf{x}_0) \otimes (\boldsymbol{Q}_0^{\dagger}\,\Delta\mathbf{x}_0)] = \boldsymbol{Q}_0^{\dagger} \circ \mathrm{E}[\Delta\mathbf{x}_0 \otimes \Delta\mathbf{x}_0] \circ \boldsymbol{Q}_0^{\dagger\top} = \boldsymbol{Q}_0^{\dagger} \circ \mathcal{P}_{\mathbf{x}}(t_0) \circ \boldsymbol{Q}_0^{\dagger\top}$$

This expression readily defines the transformation of the covariance matrix in Cartesian coordinates to the covariance matrix referred to the set $\boldsymbol{\varpi}$. The transformation is applied initially, where no time delay exists.

The covariance matrix $\mathcal{P}_{\boldsymbol{\varpi}}(\vartheta)$ is propagated like the matrix $\mathcal{P}_{\mathbf{x}}(t)$, using the corresponding state-transition matrix. For physical coherence only the t-synchronous solution is considered, *i.e.*

$$\mathcal{P}_{\boldsymbol{\varpi}}(t) = \mathcal{B}\big|_t \circ \mathcal{P}_{\boldsymbol{\varpi}}(t_0) \circ \mathcal{B}\big|_t^{\top} = \mathcal{B}\big|_t \circ \boldsymbol{Q}_0^{\dagger} \circ \mathcal{P}_{\mathbf{x}}(t_0) \circ \boldsymbol{Q}_0^{\dagger\top} \circ \mathcal{B}\big|_t^{\top} \tag{22}$$

Equation (20) defined the state-transition matrix $\mathcal{A}(t, t_0)$ in terms of $\boldsymbol{\varpi}$. Hence, the covariance of the Cartesian state vector can be propagated in terms of the set $\boldsymbol{\varpi}$ simply by introducing the expression for $\mathcal{A}(t, t_0)$ given in Eq. (20) into Eq. (21). It can also be transformed from the matrix $\mathcal{P}_{\boldsymbol{\varpi}}(\vartheta)$, considering that

$$\begin{aligned}
\mathcal{P}_{\mathbf{x}}(t) &= \mathrm{E}[\Delta\mathbf{x} \otimes \Delta\mathbf{x}] = \mathrm{E}\left[\left\{\boldsymbol{Q}\,\Delta\boldsymbol{\varpi}\big|_t - (\mathbf{f} \otimes \nabla_{\boldsymbol{\varpi}_0} t)\,\Delta\boldsymbol{\varpi}_0\right\} \otimes \left\{\boldsymbol{Q}\,\Delta\boldsymbol{\varpi}\big|_t - (\mathbf{f} \otimes \nabla_{\boldsymbol{\varpi}_0} t)\,\Delta\boldsymbol{\varpi}_0\right\}\right] \\
&= \mathrm{E}\left[\left(\boldsymbol{Q}\,\Delta\boldsymbol{\varpi}\big|_t\right) \otimes \left(\boldsymbol{Q}\,\Delta\boldsymbol{\varpi}\big|_t\right)\right] - \mathrm{E}\left[(\mathbf{f} \otimes \nabla_{\boldsymbol{\varpi}_0} t)\,\Delta\boldsymbol{\varpi}_0 \otimes \left(\boldsymbol{Q}\,\Delta\boldsymbol{\varpi}\big|_t\right) + \left(\boldsymbol{Q}\,\Delta\boldsymbol{\varpi}\big|_t\right) \otimes (\mathbf{f} \otimes \nabla_{\boldsymbol{\varpi}_0} t)\,\Delta\boldsymbol{\varpi}_0\right] \\
&\quad + \mathrm{E}\left[(\mathbf{f} \otimes \nabla_{\boldsymbol{\varpi}_0} t)\,\Delta\boldsymbol{\varpi}_0 \otimes (\mathbf{f} \otimes \nabla_{\boldsymbol{\varpi}_0} t)\,\Delta\boldsymbol{\varpi}_0\right] \\
&= \boldsymbol{Q} \circ \mathcal{P}_{\boldsymbol{\varpi}}(\vartheta(t)) \circ \boldsymbol{Q}^{\top} - (\mathbf{f} \otimes \nabla_{\boldsymbol{\varpi}_0} t) \circ \mathcal{P}_{\boldsymbol{\varpi}}(\vartheta_0) \circ \left(\mathcal{B}\big|_t^{\top} \circ \boldsymbol{Q}^{\top}\right) - \left(\boldsymbol{Q} \circ \mathcal{B}\big|_t\right) \circ \mathcal{P}_{\boldsymbol{\varpi}}(\vartheta_0) \circ (\nabla_{\boldsymbol{\varpi}_0} t \otimes \mathbf{f}) \\
&\quad + (\mathbf{f} \otimes \nabla_{\boldsymbol{\varpi}_0} t) \circ \mathcal{P}_{\boldsymbol{\varpi}}(\vartheta_0) \circ (\nabla_{\boldsymbol{\varpi}_0} t \otimes \mathbf{f})
\end{aligned}$$

That is, the transformation from the covariance matrix $\mathcal{P}_{\boldsymbol{\varpi}}(\vartheta(t))$ to the covariance matrix $\mathcal{P}_{\mathbf{x}}(t)$ reads

$$\mathcal{P}_{\mathbf{x}}(t) = \boldsymbol{Q} \circ \mathcal{P}_{\boldsymbol{\varpi}}(\vartheta(t)) \circ \boldsymbol{Q}^{\top} + \boldsymbol{C} \tag{23}$$

Matrix $\boldsymbol{C}$ is the correction to be applied when the transformation $\mathbf{q} : \boldsymbol{\varpi} \mapsto \mathbf{x}$ depends explicitly on the independent variable ϑ. It reads

$$\boldsymbol{C} = -(\mathbf{f} \otimes \nabla_{\boldsymbol{\varpi}_0} t) \circ \mathcal{P}_{\boldsymbol{\varpi}}(\vartheta_0) \circ \left(\mathcal{B}\big|_t^{\top} \circ \boldsymbol{Q}^{\top}\right) - \left(\boldsymbol{Q} \circ \mathcal{B}\big|_t\right) \circ \mathcal{P}_{\boldsymbol{\varpi}}(\vartheta_0) \circ (\nabla_{\boldsymbol{\varpi}_0} t \otimes \mathbf{f}) + (\mathbf{f} \otimes \nabla_{\boldsymbol{\varpi}_0} t) \circ \mathcal{P}_{\boldsymbol{\varpi}}(\vartheta_0) \circ (\nabla_{\boldsymbol{\varpi}_0} t \otimes \mathbf{f})$$

THE GENERAL PROCEDURE

This section summarizes the procedure for propagating the state-transition matrix and the covariance matrix using an arbitrary formulation. The formulation is defined by an invertible transformation $\mathbf{q} : \boldsymbol{\text{œ}} \mapsto \mathbf{x}$, together with the differential equations defining the evolution of the set $\boldsymbol{\text{œ}}$:

$$\mathbf{x} = \mathbf{q}(\vartheta; \boldsymbol{\text{œ}}), \qquad \boldsymbol{\text{œ}} = \mathbf{q}^{-1}(t; \mathbf{x}), \qquad \frac{\mathrm{d}\boldsymbol{\text{œ}}}{\mathrm{d}\vartheta} = \mathbf{g}(\vartheta; \boldsymbol{\text{œ}}, \mathbf{a}_\mathrm{p})$$

The state of the particle is defined at departure by the initial conditions $\mathbf{x}_0$. The initial state vector is transformed into the initial set $\boldsymbol{\text{œ}}_0$ using the transformation

$$\boldsymbol{\text{œ}}_0 = \mathbf{q}_0^{-1}(t_0; \mathbf{x}_0)$$

The initial-value problems

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{\text{œ}}}{\mathrm{d}\vartheta} = \mathbf{g}(\vartheta; \boldsymbol{\text{œ}}, \mathbf{a}_\mathrm{p}) \\ \boldsymbol{\text{œ}}(\vartheta_0) = \boldsymbol{\text{œ}}_0 \end{cases} \qquad \begin{cases} \dfrac{\partial \mathcal{B}|_\vartheta}{\partial \vartheta} = \nabla_{\boldsymbol{\text{œ}}} \mathbf{g}(\vartheta; \boldsymbol{\text{œ}}, \mathbf{a}_\mathrm{p}) \circ \mathcal{B}|_\vartheta \\ \mathcal{B}(\vartheta_0) = \mathbb{I} \end{cases}$$

are integrated simultaneously until $t(\vartheta)$ reaches the final epoch. This provides both the solution $\boldsymbol{\text{œ}}(\vartheta)$ and the ϑ-synchronous state-transition matrix, $\mathcal{B}|_\vartheta(\vartheta, \vartheta_0)$. The gradient $\nabla_{\boldsymbol{\text{œ}}} t$ is the row of $\mathcal{B}|_\vartheta(\vartheta, \vartheta_0)$ corresponding to the physical time. It has been assumed to the be the first row of the matrix, since $t \equiv \text{œ}_1$.

Three additional terms need to be computed in order to define the t-synchronous state-transition matrix and to propagate the covariance matrix: vector $\mathbf{f} = \mathbf{f}(t; \mathbf{x}, \mathbf{a}_\mathrm{p})$, matrix $\boldsymbol{Q}$, and matrix $\boldsymbol{Q}_0^\dagger$. Vector $\mathbf{f} = \mathbf{f}(t; \mathbf{x}, \mathbf{a}_\mathrm{p})$ is simply

$$\mathbf{f}(t; \mathbf{x}, \mathbf{a}_\mathrm{p}) = \begin{bmatrix} \mathbf{v} \\ -\dfrac{\mu}{r^3}\,\mathbf{r} + \mathbf{a}_\mathrm{p} \end{bmatrix}$$

Note that the state vector is given by $\mathbf{x} = \mathbf{q}(\vartheta; \boldsymbol{\text{œ}})$. Matrix $\boldsymbol{Q}$ consists of the partial derivatives of the state vector with respect to the set $\boldsymbol{\text{œ}}$. The independent variable ϑ is kept constant when computing the partial derivatives, $\boldsymbol{Q} = \nabla_{\boldsymbol{\text{œ}}} \mathbf{q}|_\vartheta$. Analytical expressions for the Jacobians of most formulations are can be found in the literature. This matrix can also be derived numerically. Finally, the linear operator $\boldsymbol{Q}_0^\dagger$ is composed by the partial derivatives of the set $\boldsymbol{\text{œ}}$ with respect to the state vector. This transformation is only needed at departure. Analytical solutions exist, although numerical differentiation is a valid approach too.

Once the aforementioned auxiliary terms have been computed the t-synchronous state-transition matrix $\mathcal{B}|_t(\vartheta, \vartheta_0)$ is solved from Eq. (13). If needed, the state-transition matrix $\mathcal{A}(t, t_0)$ can be computed from Eq. (20). The covariance matrix $\mathcal{P}_{\boldsymbol{\text{œ}}}$ is propagated by means of Eq. (22). If needed, the covariance matrix $\mathcal{P}_\mathbf{x}$ can be solved from Eq. (23).

FORMULATIONS

There are many different sets of elements and regularized formulations available in the literature. Discussing the properties, advantages and equations for the formulations is out of the scope of this paper. The present section is devoted to discussing two concepts to be considered when integrating the variational equations. First, the initialization of the independent variable. Second, the correction of the time delay when the physical time is described by a time-element.

Initializing the independent variable

The simplest generalization of the Sundman transformation reads

$$\mathrm{d}t = r^{\alpha}\,\mathrm{d}\vartheta$$

In this context the variable ϑ is usually referred to as the fictitious time. For $\alpha = 1$ the fictitious time relates to the eccentric (or hyperbolic) anomaly, and $\alpha = 2$ corresponds to the true anomaly. Nacozy[21] coined the term *intermediate anomaly* for $\alpha = 3/2$.

Typically, when the physical time is replaced by a fictitious time (or an equivalent angular variable) the new independent variable is initialized in a specific way, and many times $\vartheta_0 = \vartheta_0(\mathbf{x}_0) \neq 0$. For example, if the independent variable is the true anomaly its initial value is solved initially from the projections of the position vector in the perifocal frame. But if the initial conditions change, as it happens when the variational formulation is integrated, then the initial value of the independent variable will change too. The gradient of the set $\boldsymbol{\mathfrak{e}}$ does not account for the change in ϑ_0, this information is lost and the differentiation fails. To solve this issue the formulation must be adapted so that the independent variable is always zero at departure. This is achieved by introducing a modified independent variable, $\vartheta^* = \vartheta - \vartheta_0$. There are two systematic ways of modifying the formulation accordingly:

1. To attach ϑ_0 to the vector of elements or coordinates $\boldsymbol{\mathfrak{e}}$. It remains constant throughout the integration process, and the original value of the independent variable is recovered simply by means of $\vartheta = \vartheta^* + \vartheta_0$. This approach requires little modification of the equations of motion, but increases the dimension of system.

2. To reformulate the problem in terms of the modified independent variable ϑ^*. This approach preserves the dimension of the system at the cost of having to derive the modified equations of motion.

Examples of formulations requiring this correction are Dromo,[12] the Minkowskian formulation for hyperbolic orbits,[13,14] the Stiefel-Scheifele method,[19] or the equinoctial elements with the longitude as the independent variable.

The time-element

The concept of a time-element was proposed by Stiefel[19] for reducing the truncation error when integrating the time variable. This technique consists on separating the physical time t in a term that depends on the perturbations (the time-element, t_{te}) and a term not affected by perturbations, t_{np},

$$t(\vartheta; \boldsymbol{\mathfrak{e}}) = t_{\mathrm{te}}(\vartheta; \boldsymbol{\mathfrak{e}}) + t_{\mathrm{np}}(\vartheta; \boldsymbol{\mathfrak{e}}) \tag{24}$$

The time-element vanishes for $\mathbf{a}_{\mathrm{p}} = \mathbf{0}$. The derivative of the time-element scales with the perturbation and yields a smoother evolution in weakly perturbed problems.

The presence of a time-element complicates the definition of the time delay: the gradient $\nabla_{\boldsymbol{\mathfrak{e}}_0} t$ is no longer given explicitly by the state-transition matrix $\mathcal{B}|_{\vartheta}$, since this matrix propagates the time-element and not the physical time. In fact, taking the gradient $\nabla_{\boldsymbol{\mathfrak{e}}_0}$ in Eq. (24) provides

$$\nabla_{\boldsymbol{\mathfrak{e}}_0} t = \nabla_{\boldsymbol{\mathfrak{e}}_0} t_{\mathrm{te}}(\vartheta; \boldsymbol{\mathfrak{e}}) + \nabla_{\boldsymbol{\mathfrak{e}}_0} t_{\mathrm{np}}(\vartheta; \boldsymbol{\mathfrak{e}}) \tag{25}$$

where $\nabla_{\boldsymbol{\varpi}_0} t_{\mathrm{te}}(\vartheta; \boldsymbol{\varpi})$ is the corresponding row of the matrix $\mathcal{B}|_{\vartheta}$. The second term is required for retrieving the physical time.

Additionally, the term $\mathrm{d}t/\mathrm{d}\vartheta$ that appears in Eq. (12) is no longer the first component of $\mathbf{g}(\vartheta; \boldsymbol{\varpi}, \mathbf{a}_{\mathrm{p}})$, provided that now $g_1 = \mathrm{d}t_{\mathrm{te}}/\mathrm{d}\vartheta$. The derivative of the time is

$$\frac{\mathrm{d}\,t}{\mathrm{d}\vartheta} = \frac{\mathrm{d}t_{\mathrm{te}}}{\mathrm{d}\vartheta} + \frac{\mathrm{d}t_{\mathrm{np}}}{\mathrm{d}\vartheta}$$

so the derivative of t_{np} is required for the correction.

In order to illustrate how the time delay is solved from a time-element, the derivatives of t_{np} required for the correction using the KS transformation, the formulation in Minkowskian geometry, the SB regularization and the stabilized Cowell's method are derived in the Appendix.

NUMERICAL EXAMPLES

Three test cases are considered for testing the propagation of the state-transition matrix and the covariance matrix. Motivated by the recent flyby of Pluto performed by the New Horizons spacecraft, the two first cases are defined in the Pluto system. The inbound velocity of the actual flyby was so high that the relative orbit is close to rectilinear and may not be a good candidate for numerical analyses ($e > 3000$). In order to fully enjoy the strong perturbations from the Pluto moons (Charon, Styx, Nix, Kerberos and Hydra) the orbits of two fictitious orbiters are defined. The first case (Case 1) is a strongly-perturbed elliptical orbit, and the second case (Case 2) is a weakly-perturbed quasi-circular orbit. The third and last example (Case 3) is the geocentric flyby of the NEAR spacecraft on January 23, 1998. The entire trajectory of the spacecraft inside the sphere of influence of the Earth is propagated. Geocentric orbits allow to use a more detailed force model. Table 1 shows the initial conditions for the described orbits and the initial separation used to test the linear propagation of the relative dynamics.

Table 1: **Definition of the numerical test cases**

	a (km)	e (−)	i (deg)	Ω (deg)	ω (deg)	Osculation	Span (days)	Center
Case 1	10000	0.400	137.6	253.9	247.9	2015-Jul-08	20	Pluto
Case 2	1400	0.010	13.8	253.9	247.9	2015-Jul-08	20	Pluto
Case 3	8496	1.813	108.0	88.3	145.1	1998-Jan-23	4	Earth

Note: The orbits are defined by means of the semimajor axis, a, eccentricity, e, right ascension of the ascending node, Ω and argument of periapsis, ω. The orbital elements correspond to the epoch of osculation, shown in the table. Angles are referred to the ICRF/J2000.0 reference frame, and the reference plane is the Earth Mean Equator and Equinox of reference epoch.

	Δx_0 (m)	Δy_0 (m)	Δz_0 (m)	Δv_{x0} (m/s)	Δv_{y0} (m/s)	Δv_{z0} (m/s)
Case 1	6759.3	-2823.5	6523.0	0.133	-0.051	0.296
Case 2	-2267.4	-1520.6	-870.5	1.445	-0.757	0.244
Case 3	1496.0	0.0	0.0	-0.069	0.000	0.000

The perturbations from all the planets and the Sun are computed from the DE430 ephemeris model. The orbits around Pluto also include the perturbations from Charon, Styx, Nix, Kerberos

and Hydra. Perturbations from Charon are over six orders of magnitude stronger than those from the rest of moons. For the geocentric flyby a 20×20 gravitational field grid is considered, together with the IAU2006/2000A CIO based (X-Y series) Earth rotation model using the IERS Earth Orientation Parameters. Relativistic corrections to the flyby trajectory are applied.

Figure 1 shows the configuration of the Pluto system between July 8 and July 28, 2015. The path of the Pluto moons during that interval is plotted in black. The green line represents the New Horizons flyby, passing inside the orbit of Charon. The blue orbits correspond to Cases 1 and 2.

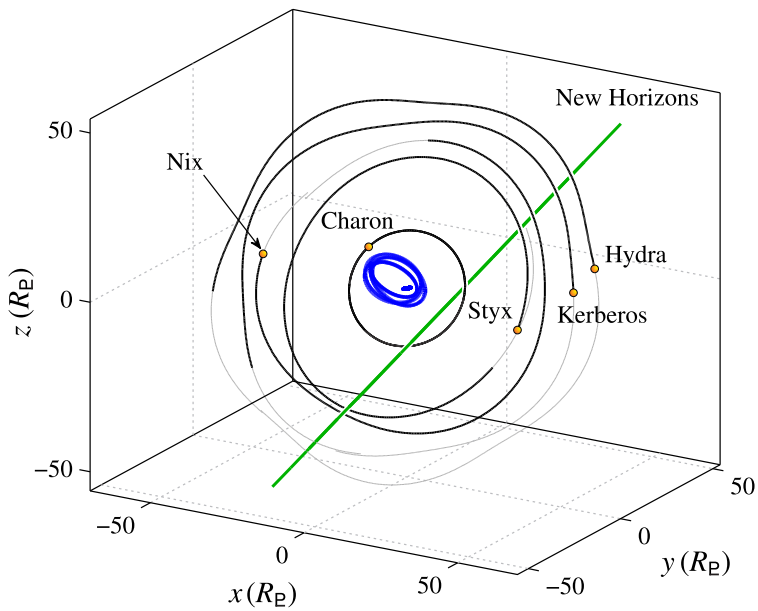


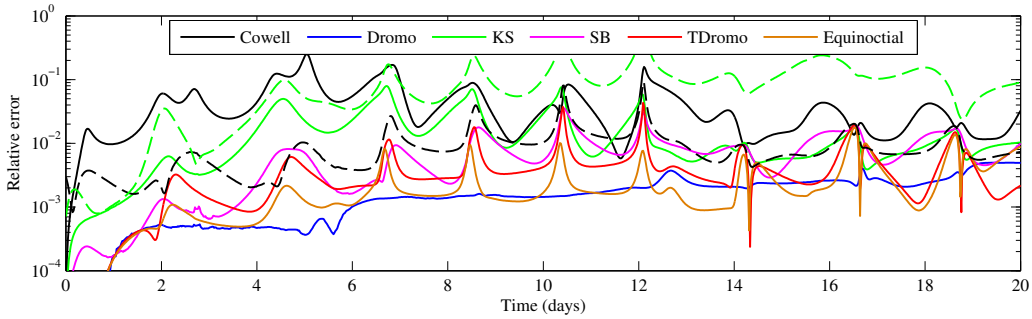
Figure 1: **Configuration of the Pluto system between 08-Jul-2015 and 28-Jul-2015. Distances appear in Pluto radii** ($R_{\mathrm{P}} = 1184\,\mathrm{km}$)**. The spheres represent the final position of the moons (Source: JPL Horizons)**

In order to validate the propagation of the state-transition matrix using the proposed theory the error of the linear propagation is considered. A neighbor orbit is generated, applying a small $\Delta\boldsymbol{\alpha}_0$ to the initial set of elements or coordinates. The exact neighbor trajectory is propagated numerically and the difference with respect to the nominal trajectory is considered the reference solution to the relative motion, $\Delta\boldsymbol{\alpha}_{\mathrm{num}}$. Alternatively, the initial separation is propagated using the state-transition matrix according to Eq. (14), which provides $\Delta\boldsymbol{\alpha}_{\mathrm{lin}}$. The relative error is measured at every time step as

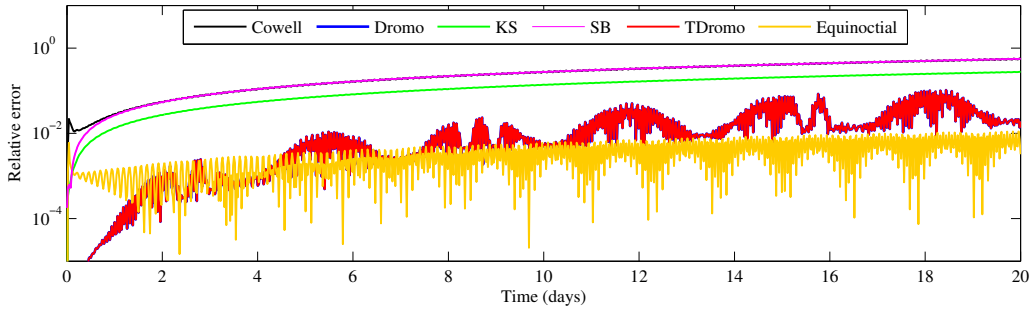
$$\mathrm{Error}_j = \frac{\|\Delta\boldsymbol{\alpha}_{\mathrm{num},j} - \Delta\boldsymbol{\alpha}_{\mathrm{lin},j}\|}{\|\Delta\boldsymbol{\alpha}_{\mathrm{num},j}\|}$$

The solution corresponds to the t-synchronous case. The propagation in Cartesian coordinates (Cowell's method) is compared against Dromo formulation,[12] the Kustaanheimo-Stiefel transformation,[5] the equinoctial elements[22] and the time explicit version of Dromo (TDromo). In addition, the hyperbolic orbit is also described using the formulation in Minkowskian geometry.[13,14] This formulation is replaced by the Sperling-Burdet regularization[6] in the elliptic cases. The time-element version of the methods is also considered, but represented only when the differences are of interest. The stabilized version of Cowell's method[11] is referred to as its time-element version.

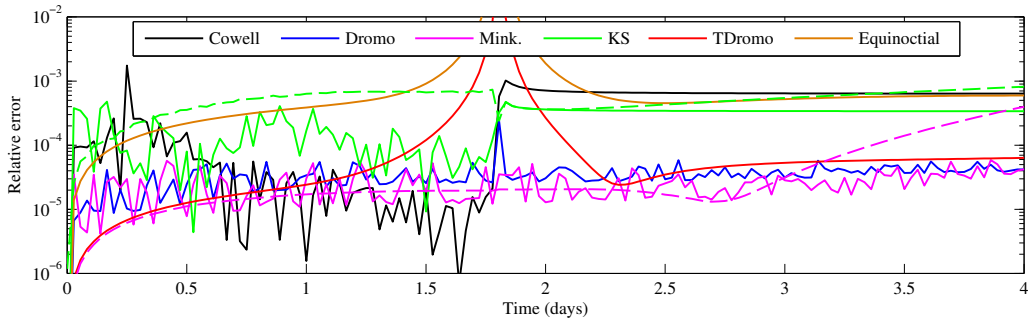
The relative error of the linear propagation is displayed in Fig. 2. It is clear from Fig. 2b that element-based formulations are better behaved than coordinate-based formulations when the perturbations are small, because they change in a slower time scale and the derivatives scale with the perturbations. In this quasi-circular case the introduction of a Sundman transformation provides little advantages. The fact that Dromo and TDromo coincide exactly proves this statement. In Case 1 the perturbations are stronger and Fig. 2a shows that the difference in performance between element-based and coordinate-based formulations is reduced. The frequency of the error is mostly governed by the perturbation from Charon, and error peaks correspond to close approaches. The analytic step-size adaption provided by the Sundman transformation is beneficial for the numerical integration,



(a) Case 1: Pluto system, strongly-perturbed



(b) Case 2: Pluto system, weakly-perturbed



(c) Case 3: NEAR flyby

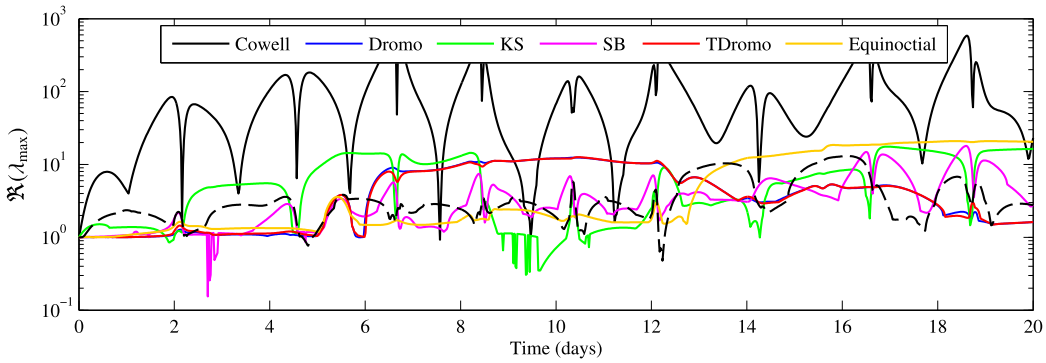
Figure 2: Relative error of the linear propagation of the relative dynamics. Dashed lines represent the time-element version of the method. It is included when the difference with respect to the regular method is relevant.

but has no effect on the linearized dynamics. The larger error of TDromo when compared to Dromo is caused by the error in the ideal anomaly. Excluding this variable the difference in performance in negligible. This phenomenon is amplified in Case 3. The NEAR flyby characterizes by the error increment at the perigee. The error peak that appears when using both the equinoctial elements and TDromo is quite noticeable. It is caused by the error in the longitude and anomaly, respectively. The integration of the angular variable is extremely sensitive to errors in the energy of the orbit. This is because the corresponding component of the state-transition matrix, defining the partial derivative of the angle with respect to the energy, grows rapidly. The original Dromo and the Minkowskian propagator do not suffer this issue because they rely on the Sundman transformation. Thanks to it, they are not affected by perigee passage. The rest of coordinate-based formulations yield similar results.

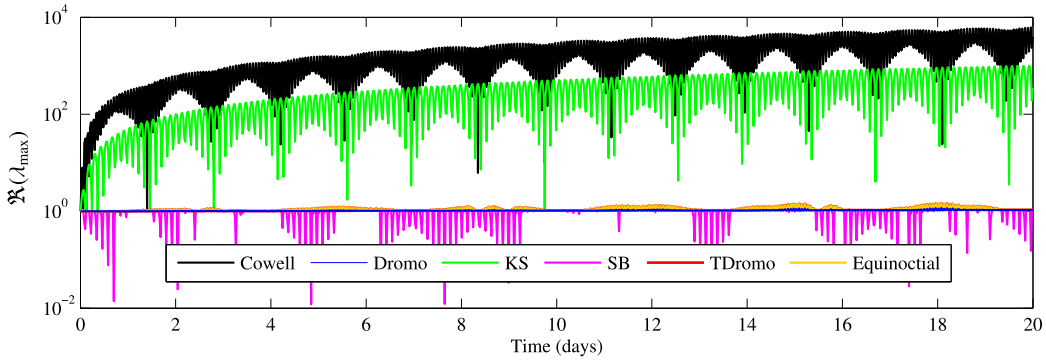
The results from this figure yield an important conclusion about the use of the Sundman transformation. Not only the analytic step-size adaption helps the integration during the propagation, but it also makes the formulation more robust due to initialization errors. Angle variables turn out to be especially sensitive to errors in the orbital energy during flybys. Replacing the angle variable by a time variable by means of the Sundman transformation compensates for this issue.

An illustrative to way to analyze the state-transition matrix of the system is by monitoring its largest eigenvalue,[2] $\lambda_{\max}$. The slow growth of the largest eigenvalue indicates a smooth, typically stable behavior of the system. An exponential growth anticipates rather unstable dynamics, sometimes motivated by critical events such as the flyby shown in Fig. 3c. This figure shows how the maximum eigenvalue increases during the flyby. Element-based methods exhibit the more stable dynamics, except for the already discussed problem with the angular variable in the equinoctial elements and TDromo. Dromo, and especially the Minkowskian propagator, yield the most stable dynamical behavior. In the circular case, presented in Fig. 3b, it is clear than coordinate-based formulations (Cowell and KS) are not as well-suited for linearization as element-based formulations. This is due to the fact that the relative separation grows much faster, and hence nonlinear effects soon appear. The relative dynamics in Case 1 are rather unpredictable due to the strong perturbations and overlapping resonances from Charon. The propagation in Cartesian coordinates shows the most remarkable variations on the largest eigenvalue, which is greatly reduced when switching to the stabilized version of the method. The stabilization is achieved by introducing the Keplerian energy in the state vector. The advantages of introducing first integrals for the numerical propagation of the equations of motion is widely discussed by Baumgarte.[23] In both Case 1 and 2 the behavior of Dromo and TDromo is comparable: the flow of the equations is not subject to the divergence of the flyby, and the performance of the angular variable is not jettisoned. Despite the more adequate discretization provided by the Sundman transformation it has little impact on the eigenvalues of the state-transition matrix.

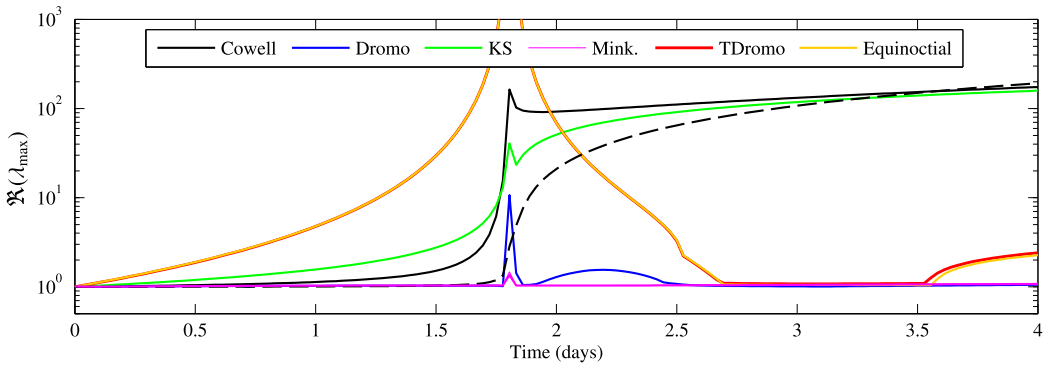
The state-transition matrix is required for propagating the covariance matrix of the system, defining the uncertainties on the corresponding elements. Although the analysis of its largest eigenvalue leads to important conclusions, the study of the evolution of the covariance requires to consider additional factors. The most important is possibly how a covariance matrix in Cartesian coordinates maps to the covariance matrix for a certain set of elements (or coordinates). How the covariance propagates in time is studied by following the time evolution of the largest eigenvalue of the covariance matrix. In the example an exaggerated covariance matrix is defined initially in the orbital frame, with $\sigma_x = \sigma_y = \sigma_z = 85\,\mathrm{km}$ and $\sigma_{vx} = \sigma_{vy} = \sigma_{vz} = 20\,\mathrm{m/s}$ for the case of NEAR, and $\sigma_x = \sigma_y = \sigma_z = 40\,\mathrm{km}$ and $\sigma_{vx} = \sigma_{vy} = \sigma_{vz} = 2\,\mathrm{m/s}$. Figure 4b clearly shows the behavior of the



(a) Case 1: Pluto system, strongly-perturbed



(b) Case 2: Pluto system, weakly-perturbed



(c) Case 3: NEAR flyby

Figure 3: **Evolution of the maximum eigenvalue of the state-transition matrix. The state-transition matrix propagates the corresponding set of elements or coordinates**

largest eigenvalue: the most rapid growth of the covariance corresponds to Cartesian coordinates, which is strongly coupled. The covariance associated to TDromo and the equinoctial elements coincide, and its rapid growth is caused by the uncertainty in the anomaly/longitude. The evolution of the covariance in terms of the KS transformation and the SB regularization improves with respect to the Cartesian propagation, thanks to transforming the system into a perturbed oscillator. It is interesting to compare TDromo and the equinoctial elements in Figs. 3b and 4b. The evolution of the largest eigenvalue of the state-transition matrix does not anticipate the rapid growth of the covariance. The component which grows rapidly is the partial derivative of the angular variable with respect to the energy; it is responsible for the rapid growth of the covariance and is not captured by the analysis of the maximum eigenvalue. Dromo corrects this behavior and yields a slowly changing covariance. In strongly-perturbed environments the performance of element-based formulations is affected, and the evolution of the covariance is comparable to that of coordinate-based formulations. The NEAR flyby, shown in Fig. 4c, is a good example of how the covariance of coordinate-based method grows rapidly after periapsis passage. Element-based methods, however, are not so sensitive to the flyby.

Table 2 shows the mean and standard deviation of the error in the relative position and velocity. All formulations yield similar results to the Cartesian solution. These results validate the transformation defined in Eq. (20), and show that the potential improvements in accuracy obtained when propagating the linearized dynamics with alternative sets of elements might be lost when transforming to the state vector.

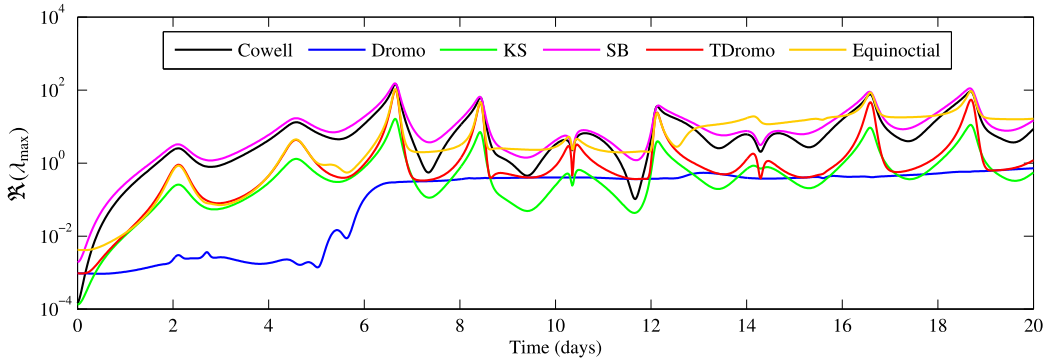
Table 2: **Error in the propagation of the relative state vector**

Formulation	$\mu_{\mathbf{r}}$ (m)			$\mu_{\mathbf{v}}$ (m/s)			$\sigma_{\mathbf{r}}$ (m)			$\sigma_{\mathbf{v}}$ (m/s)		
	C1	C2	C3	C1	C2	C3	C1	C2	C3	C1	C2	C3
Cowell	954.30	275374	162.42	0.0402	164	0.0017	687.61	244556	191.66	0.0607	146	0.0015
Dromo	949.33	275374	162.56	0.0507	164	0.0428	686.62	244556	191.69	0.0617	146	0.0684
KS	1083.95	275366	162.49	0.0451	164	0.0017	789.19	244548	191.73	0.0721	146	0.0015
SB	847.15	275373	207.84	0.0370	164	0.0023	596.90	244556	245.72	0.0580	146	0.0021
Mink.	–	–	163.09	–	–	0.0017	–	–	188.94	–	–	0.0016
TDromo	952.52	275373	162.26	0.0401	164	0.0017	686.52	244556	191.15	0.0606	146	0.0015
Equinoctial	953.55	275373	209.30	0.0402	164	0.0017	687.17	244556	248.42	0.0606	146	0.0015
Stab. Cow.	938.49	275371	163.66	0.0400	164	0.0017	674.54	244554	193.13	0.0618	146	0.0015

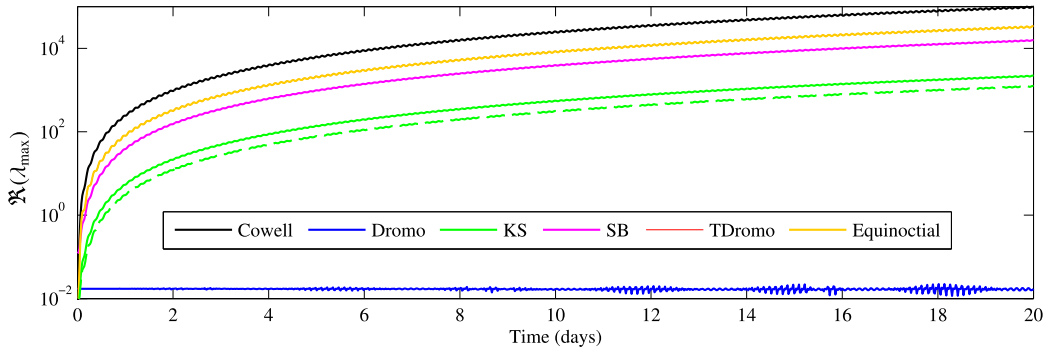
CONCLUSIONS

Regularized formulations of motion are not restricted to propagating the trajectory of a particle. They can be extended to account for the partial derivatives of the corresponding elements or coordinates, being applicable to orbit determination, optimization, and many other problems. The main difficulty comes from the *synchronism* of the solution. Having introduced an independent variable different from time, the derivatives require an additional transformation to correct the time delay.

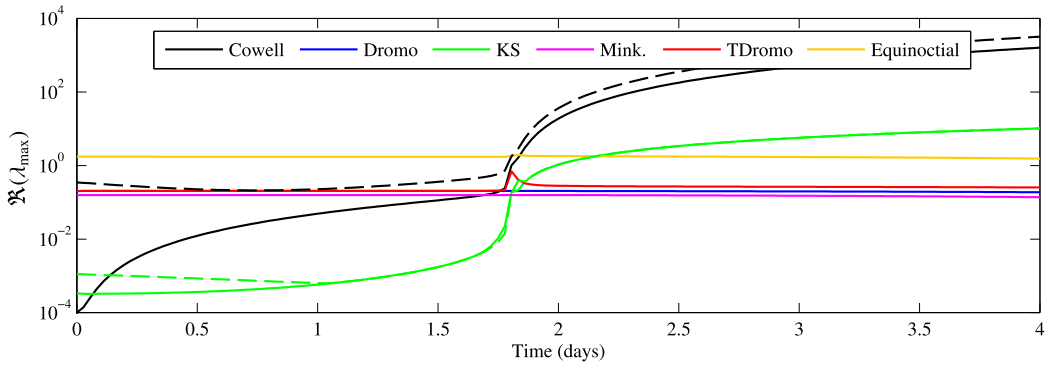
- Element-based formulations are less sensitive to initialization errors. The state-transition matrix grows in a slower time-scale when compared to coordinate-based formulations.
- The introduction of a fictitious time through the Sundman transformation improves the robustness of the formulations and reduces the growth rate of the covariance. This advantage does not come from the regularization nor the analytic step-size adaption; it replaces the propagation of an angular variable (such as the true anomaly or the longitude) by the propagation



(a) Case 1: Pluto system, strongly-perturbed



(b) Case 2: Pluto system, weakly-perturbed



(c) Case 3: NEAR flyby

Figure 4: **Evolution of the maximum eigenvalue of the covariance matrix. The covariance matrix defines the uncertainties on the corresponding set of elements or coordinates**

of the time or a time element. Numerical results show that angular variables are especially sensitive to initialization errors, in particular to those not preserving the energy. Replacing these variables enhances the performance of the methods.

- Flybys have a greater impact on coordinate-based formulations: the rate of change of the coordinates is faster than that of the elements. The derivatives take larger values, which make them more sensitive to errors in the state definition. The divergence of the flow is stronger during the flyby and magnifies the sensitivity of the propagation. Sets of elements might be preferable for describing the orbit thanks to a smoother evolution.

The introduction of a time-element has well-known numerical advantages, although no clear benefits in the propagation of the partial derivatives have been observed. However, we present a general procedure for correcting the time delay in formulations including time-elements.

ACKNOWLEDGMENT

This work has been carried out at the Jet Propulsion Laboratory within the framework of the JVSRP program. It is part of the research project entitled "Dynamical Analysis, Advanced Orbit Propagation and Simulation of Complex Space Systems" (ESP2013-41634-P) supported by the Spanish Ministry of Economy and Competitiveness. Authors thank J. Senent for motivating this work, the Spanish Government for its support and J. Roa especially thanks "La Caixa" for his doctoral fellowship.

APPENDIX: ADJUSTING THE TIME DELAY WITH TIME-ELEMENTS

Minkowskian formulation

Roa and Peláez[13,14] derived the set of elements $\boldsymbol{\varpi} = (t_{\rm te}, \psi_1, \psi_2, \psi_3, \chi_1, \chi_2, \chi_3, \chi_4)^\top$ for describing arbitrarily perturbed hyperbolic orbits. A modified independent variable $\vartheta \equiv u^*$ needs to be introduced, $u^* = u - H_0$, replacing the original independent variable u. Here H_0 is the initial hyperbolic anomaly. The physical time decomposes in

$$t = t_{\rm te} + \psi_3^{3/2}(\hat{r} + 1 - u^* - H_0) - \psi_{30}^{3/2}(\psi_{10}\sinh H_0 - H_0), \qquad \hat{r} = \psi_1\cosh(u^* + H_0) + \psi_2\sinh(u^* + H_0) - 1$$

The definition of the time delay involves the gradient $\nabla_{\boldsymbol{\varpi}_0} t_{\rm np}$. Considering the function

$$d(u, \xi_0) = \psi_3^{3/2}\left(\sinh u\,\frac{\partial\psi_1}{\partial\xi_0} + \cosh u\,\frac{\partial\psi_2}{\partial\xi_0}\right) + \frac{3}{2}\sqrt{\psi_3}(\hat{r} + 1 - u)\,\frac{\partial\psi_3}{\partial\xi_0}$$

the partial derivatives constituting the gradient $\nabla_{\boldsymbol{\varpi}_0} t_{\rm np}$ take the form:

$$\begin{aligned}
\frac{\partial\, t_{\rm np}}{\partial\psi_{10}} &= d(u, \psi_{10}) - \psi_{30}^{3/2}\sinh H_0 \\
\frac{\partial\, t_{\rm np}}{\partial\psi_{30}} &= d(u, \psi_{10}) - \frac{3}{2}\sqrt{\psi_{30}}\,(\hat{r}_0 + 1 - H_0) \\
\frac{\partial\, t_{\rm np}}{\partial H_0} &= d(u, H_0) - \frac{3}{2}\sqrt{\psi_{30}}\,(\psi_{10}\cosh H_0 - 1) \\
\frac{\partial\, t_{\rm np}}{\partial\xi_{i,0}} &= d(u, \xi_{i,0}), \qquad \boldsymbol{\xi}_0 = (t_{\rm te0}, \psi_{20}, \chi_{10}, \chi_{20}, \chi_{30}, \chi_{40})
\end{aligned}$$

These derivatives complete Eq. (25). The partial derivatives of the elements with respect to the initial conditions that appear in the definition of the function $d(u, \xi_0)$ are contained in the state-transition matrix $\mathcal{B}|_{u^*}$. Finally and denoting by $'$ the derivatives with respect to u^*, the derivative of t_{np} reads

$$t'_{\mathrm{np}} = \frac{3}{2}\psi'_3\sqrt{\psi_3}(\hat{r} + 1 - u^* - H_0) + \psi_3^{3/2}(\hat{q} - 1), \qquad \hat{q} = \psi_1 \sinh(u^* + H_0) + \psi_2 \cosh(u^* + H_0)$$

Kustaanheimo-Stiefel transformation

Kustaanheimo and Stiefel[5] published an extension of the Levi-Civita transformation in $\mathbb{R}^4$. Relying on the Sundman transformation, the set of KS variables is $\boldsymbol{\text{œ}} = (t_{\mathrm{te}}, \mathbf{u}, \mathbf{u}', \alpha)$. Here $\mathbf{u}, \mathbf{u}' \in \mathbb{R}^4$ are the representation of the position and velocity vectors in the KS space, derivatives with respect to the fictitious time are denoted by $'$ and α is equivalent to the Keplerian energy:

$$\alpha = \frac{1}{r}\left(\frac{1}{2} - \|\mathbf{u}'\|^2\right), \qquad r = \mathbf{u}\cdot\mathbf{u}$$

The inclusion of α in the vector $\boldsymbol{\text{œ}}$ is optional.

The gradient $\nabla_{\boldsymbol{\text{œ}}} t_{\mathrm{np}}$ required for the definition of the time delay takes different expressions depending on whether α is integrated directly or computed from the KS coordinates. If α is not integrated as part of $\boldsymbol{\text{œ}}$ the gradient $\nabla_{\boldsymbol{\text{œ}}} t_{\mathrm{np}}$ decomposes in

$$\begin{aligned}
\nabla_{\mathbf{u}_0} t_{\mathrm{np}} &= -\frac{1}{2r\alpha}\left(r\mathbf{u}' + 2r'\mathbf{u}\right) \circ \nabla_{\mathbf{u}_0}\mathbf{u} - \frac{1}{2r\alpha^2}(2r'\mathbf{u}' + r\alpha\mathbf{u}) \circ \nabla_{\mathbf{u}_0}\mathbf{u}' \\
\nabla_{\mathbf{u}'_0} t_{\mathrm{np}} &= -\frac{1}{2r\alpha}\left(r\mathbf{u}' + 2r'\mathbf{u}\right) \circ \nabla_{\mathbf{u}'_0}\mathbf{u} - \frac{1}{2r\alpha^2}(2r'\mathbf{u}' + r\alpha\mathbf{u}) \circ \nabla_{\mathbf{u}'_0}\mathbf{u}'
\end{aligned}$$

Typically the energy α is integrated together with the time and the coordinates $\mathbf{u}$ and $\mathbf{u}'$. In such a case the gradients $\nabla_{\mathbf{u}_0}\alpha$ and $\nabla_{\mathbf{u}'_0}\alpha$ are given by the corresponding row of the state-transition matrix. The previous equations then reduce to

$$\begin{aligned}
\nabla_{\mathbf{u}_0} t &= \nabla_{\mathbf{u}_0} t_{\mathrm{te}} - \frac{1}{2\alpha}(\mathbf{u} \circ \nabla_{\mathbf{u}_0}\mathbf{u}' + \mathbf{u}' \circ \nabla_{\mathbf{u}_0}\mathbf{u}) + \frac{(\mathbf{u}\cdot\mathbf{u}')}{2\alpha^2}\nabla_{\mathbf{u}_0}\alpha \\
\nabla_{\mathbf{u}'_0} t &= \nabla_{\mathbf{u}'_0} t_{\mathrm{te}} - \frac{1}{2\alpha}(\mathbf{u} \circ \nabla_{\mathbf{u}'_0}\mathbf{u}' + \mathbf{u}' \circ \nabla_{\mathbf{u}'_0}\mathbf{u}) + \frac{(\mathbf{u}\cdot\mathbf{u}')}{2\alpha^2}\nabla_{\mathbf{u}'_0}\alpha
\end{aligned}$$

Finally, the term t'_{np} reads

$$t'_{\mathrm{np}} = -\frac{1}{2r\alpha}\left(r\mathbf{u}' + 2r'\mathbf{u}\right)\cdot\mathbf{u}' - \frac{1}{2r\alpha^2}(2r'\mathbf{u}' + r\alpha\mathbf{u})\cdot\mathbf{u}''$$

Sperling-Burdet regularization and stabilized Cowell's method

Both the Sperling-Burdet[6] regularization and the stabilized Cowell's method[11] introduce a fictitious time $\vartheta \equiv s$ by means of the Sundman transformation $\mathrm{d}t = r\,\mathrm{d}s$. For the former it is $\boldsymbol{\text{œ}} = (t_{\mathrm{np}}, \mathbf{r}, \mathbf{r}', r, r', \alpha, \mu\mathbf{e})^\top$, and for the latter $\boldsymbol{\text{œ}} = (t_{\mathrm{np}}, \mathbf{r}, \mathbf{r}', \alpha)^\top$. Considering the auxiliary terms

$$r' = \frac{\mathbf{r}\cdot\mathbf{r}'}{r}, \qquad \alpha = \frac{1}{r}\left(2 - \frac{\|\mathbf{r}'\|^2}{r}\right)$$

the gradient $\nabla_{\boldsymbol{\varpi}_0} t_{\mathrm{np}}$ consists of

$$\nabla_{\mathbf{r}_0} t_{\mathrm{np}} = \frac{1}{r^4\alpha^2}\Big[(r'\|\mathbf{r}'\|^2\mathbf{r} - r^3\alpha\,\mathbf{r}') \circ \nabla_{\mathbf{r}_0}\mathbf{r} - r^2(2r'\mathbf{r}' + \alpha r\,\mathbf{r}) \circ \nabla_{\mathbf{r}_0}\mathbf{r}'\Big]$$

$$\nabla_{\mathbf{r}_0'} t_{\mathrm{np}} = \frac{1}{r^4\alpha^2}\Big[(r'\|\mathbf{r}'\|^2\mathbf{r} - r^3\alpha\,\mathbf{r}') \circ \nabla_{\mathbf{r}_0'}\mathbf{r} - r^2(2r'\mathbf{r}' + \alpha r\,\mathbf{r}) \circ \nabla_{\mathbf{r}_0'}\mathbf{r}'\Big]$$

and the derivative with respect to the fictitious time reads

$$t_{\mathrm{np}}' = \frac{1}{r^4\alpha^2}\Big[(r'\|\mathbf{r}'\|^2\mathbf{r} - r^3\alpha\,\mathbf{r}') \cdot \mathbf{r}' - r^2(2r'\mathbf{r}' + \alpha r\,\mathbf{r}) \cdot \mathbf{r}''\Big]$$

REFERENCES

[1] Andrea Milani and Giovanni Gronchi. *Theory of orbit determination*. Cambridge University Press, 2010.

[2] William E Wiesel. *Modern Orbit Determination*. Aphelion Press, 2003.

[3] David A Vallado. Covariance transformations for satellite flight dynamics operations*. 2004.

[4] David A Vallado and John H Seago. Covariance realism. *Advances in the Astronautical Sciences*, 135(1):49–67, 2009.

[5] P Kustaanheimo and E Stiefel. Perturbation theory of kepler motion based on spinor regularization. *J Reine Angew Math*, 218:204–219, 1965.

[6] Victor R Bond and Mark C Allman. *Modern Astrodynamics: Fundamentals and perturbation methods*. Princeton University Press Princeton, 1996.

[7] André Deprit, Antonio Elipe, and Sebastian Ferrer. Linearization: Laplace vs. stiefel. *Celest Mech Dyn Astr*, 58(2):151–201, 1994.

[8] AE Roy and PE Moran. Studies in the application of recurrence relations to special perturbation methods. *Celestial Mech*, 7(2):236–255, 1973.

[9] Aaron J. Rosengren and Daniel J. Scheeres. On the milankovitch orbital elements for perturbed keplerian motion. *Celest Mech Dyn Astr*, 118:197–220, 3 2014.

[10] J Baumgarte and E Stiefel. Examples of transformations improving the numerical accuracy of the integration of differential equations. In *Proceedings of the Conference on the Numerical Solution of Ordinary Differential Equations*, pages 207–236. Springer, 1974.

[11] Guy Janin. Accurate computation of highly eccentric satellite orbits. *Celestial Mech*, 10(4):451–467, 1974.

[12] Jesús Peláez, José Manuel Hedo, and Pedro Rodríguez de Andrés. A special perturbation method in orbital dynamics. *Celest Mech Dyn Astr*, 97(2):131–150, 2007.

[13] Javier Roa and Jesús Peláez. Orbit propagation in minkowskian geometry. *Celest Mech Dyn Astr*, pages 1–31, 2015.

[14] Javier Roa and Jesús Peláez. Orbit propagation in minkowskian geometry. In *25th Spaceflight Mechanics Meeting*, pages AAS 15–209, 2015.

[15] Javier Roa and Jesus Peláez. The elliptic rendezvous problem in dromo formulation. *Adv Astronaut Sci*, 152:2531–2546, 2014.

[16] Javier Roa and Jesús Peláez. Frozen-anomaly transformation for the elliptic rendezvous problem. *Celest Mech Dyn Astr*, 121:61–81, 2015.

[17] Javier Roa and Jesús Peláez. Regularized formulations in relative motion. In *25th Spaceflight Mechanics Meeting*, pages AAS 15–210, 2015.

[18] Javier Roa, José Ignacio Gómez-Mora, and Jesús Peláez. Error propagation in relative motion. In *25th Spaceflight Mechanics Meeting*, pages AAS 15–272, 2015.

[19] Eduard L. Stiefel and Gerhard Scheifele. *Linear and regular celestial mechanics*. Springer-Verlag, New York, Heidelberg, Berlin, 1971.

[20] Dino Boccaletti and Giuseppe Pucacco. *Theory of Orbits (Vol. I): Integrable Systems and Non-perturbative Methods*, volume 1. Springer, 1996.

[21] Paul Nacozy. The intermediate anomaly. *Celestial Mech*, 16(3):309–313, 1977.

[22] Paul J Cefola. Equinoctial orbit elements-application to artificial satellite orbits. In *AIAA/AAS Astrodynamics Conference*, pages AIAA 72–937, 1972.

[23] Joachim Baumgarte. Stabilization of constraints and integrals of motion in dynamical systems. *Comput Method Appl M*, 1(1):1–16, 1972.