Partial Derivatives of the Lambert Problem

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A procedure for deriving analytic partial derivatives of the Lambert problem is presented. Using the universal, cosine based Lambert formulation; first order partial derivatives of the velocities with respect to the positions and times are developed. Taking advantage of inherent symmetries and intermediate variables, the derivatives are expressed in a computationally efficient form. The added cost of computing these partials is found to be ∼10% to ∼60% of the Lambert compute cost. The availability of analytic partial derivatives increases optimization speed, efficiency and allows for trajectory optimization formulations that implicitly enforce continuity constraints via embedded Lambert problems.

Nomenclature

q = Scalar function
k = Universal variable
θ = Transfer angle
d = Transfer angle parameter (+1 for $0 < \theta < \pi$, -1 for $\pi < \theta < 2\pi$)
f, g = Lagrange f and g functions
τ = First Lambert geometry parameter
S = Second Lambert geometry parameter
μ = Gravitational parameter of the primary(GM)
T^* = Target time of flight
T_p = Parabolic time of flight
k^* = Value of k corresponding to T^*
N_{rev} = Number of current the Lambert revolution (short period)
TOF = Time of flight
$TOF^{'}$ = First derivative of TOF function with respect to k
$\vec{r_1}$ = Initial position vector (column vector)
$\vec{r_2}$ = Final position vector (column vector)
$\vec{v_1}$ = Initial velocity vector (column vector)
$\vec{v_2}$ = Final velocity vector (column vector)
r_1 = Magnitude of the initial position vector
r_2 = Magnitude of the final position vector
$\vec{p}$= $[\vec{r_1}, \vec{r_2}, T^*]^T$
$\vec{x}$ = Fundamental Lambert parameters; $[r_1, r_2, \theta, T^*]^T$
W = Auxiliary function
wrt = With Respect To

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I. Introduction

Lambert's problem is one of the most extensively studied problems in space-flight mechanics and enjoys a large volume of research, spanning over several decades. The solution to the Lambert problem and its derivatives acts as a building-block algorithm for various problems like trajectory design and optimization,[1,2,3] tour design,[4,2,5,6,7,8,9] and orbit determination.[10,11]

The Lambert solution results from a one dimensional root-solve, and therefore the sensitivities of the outputs to the inputs are not as straight forward as computing the partials of an explicit function. There is a rich literature on the Lambert problem in general but a notable absence exists on the topic of the Lambert problem partial derivatives. The partials can be formed from partitions of the Kepler state transition matrix,[14] and they also can be formed with traditional numerical differencing or complex step differentiation, or automatic differentiation. In this paper, the partials are given using the Lambert problem directly and efficiently computed without extra iterations assuming the user already must solve the Lambert problem itself. The technique can enable new optimization strategies that automatically enforce continuity in position without resorting to constrained non-linear programming and numerical solutions to the lambert boundary value problem.

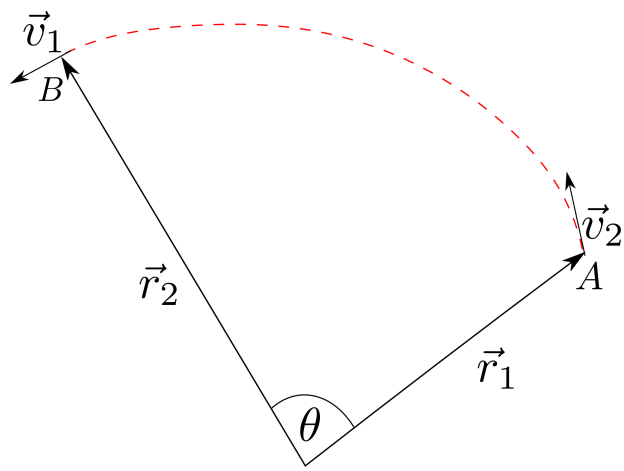
In this study, a procedure is presented for deriving partial derivatives of the Lambert problem. These partials are expressed as derivatives of the two velocity vectors with respect to the two position vectors and the times. The partial derivatives are developed based upon the recently proposed cosine based Lambert formulation.[12] The formulation takes advantage of a geometry-based parameters (S, τ) to simplify the universal formulation of the Lambert equation. This equation, defined as a function of the universal variable k, is shown to have simplified derivative expressions and requires only a single transcendental function (W) evaluation. Apart from enjoying 40% to 60% reduction in runtime over the current state-of-the art Gooding's method, the formulation also leads to an efficient representation of the partial derivatives. Note that the partial derivatives can also be computed by computing the Keplerian state transition matrix (STM's) and its adjoint, provided we already know the complete state vector (position and velocity) at one of the end points.[13,14]

The general procedure of computing the partials involves taking derivatives of the f and g functions (see pgs. 218-219 in Bate et al.[15]) and applying the chain rule, taking into account various implicit dependencies. The decoupling of the geometric (S, τ) and iterative terms (k) in the Lambert formulations simplifies the derivatives of the universal variable k with respect to the fundamental Lambert parameters (r_1, r_2, θ, T^*).

In this study all first order partial derivatives of the Lambert problem are derived and documented. Taking advantage of inherent symmetries and intermediate variables, the derivatives are be developed and expressed in a computationally efficient form. A large scale simulation is performed to quantify the cost of computing these partials partial derivatives. The next section gives a brief summary of the adopted Lambert formulation.

II. Multiple Revolution Lambert Formulation

This section summarizes, the Lambert formulation based on the new universal variable,[12] "k". Figure 1 depicts a general diagram of the Lambert problem. Vectors $\vec{r_1}$ and $\vec{r_2}$ are the bounding position vectors; TOF stands for the time of flight for the transfer and θ is the transfer angle between the two position vectors. All possible arcs connecting the two points A and B with the required TOF and which satisfy Keplerian motion are possible Lambert solutions. There are $2N_{ub}^D + 1$ direct solutions and $2N_{ub}^R + 1$ retrograde solutions where N_{ub}^D and N_{ub}^R is the maximum number of revolutions possible for direct and retrograde transfers, respectively, given a set of input parameters.



$\vec{v}_1$ and $\vec{v}_2$ are obtained after solving the Lambert problem

Figure 1. General problem geometry

As Keplerian motion is confined in a plane, the vectors $\vec{r_2}$ and $\vec{v_2}$ can be expressed as a function of $\vec{r_1}$ and $\vec{v_1}$, given by Eqs. 1 and 2.

$$\vec{r_2} = f\vec{r_1} + g\vec{v_1} \tag{1}$$
$$\vec{v_2} = \dot{f}\vec{r_1} + \dot{g}\vec{v_1} \tag{2}$$

where f, g, are the Lagrange f and g functions, respectively. The f and g functions assume vectors $\vec{r_1}$ and $\vec{v_1}$ form the basis of the resulting solution space. The formulation as defines two geometry based parameters, S and τ. The parameter τ is a non-dimensional function of r_1, r_2 (magnitude of positions, see nomenclature) and θ and varies between $\left[\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$. It is given as follows:

$$\tau = d\frac{\sqrt{r_1 r_2[1+\cos(\theta)]}}{r_1 + r_2} \quad , \quad d = \begin{cases} +1 & 0 \leq \theta \leq \pi \\ -1 & \pi \leq \theta \leq 2\pi \end{cases} \tag{3}$$

The parameter S has a unit of time and is defined as

$$S = \sqrt{\frac{(r_1 + r_2)^3}{\mu}} \tag{4}$$

Next, the Lambert TOF equation is defined as a function of S, τ and the universal variable k, summarized as follows:

$$TOF(k, S, \tau) = S\sqrt{1 - k\tau}\,[\tau + (1 - k\tau)W] \tag{5}$$

where W is given by the following expression:

$$W(k) = \begin{cases} \frac{(1-\mathrm{sgn}(k))\pi + \mathrm{sgn}(k)\cos^{-1}(1-m) + 2\pi N}{\sqrt{m^3}} - \frac{k}{m} & -\sqrt{2} \leq k < \sqrt{2} - \epsilon \text{ (elliptical orbits)} \\ \frac{-\cosh^{-1}(1-m)}{\sqrt{-m^3}} - \frac{k}{m} & k > \sqrt{2} + \epsilon \text{ (hyperbolic orbits)} \\ W_s & \sqrt{2} - \epsilon \leq k \leq \sqrt{2} + \epsilon \ (N = 0) \end{cases} \tag{6}$$

The expression for W_s is given in ref.[12] For a given value of k, f and g functions are used to

calculate the final $\vec{v_1}$ and $\vec{v_2}$ velocity vectors as follows:

$$f = 1 - \frac{(1-k\tau)(r_1+r_2)}{r_1} \tag{7}$$

$$g = S\tau\sqrt{(1-k\tau)\mu} \tag{8}$$

$$\dot{g} = 1 - \frac{(1-k\tau)(r_1+r_2)}{r_2} \tag{9}$$

$$\vec{v_1} = \frac{\vec{r_2} - f\vec{r_1}}{g} \tag{10}$$

$$\vec{v_2} = \frac{\dot{g}\vec{r_2} - \vec{r_1}}{g} \tag{11}$$

Note that the above equations as given in ref.[12] have a typographical error, which has been corrected here.

III. Partials of the Lambert Problem

The partials of the Lambert problem are defined as partials of $\vec{y}$ wrt $\vec{p}$, where $\vec{y}$ and $\vec{p}$ are given as follows:

$$\vec{y} = [\vec{v_1}, \vec{v_2}]^T \tag{12}$$

$$\vec{p} = [\vec{r_1}, \vec{r_2}, T^*]^T \tag{13}$$

To obtain $\frac{\partial \vec{y}}{\partial \vec{p}}$ we need to compute $\frac{\partial q}{\partial \vec{p}}$, where q can be either f, g or $\dot{g}$ (see Eq. 10 and 11).

$$q \in \{f, g, \dot{g}\} \tag{14}$$

The scalar q itself is a function of a vector $\vec{p}$ and the Lambert iteration variable k. Using the chain rule, $\frac{\partial q}{\partial \vec{p}}$ can be computed as follows:

$$\frac{\partial q}{\partial \vec{p}} = q_{\vec{p}}|_{k=const.} + q_k k_{\vec{p}} \tag{15}$$

where $k_{\vec{p}}$ is given as follows:

$$\frac{\partial k}{\partial \vec{p}} = k_{\vec{x}} \vec{x}_{\vec{p}} \tag{16}$$

$$\vec{x} = [r_1, r_2, \theta, T^*]^T \tag{17}$$

Next, the partials required to compute Eq. 16 are derived in the next section.

A. Fundamental Lambert problem partials: $k_{\vec{x}}$

The partials given by $k_{\vec{x}}$ represent the fundamental partials of the Lambert problem and are also the most difficult to obtain. A systematic procedure based on the Lambert TOF equations (Eq. 5) is followed. Given $\vec{x}$; the solution to the Lambert problem is found via a one-dimensional root-solve of the function L, given below:

$$L(\vec{x}) = TOF(k(\vec{x}), S(\vec{x}), \tau(\vec{x})) - T^*(\vec{x}) \tag{18}$$

Next, taking the partial of the above wrt to the vector $\vec{x}$, we get the following:

$$\frac{\partial TOF}{\partial \vec{x}} - \frac{\partial T^*}{\partial \vec{x}} \tag{19}$$

At the solution $\partial L = 0$, where k^* is the root-solved value of k. The first term on RHS of Eq. 19 can be written as:

$$\frac{\partial TOF}{\partial \vec{x}} = \frac{\partial k}{\partial \vec{x}} \left.\frac{\partial TOF}{\partial k}\right|_{\tau,S=const.} + \frac{\partial S}{\partial \vec{x}} \left.\frac{\partial TOF}{\partial S}\right|_{\tau,k=const.} + \frac{\partial \tau}{\partial \vec{x}} \left.\frac{\partial TOF}{\partial \tau}\right|_{S,k=const.} \tag{20}$$

$$\left.\frac{\partial TOF}{\partial S}\right|_{\tau,k=const.} = \frac{TOF|_{k=k^*}}{S}$$

$$\left.\frac{\partial TOF}{\partial \tau}\right|_{\tau,k=const.} = \frac{TOF|_{k=k^*}}{\tau\gamma},\ \gamma = \frac{1+cW}{1-\frac{k^*}{2c}\left[3cW+1\right]},\ c = \frac{1-k^*\tau}{\tau}$$

$$\left.\frac{\partial TOF}{\partial k}\right|_{\tau,k=const.} = TOF^{'}$$

$$TOF|_{k=k^*} = T^*$$

$$TOF^{'} = \frac{-T^*}{2c} + S\tau\sqrt{c\tau}(W^{'}c - W)$$

where $W^{'}$ is the partial of the auxiliary Lambert variable wrt k, expressed as follows:

$$W^{'} = \begin{cases} \frac{-2+3Wk}{m} & k < \sqrt{2}-\epsilon, k > \sqrt{2}+\epsilon \\ \frac{\partial W_s}{\partial k} & \sqrt{2}-\epsilon < k < \sqrt{2}+\epsilon \end{cases} \tag{21}$$

Note that $TOF^{'}$ and $W^{'}$ are computed and saved during the root solve of L. For all subsequent sections we drop the superscript $*$ over k for presentation purposes. Substituting the above expressions (Eq. 20) in Eq. 19 and equating it to zero, the following expression for $k_{\vec{x}}$ is obtained:

$$k_{\vec{x}} = \frac{\partial k}{\partial \vec{x}} = \frac{-T^*}{TOF^{'}}\left[-\frac{T^*_{\vec{x}}}{T^*} + \frac{S_{\vec{x}}}{S} + \frac{\tau_{\vec{x}}}{\tau\gamma}\right] \tag{22}$$

$$T^*_{\vec{x}} = \frac{\partial T^*}{\partial \vec{x}},\ S_{\vec{x}} = \frac{\partial S}{\partial \vec{x}},\ \tau_{\vec{x}} = \frac{\partial \tau}{\partial \vec{x}}$$

The expressions for the partials $T^*_{\vec{x}}$, $S_{\vec{x}}$ and $\tau_{\vec{x}}$ are given as follows:

$$T^*_{\vec{x}} = [0,0,0,1] \tag{23}$$

$$S_{\vec{x}} = [S_{r1}, S_{r2}, 0, 0]$$

$$\tau_{\vec{x}} = [\tau_{r1}, \tau_{r2}, \tau_\theta, 0]$$

$$S_{r1} = S_{r2} = \frac{3S}{2(r_1+r_2)}$$

$$\tau_{r1} = \tau\left[\frac{1}{2r_1} - \frac{1}{r_1+r_2}\right],\ \tau_{r2} = \tau\left[\frac{1}{2r_2} - \frac{1}{r_1+r_2}\right]$$

$$\tau_\theta = \frac{-\tau\sin(\theta)}{2\left(1+\cos(\theta)\right)}$$

B. Self partials: $\vec{x}_{\vec{p}}$, $S_{\vec{p}}$ and $\tau_{\vec{p}}$

The self partials, $\vec{x}_{\vec{p}}$, aid to map the scalar partial derivatives of k to their corresponding vector versions. They are simply defined as follows:

$$\vec{x}_{\vec{p}} = \left[\frac{\partial r_1}{\partial \vec{p}}, \frac{\partial r_2}{\partial \vec{p}}, \frac{\partial \theta}{\partial \vec{p}}, \frac{\partial T^*}{\partial \vec{p}}\right]^T \tag{24}$$

The resulting expression is matrix of size 4×7. The individual "row" partials in the above expression can be a calculated as:

$$\begin{aligned}
\frac{\partial r_1}{\partial \vec{p}} &= \left[\hat{r_1}^T, 0, 0, 0, 0\right] \\
\frac{\partial r_2}{\partial \vec{p}} &= \left[0, 0, 0, \hat{r_2}^T, 0\right] \\
\frac{\partial \theta}{\partial \vec{p}} &= \left[\frac{\hat{r_1}^T \cos(\theta) - \hat{r_2}^T}{r_1 \sin(\theta)}, \frac{\hat{r_2}^T \cos(\theta) - \hat{r_1}^T}{r_2 \sin(\theta)}, 0\right] \\
\frac{\partial T^*}{\partial \vec{p}} &= [0, 0, 0, 0, 0, 0, 1]
\end{aligned} \tag{25}$$

where $\hat{r_1}^T$ and $\hat{r_2}^T$ are unit vectors corresponding to $\vec{r_1}$ and $\vec{r_2}$,respectively. The self partials, $\frac{\partial S}{\partial \vec{p}}$ and $\frac{\partial \tau}{\partial \vec{p}}$ are required for computing $\frac{\partial q}{\partial \vec{p}}$ and are given as:

$$\begin{aligned}
S_{\vec{p}} &= \frac{\partial S}{\partial \vec{p}} = S_{\vec{x}} \vec{x}_{\vec{p}} \\
\tau_{\vec{p}} &= \frac{\partial \tau}{\partial \vec{p}} = \tau_{\vec{x}} \vec{x}_{\vec{p}}
\end{aligned} \tag{26}$$

where the partials $S_{\vec{x}}$ and $\tau_{\vec{x}}$ are given in Eq. 23. The next section summarizes the partials required to compute the chain rule.

C. Chain rule partials: $q_{\vec{p}}$ and q_k

The partials to complete the chain rule in Eq. 15, are given as follows:

$$\begin{aligned}
q_{\vec{p}}|_{k=const.} &\in \left\{f_{\vec{p}}, g_{\vec{p}}, \dot{g}_{\vec{p}}\right\}|_{k=const.} \\
q_k &\in \{f_k, g_k, \dot{g}_k\}
\end{aligned} \tag{27}$$

Obtaining the partials in Eq. 27 require the partials of the f and g functions wrt $\vec{p}$, given by:

$$\begin{aligned}
f_{\vec{p}}|_{k=const.} &= \left(f_{r_1}\frac{\partial r_1}{\partial \vec{p}} + f_{r_2}\frac{\partial r_1}{\partial \vec{p}} + f_\tau \frac{\partial \tau}{\partial \vec{p}}\right)^T \\
g_{\vec{p}}|_{k=const.} &= \left(g_S \frac{\partial S}{\partial \vec{p}} + g_\tau \frac{\partial \tau}{\partial \vec{p}}\right)^T \\
\dot{g}_{\vec{p}}|_{k=const.} &= \left(\dot{g}_{r_1} r_{1\vec{p}} + \dot{g}_{r_2} r_{2\vec{p}} + \dot{g}_\tau \tau_{\vec{p}}\right)^T \\
f_{r_1} = \frac{(1-f)(r_1 - r_2)}{r_1},\ f_{r_2} &= \frac{(1-f)r_2}{r_1 + r_2},\ \dot{g}_{r_1} = \frac{(1-\dot{g})r_1}{r_1 + r_2},\ \dot{g}_{r_2} = \frac{(1-\dot{g})(r_2 - r_1)}{r_2} \\
f_\tau &= \frac{k(r_1 + r_2)}{r_1},\ \dot{g}_\tau = \frac{k(r_1 + r_2)}{r_2} \\
g_\tau &= g\left[\frac{1}{\tau} - \frac{k}{2c\tau}\right],\ g_S = \frac{g}{S} \\
f_{\vec{p}} &= \frac{\partial f}{\partial \vec{p}},\ g_{\vec{p}} = \frac{\partial g}{\partial \vec{p}},\ \dot{g}_{\vec{p}} = \frac{\partial \dot{g}}{\partial \vec{p}}
\end{aligned} \tag{28}$$

where c has been computed before in Eq. 22. Computing q_k require the partials of the f and g functions wrt k, which are given as:

$$\begin{aligned}
f_k = \frac{\tau(r_1 + r_2)}{r_1},\ \dot{g}_k &= \frac{\tau(r_1 + r_2)}{r_2} \\
g_k &= \frac{g}{2c}
\end{aligned} \tag{29}$$

Finally, the derivatives defined in Eq. 15 are given as follows:

$$\begin{gathered} q_{\vec{p}} = q_{\vec{p}}|_{k=const.} + q_k k_{\vec{p}} \\ q_{\vec{p}} \in \left\{ f_{\vec{p}}, g_{\vec{p}}, \dot{g}_{\vec{p}} \right\} \\ k_{\vec{p}} = k_{\vec{x}} \vec{x}_{\vec{p}} \end{gathered} \tag{30}$$

Equation 30 can be used to generate complete partials of f,g and $\dot{g}$ wrt the vector $\vec{p}$. We are now in a position to compute the partials of the velocity vectors.

D. Partials of the Velocity Vectors

The partial of the velocities $\vec{v_1}$ and $\vec{v_2}$ wrt to the vector $\vec{p}$ can be computed by differentiating Eqs. 10 and 11 along with the Eq. 30 and are given as follows:

$$\begin{aligned} \frac{\partial \vec{v_1}}{\partial \vec{p}} &= \frac{\frac{\partial \vec{r_2}}{\partial \vec{p}} - \left[f_{\vec{p}} \vec{r_1} + f \frac{\partial \vec{r_1}}{\partial \vec{p}} \right] - \vec{v_1} g_{\vec{p}}}{g} \\ \frac{\partial \vec{v_2}}{\partial \vec{p}} &= -\frac{\frac{\partial \vec{r_1}}{\partial \vec{p}} - \left[\dot{g}_{\vec{p}} \vec{r_2} + \dot{g} \frac{\partial \vec{r_2}}{\partial \vec{p}} \right] - \vec{v_2} g_{\vec{p}}}{g} \end{aligned} \tag{31}$$

Algorithm 1 outlines the steps required for computing the partials, given a solution to the Lambert problem.

Algorithm 1 Pseudo Algorithm for computing the Lambert partials

1: **procedure** PARTIALS_KLAM($\mu, k, S, \tau, \vec{y}, \vec{x}, \vec{p}, W^{'}, TOF'$) ▷ Input parameters from a Lambert solution
2: Compute the fundamental partials $\frac{\partial k}{\partial x}$ using Eq. 22.
3: Store the intermediate partials $\frac{\partial S}{\partial x}$ and $\frac{\partial \tau}{\partial x}$ for the next step.
4: Compute the self partials $\frac{\partial \vec{x}}{\partial \vec{p}}$, $\frac{\partial S}{\partial \vec{p}}$ and $\frac{\partial \tau}{\partial \vec{p}}$ from Eq. 24 and 26, respectively.
5: Compute the chain partials $\frac{\partial f}{\partial \vec{p}}$, $\frac{\partial g}{\partial \vec{p}}$ and $\frac{\partial \dot{g}}{\partial \vec{p}}$ from Eq. 30 using Eqs. 28 and the already computed partials in steps 2 and 4.
6: Compute the required partials $\frac{\partial \vec{y}}{\partial \vec{p}}$ using Eq. 31.
7: **end procedure**

IV. Performance and Accuracy Benefits

For estimating the cost of computing the partials in Eq. 31, a computationally efficient form is implemented in Fortran alongside the original code from ref.[12] Figure 2 shows the variation of ratio of runtimes for the Lambert plus partial computation vs. the Lambert only case, as a function of $\frac{T^*}{T_p}$, where T_p is the parabolic time for the input geometry. Each data point represents 200,000 Lambert cases with randomly varying $\vec{r_1}$ and $\vec{r_2}$. The ratio $\frac{TOF}{T_p}$ is varied from 0.05 to ∼20, thereby allowing hyperbolic ($\frac{T^*}{T_p} < 1$), parabolic ($\frac{T^*}{T_p}$=1) and elliptical multi-rev ($\frac{T^*}{T_p} > 1$) solutions. The target flight time (T^*) is varied from ∼0.01 TU to ∼1000 TU, when expressed in normalized units ($r_1 = 1$ and $\mu = 1$) . The variable N_{rev} denotes the fixed number of Lambert revolutions.

The cost of computing the partials is fixed while the cost of computing the Lambert solution varies as a function of $\frac{T^*}{T_p}$. This discrepancy is evident from the results in Fig. 2 near the parabolic and for the multi-rev ($N_{rev} > 0$) case. For elleptical zero-rev case and hyperbolic cases, the extra cost of computing the partials is around 24% and 26%, respectively. For the multi-rev this number varies between 10% and 60% depending upon the ratio $\frac{T^*}{T_p}$ and N_{rev} (see Fig. 2). The Lambert

routine itself varies in runtime because the number of iterations necessary to converge varies from case to case (usually between 2 and 4 iterations).

On the other hand the extra computational cost of computing the partials via the forward and central difference method is 700% and 1,400%, respectively. Recall that computing $\frac{\partial \vec{y}}{\partial \vec{p}}$ via forward or central differencing requires 7 and 14 additional lambert calls, respectively. Furthermore, the finite difference methods are limited in accuracy due to precision and roundoff error, and require tuning for step sizes, while our analytic approach achieves accuracy near to that of the computed velocity vectors. Hence, the analytic partials, presented in this paper, provide up to an order of magnitude in speed benefit while achieving high accuracy.

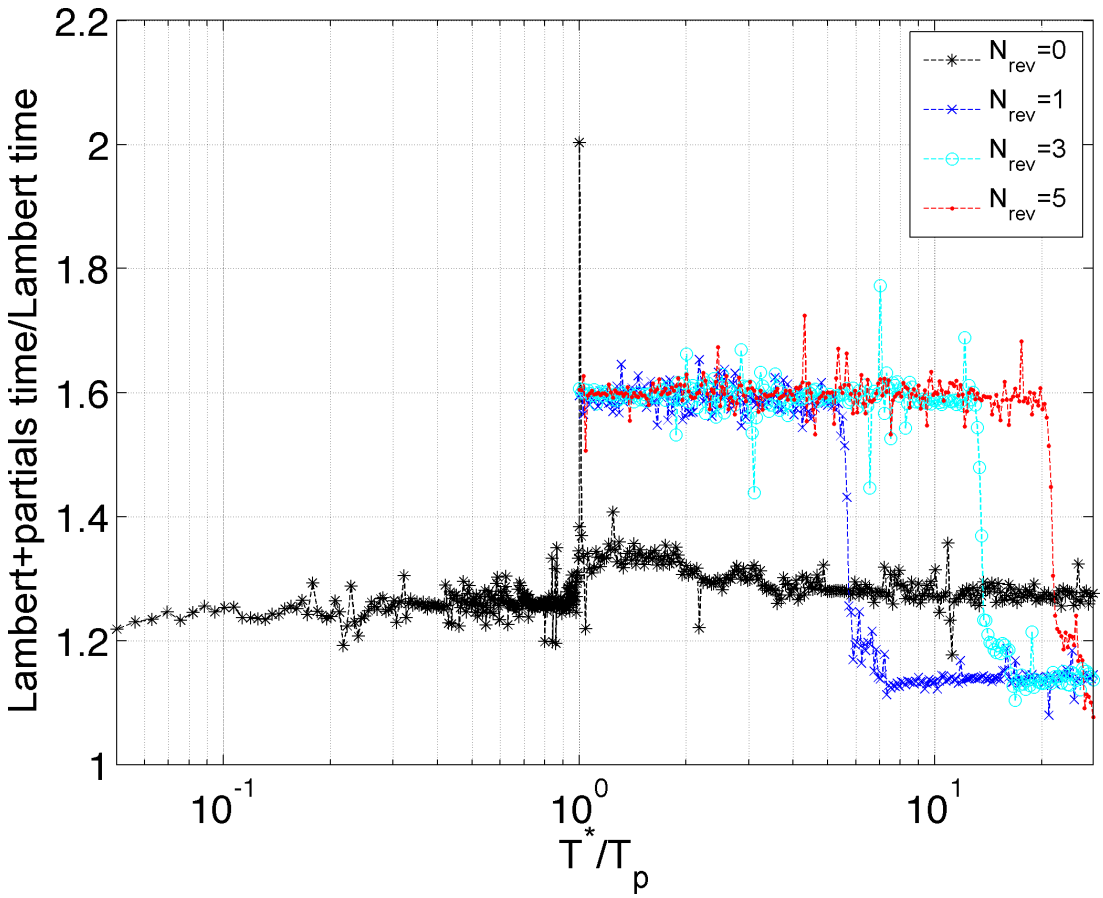


Figure 2. Runtime ratio vs. $\frac{T^*}{T_p}$

V. Conclusion

In this paper an approach to derive analytic partial derivatives of the Lambert problem, is presented. Based on a recently proposed cosine based Lambert formulation, first order partials of the two velocity vectors with respect to the two position vectors and the times, are derived and documented. Various other intermediate partials relevant to the Lambert problem are also derived. The partials, expressed in a computationally efficient form, requires ∼10% to ∼60% more computation time over the Lambert problem. This results in up to an order of magnitude in speed benefit over the finite difference approach. Being analytic in nature, the accuracy of the partials is limited only by the accuracy of the underlying Lambert solution.

Future work will focus on deriving and documenting the second order partial derivatives. De-

signing trajectory optimization algorithms which exploit these partials will also be investigated. The availability of analytic partial derivatives allows for novel algorithmic improvements and could result in increased speed and efficiency of the trajectory design and optimization process.

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References

[1]Abdelkhalik, O. and Mortari, D., "N-Impulse Orbit Transfer Using Genetic Algorithms," *Journal of Spacecraft and Rockets*, Vol. 44, 2007.

[2]Longuski, J. M. and Williams, S. N., "Automated design of gravity-assist trajectories to Mars and the outer planets," *Celestial Mechanics and Dynamical Astronomy*, Vol. 52, 1991, pp. 207–220.

[3]Russell, R. P. and Ocampo, C. A., "Geometric Analysis of Free-Return Trajectories Following a Gravity-Assisted Flyby," *Journal of Spacecraft and Rockets*, Vol. 42, No. 1, 2005, pp. 138–151.

[4]Uphoff, C., Roberts, P. H., and Friedman, L. D., "Orbit Design Concepts for Jupiter Orbiter Missions," *Journal of Spacecraft and Rockets*, Vol. 13, No. 6, 1976, pp. 348–355.

[5]Russell, R. P., *Global Search and Optimization for Free-Return Earth-Mars Cyclers*, Ph.D. thesis, The University of Texas at Austin, august 2004.

[6]Strange, N. J. and Longuski, J. M., "Graphical Method for Gravity-Assist Trajectory Design," *Journal of Spacecraft and Rockets*, Vol. 39, No. 1, 2002.

[7]Labunsky, A. V., Papkov, O. V., and Sukhanov, K. G., *Multiple gravity assist interplanetary trajectories*, ESI book series, 1998.

[8]Heaton, A. F., Strange, N. J., Longuski, J. M., and Bonfiglio, E. P., "Automated Design of the Europa Orbiter Tour," *Journal of Spacecraft and Rockets*, Vol. 39, No. 1, 2002, pp. 17–22.

[9]Buffington, B. and Strange, N., "Patched-Integrated Gravity-Assist Trajectory Design," *AAS/AIAA Astrodynamics Specialist Conference, AAS 07-276*, 2007.

[10]Marsden, B. G., "Initial orbit determination - The pragmatist's point of view," *Astronomical Journal*, Vol. 90, 1985, pp. 1541–1547.

[11]Beley, K. B., "Analytic Determination of Perigee Passage Using Lambert's Therorem," *Journal of Spacecraft and Rockets*, Vol. 4, No. 8, 1967, pp. 1101–1103.

[12]Arora, N. and Russell, R. P., "A Fast and Robust Multiple Revolution Lambert Algorithm Using a Cosine Transformation," *AAS/AIAA Astrodynamics Specialist Conference, Hilton Head, AAS 13-728*, 2013.

[13]Pitkin, E. T., "Second Transition Partial Derivatives Via Universal Variables," *Journal of the Astronautical Sciences*, Vol. 13, 1966, pp. 204.

[14]Battin, R. H., "An Introduction to the Mathemat-ics and Methods of Astrodynamics," *AIAA Education Series*, 1999.

[15]Bate, R. R., Mueller, D. D., and White, J. E., *Fundamentals of astrodynamics*, Dover, 1971.

[16]Wchter, A. and Biegler, L. T., "On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming," *Mathematical Programming*, Vol. 106, No. 1, 2006, pp. 25–57.