A Study of Cross Polarization Effects in Reflector Antenna Arrays

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Abstract—This paper addresses the issue of cross-polarized field components of an array of antennas as compared to that of individual elements of the array. For a single antenna, the co- and cross-polarization components are completely correlated in terms of phase and amplitude in a given direction. The co/cross-pol relations vary as a function of angular position from peak of the beam, which is important when there is pointing error. More specifically, for the reflector antennas, this relation might vary as the antenna points in different directions in azimuth and elevation, due to the changes in gravity profile, wind effects, temperature changes, etc., on the surface and feed/sub/main reflector alignment. In an array environment, these changes will vary among various antennas in the array, and indeed very small mechanical and design variations in the antenna elements (in terms of feed horns, feed/reflector misalignments, surface variations, etc., will contribute to the cross polarization variations.

Here we present a study of the effects of the variation of individual antennas on the overall polarization of an array. We first provide a general introduction to the polarization concept and formulation and then provide the results of a statistical study. Ample plots are provided to illustrate the effects. We show that the co/cross polarization ratio for the array is no higher than the worst of the individual elements and indeed is much lower in the majority of cases. This study is useful in the general design of large arrays of small reflector antennas in many instances and specifically for the NASA/JPL Deep Space Network (DSN). [1, 2]

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1. INTRODUCTION

JPL/NASA has been studying and performing experiments in the use of arrays of relatively small reflector antenna of the order of a few meters in diameter instead of single large antennas of the order of tens of meter which are presently used [1-7]. Among many issues of concern, the effects of random phase and amplitude errors among the elements and their overall effect on the performance of the array can be mentioned and has been studied in some detail.

Another issue of concern is the behavior of co- and cross polarization components among the various elements of the array and the effect of their random variations on the overall polarization of the array. For a given element, the co- and cross-pol components are completely correlated in terms of phase and amplitude in a given direction. The co/cross-pol relation varies as a function of angle from peak of the beam and around the beam, which is of significance when there is pointing error. Also, this relation might vary as the antenna points in different directions in azimuth and elevation, due to the changes in gravity profile, wind effects, temperature changes, etc. and the corresponding changes on the surface and alignments. The co-pol might vary only slightly, but due to the very low amplitude level of the cross-pol with respect to the co-pol, and its sensitivity to minor physical changes, the cross-pol may be affected more substantially.

In an array environment, due to the low levels of the cross-pol with respect to the co-pol, the very small mechanical and design variations in each antenna (in terms of feed horns, feed/reflector misalignments, and the surface variations, etc.) will have relatively larger effects (both in phase and amplitude) on the cross-pol than the co-pol components of the electromagnetic (EM) field which will vary among various antenna elements in the array.

Antennas for deep space communication are usually circularly polarized in order to combat the effects of Faraday rotations through the ionosphere. In an array environment, if

the array elements are separated by large distances (of the order of hundreds of meters or more) the Faraday rotation might be different for different elements and its impact on the array performance might be of significance and requires further study. In this paper, however, we assume that the array elements are spaced close enough such that the Faraday rotation effect is nearly identical for all the elements.

As we will show in the following sections, when the co-pol components of the array elements are made to align and be in-phase, the cross-pol components will have different phase and orientations with the overall effect of producing the same or a smaller total cross-pol for the array as compared to a single element, and in any case, will always be better than that of the worst element of the array.

2. TRANSMISSION OF THE FIELD

The most useful parameter of the antenna in the transmit mode (see Appendices I and II for details) is the electric field vector given as

$$\bar{E}(\theta,\phi) = \sqrt{\frac{P_t Z_0}{4\pi r^2}} g(\theta,\phi)\hat{p}(\theta,\phi) = E(\theta,\phi)\hat{p}(\theta,\phi)$$

In which r, is the distance from antenna phase center to field point, $Z_0 \cong 377$ ohm is the free space wave impedance, $g(\theta,\phi) = \sqrt{G(\theta,\phi)}$ is the voltage gain function related to the power gain function, and $\hat{p} = p_r\hat{r} + p_l\hat{l}$ is Complex unit polarization vector written in terms of the Right-handed Circular polarization (RCP) and Left-handed Circular Polarization (LCP) complex unit vectors, as explained in Appendix II. Thus, the complex polarization vector completely describes the state of the polarization of the field. Ideally, the polarization for ground-to-space applications should be either purely RCP or LCP. However, due to various errors, particularly at the feeding stage, as will be described later, a certain amount of the cross-pol is generated which will not be received by a corresponding RCP or LCP receiving antenna and becomes a cause of gain reduction and/or interference with other receiving systems.

The electric field of the array can then be written as

$$\begin{aligned}\bar{E}(\theta,\phi) &= \sum_n E_n(\theta,\phi)\hat{p}_n(\theta,\phi) \\ &= \sum_n E_n(\theta,\phi)(p_{rn}\hat{r} + p_{\ln}\hat{l}) \\ &= \sum_n E_n(\theta,\phi)p_{rn}\hat{r} + \sum_n E_n(\theta,\phi)p_{ln}\hat{l}\end{aligned}$$

The power density at a given observation point in space can be written as

$$\begin{aligned}P &= \frac{1}{Z_0}\bar{E}(\theta,\phi)\bar{E}^*(\theta,\phi) \\ &= \frac{1}{Z_0}\left\{\sum_n E_n(\theta,\phi)(p_{rn}\hat{r} + p_{\ln}\hat{l})\right\}\left\{\sum_n E_n(\theta,\phi)(p_{rn}\hat{r} + p_{\ln}\hat{l})\right\}^* \\ &= \frac{1}{Z_0}\sum_n |E_n(\theta,\phi)|^2 (|p_{rn}|^2 + |p_{\ln}|^2)\end{aligned}$$

Notice that this is the total power density transmitted by the array at a given direction. The actual amount absorbed by the receiving antenna will be discussed below.

3. RECEPTION OF THE FIELD

The most useful parameter of an antenna in the receive mode is the "length" vector which is given as

$$\bar{L}(\theta,\phi) = \lambda\sqrt{\frac{R_a}{\pi Z_0}} g(\theta,\phi)\hat{p}(\theta,\phi) = L(\theta,\phi)\hat{p}(\theta,\phi)$$

in which

$$L(\theta,\phi) = \lambda\sqrt{\frac{R_a}{\pi Z_0}} g(\theta,\phi)$$

is defined as antenna "length" , λ is the wavelength, and R_a is the real part of the receiving antenna impedance, $Z_a = R_a + jX_a$.

Then the open circuit voltage at the receiving antenna due to an incoming field is given very appropriately by

$$V = \bar{E}_t \cdot \bar{L}_r^* = E_t L_r(\hat{p}_t \cdot \hat{p}_r^*)$$

and the power delivered to the receiver can be simply calculated. For example, the power delivered to a matched load impedance $Z_l = R_a - jX_a$ is given by

$$P_r = \frac{|V|^2}{4R_a} = \frac{(E_t L_r)^2}{4R_a}\eta_p = \frac{P_t G_t G_r}{(4\pi r/\lambda)^2}\eta_p$$

In which

$$\eta_p = \left|(\hat{p}_t \cdot \hat{p}_r^*)\right|^2$$

is the polarization efficiency. The received power equation above is the familiar Friis transmission formula in which the polarization properties of the transmitting and receiving antennas have also been incorporated.

For a transmitting array impinging on a single receiving antenna, the polarization efficiency is found in the following manner. Upon expanding the term

$$\hat{p}_t \cdot \hat{p}_r^* = \left\{ \left[\frac{\sum_n p_{trn}}{n} \right] \hat{r} + \left[\frac{\sum_n p_{tln}}{n} \right] \hat{l} \right\} \cdot (p_{rr}^* \hat{r} + p_{rl}^* \hat{l})$$

$$= \left[\frac{\sum_n p_{trn}}{n} \right] p_{rr}^* + \left[\frac{\sum_n p_{tln}}{n} \right] p_{rl}^*$$

We obtain

$$\eta_p = \left| \left[\frac{\sum_n p_{trn}}{n} \right] p_{rr}^* + \left[\frac{\sum_n p_{tln}}{n} \right] p_{rl}^* \right|^2$$

4. Sources of Polarization Error

In an antenna designed for producing a perfect circularly polarized wave (RCP or LCP), there are always some errors that cause the polarization to be slightly elliptic, which is a combination of LCP and RCP. The primary error is caused at the generating point of the field, e.g., the design variation of the feed horn of a reflector antenna. Other elements and components of the antenna system such as mirrors, frequency selective surfaces (FSS), etc., may also contribute to depolarization at later stages before radiation into space.

At the feed level, the RCP field generation is usually achieved by combining two linearly polarized sources of equal strength which are normal to each other (90° physical rotation) and temporally out of phase by 90° as well. The error is caused primarily in one of the following three ways:

i) Amplitude error, α: Ratio of the vertical to horizontal amplitude is different from unity.

ii) Phase error, δ: Phase between vertical and horizontal components is different from 90°.

iii) Angle error, γ: Angle between vertical and horizontal components is different from 90°.

These errors are graphically shown in Figure 1.

Additional cross-pol errors in final radiated field are introduced due to the geometry of the feed/sub-reflector(s)/main reflector, as well as support struts for the subreflectors. Of course, these errors may not be identical across the various elements of the array and will contribute to the final discrepancy in the phases and amplitudes of the cross-pol components across the array.

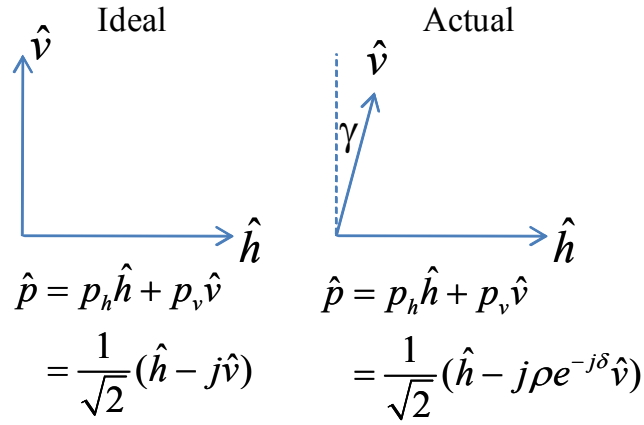


Fig. 1– Comparing ideal and actual polarization vectors

5. Array Cross-Polarization Errors

Now to assess the effects of the cross-polarization errors of the array (assuming RCP as co-pol), we write the co- and cross-pol components of the array polarization as

$$\frac{1}{n}\sum_n p_{trn} = |p_{tr}| e^{j\phi_{tr}} \quad \text{and} \quad \frac{1}{n}\sum_n p_{tln} = |p_{tl}| e^{j\phi_{tl}}$$

And that of the receiving antenna as

$$p_{rr} = |p_{rr}| e^{j\phi_{rr}} \quad \text{and} \quad p_{rl} = |p_{rl}| e^{j\phi_{rl}}$$

The polarization efficiency can, upon some manipulation, be written as

$$\eta_p = |p_{tr}|^2 |p_{rr}|^2 \left[1 + X_t^2 X_r^2 + 2X_t X_r \cos(\phi_{xt} - \phi_{xr}) \right]$$

In which cross-pol ratios and relative phases are defined as

$$X_t = \left| \frac{p_{tl}}{p_{tr}} \right|, \; X_r = \left| \frac{p_{rl}}{p_{rr}} \right|, \; \phi_{tx} = \phi_{tr} - \phi_{tl}, \; \phi_{rx} = \phi_{rr} - \phi_{rl}$$

The efficiency in general will vary in the minimum and maximum range provided by

$$\eta_p = |p_{tr}|^2 |p_{rr}|^2 (1 \pm X_t X_r)^2$$

However, in cases that receiving antenna has zero cross-pol, $p_{rl} = 0$, or cross-pol components of the elements of transmit array cancel due to the phase variations such that $p_{tl} = 0$, the above equation will be reduced to the simple expression

$$\eta_p = |p_{tr}|^2 |p_{rr}|^2$$

When co-pol components of the elements of the array are aligned in phase, we have

$$\left|\frac{\sum p_{trn}}{n}\right| = \frac{\sum |p_{trn}|}{n} \approx |p_{tr}| \quad \text{and} \quad \left|\frac{\sum p_{ln}}{n}\right| \le \frac{\sum |p_{ln}|}{n}$$

These formulas indicate that due to very small values of cross-pol for all elements, the co-pols are nearly identical and magnitude of the array co-pol component is the average of the co-pol magnitudes of all array elements, and the magnitude of the cross-pol of the array is in general less than the average of the magnitudes of all individual elements.

In the event of pointing misalignment of one or more elements, the individual cross-pols of the elements will vary but the above observations are still valid. The total co-pol component, however, might become slightly less which is the pointing error loss of the array.

It should also be noted that the reduction of the cross-pol at a given receiving point does not imply a power transfer into the co-pol component or a gain increase. However, it does imply less interference into non-target antennas which operate at opposite polarization. The actual received power into the desired target antenna, however, depends on its polarization properties as discussed above.

6. Numerical Examples

At this point we provide a number of examples to illustrate the effects of the polarization errors and the change in the array cross polarization due to the variations of that of the individual elements of the array.

1- *Role of amplitude and phase variations on Cross Polarization*. Figures 2(a, b) show the plots of the RCP and LCP components when there are amplitude errors in the two linear components as stated in Section 4. It can be observed that due to an amplitude error ratio such that the vertical components is twice the horizontal component or vice versa, we get an elliptically polarized field. Similarly, in Figures 3(a, b) the effect of phase error between the two linear components (ellipticity and tilt) are clearly observed. Similar results have been obtained due to angular tilt of the two linear components.

Notice that in all figures, the total polarization is shown by green ellipses, the co-pol by blue circles, the unwanted cross-pol by red circles, and finally dashed black circles represents the ideal circular polarization cases.

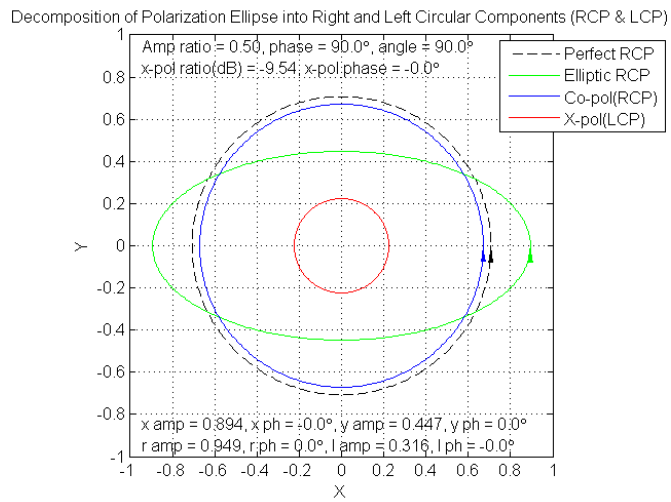


Fig. 2a– Depolarization due to amplitude ratio error=0.5

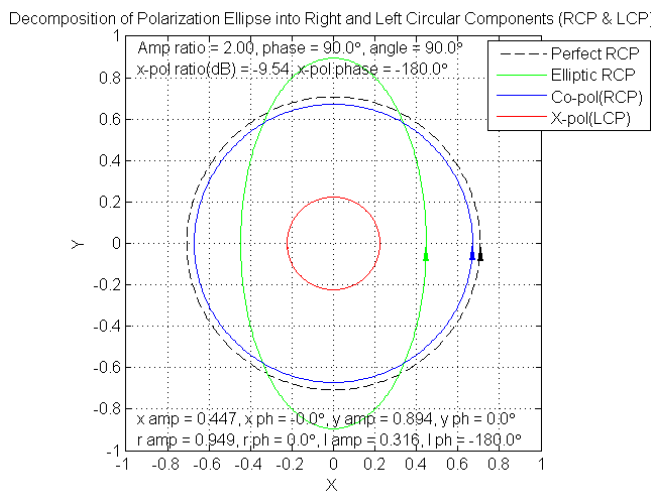


Fig. 2b- Depolarization due to amplitude ratio error=2

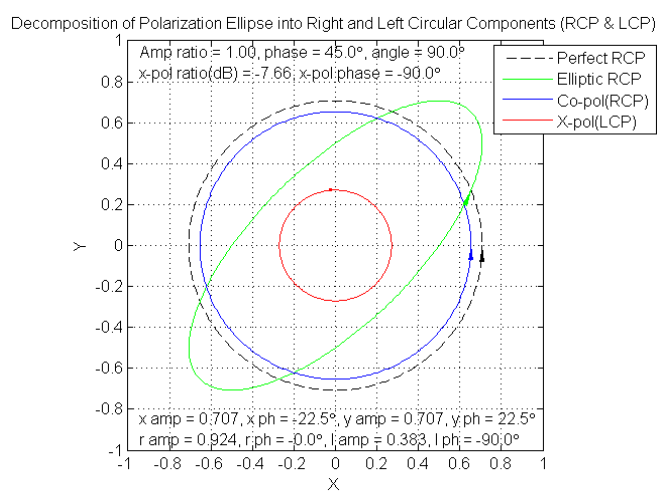


Fig. 3a - Depolarization due to phase error (45° vs 90°)

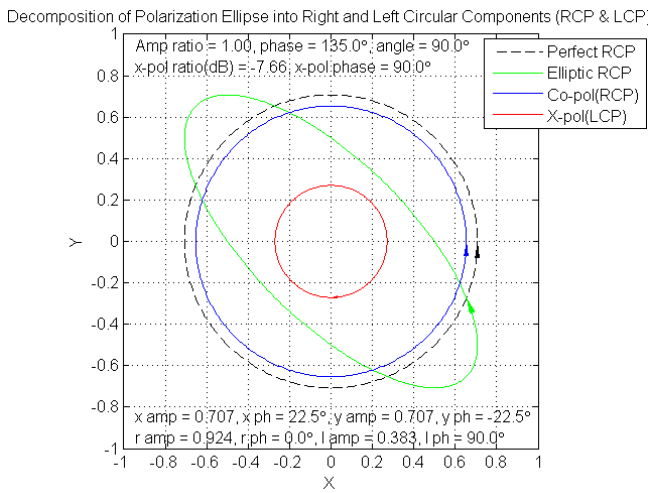


Fig. 3b - Depolarization due to phase error (135° vs 90°)

2- We present the reduction of cross-pol in a two-element array in Figures 4(a, b, c). One element has a phase error of -20° (70° between the vertical and horizontal components instead of the ideal 90°) and the other an error of +10° (100° instead of the ideal 90°). The reduction of the unwanted cross-pol is quite evident in this case. Similar results have been obtained for other errors but will not be presented here.

3- Next, the case of a 10-element array is considered. Assuming that the cross-pol phase errors among the elements obey a Gaussian distribution with an rms = 20°, the results for the array polarization are shown in Figure 5. The reduction in cross pol can be clearly seen. Similar results are obtained for amplitude and angle errors. As a final example we show the case with rms errors of 20° for both the phase as well as the rotation angle errors in the 10 element array. Figure 6 shows the results which indicate even more reduction in the cross-pol level of the array.

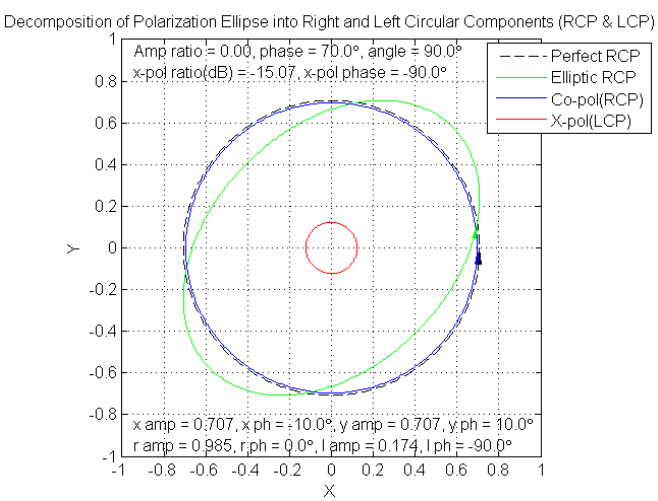


Fig. 4a – Polarization for 1st element with -20° phase error

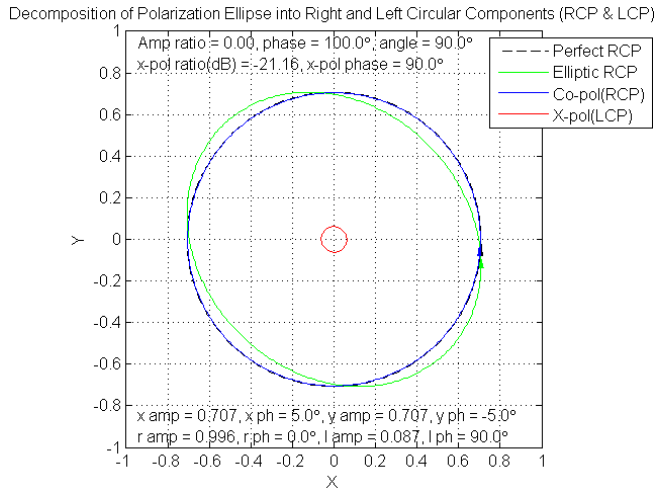


Fig. 4b – Polarization for 2nd element with +10° phase error

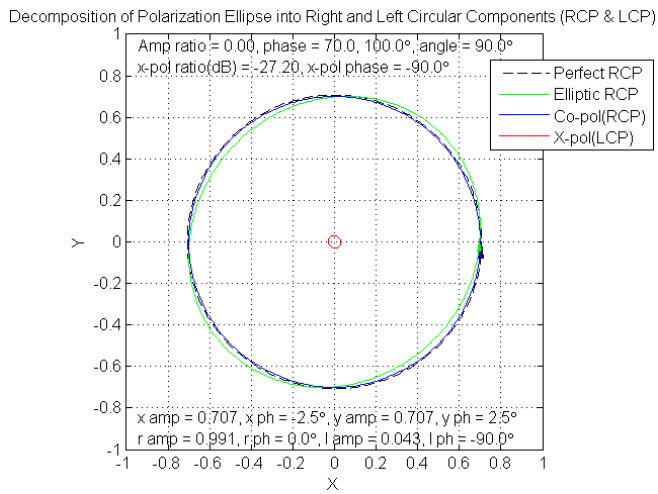
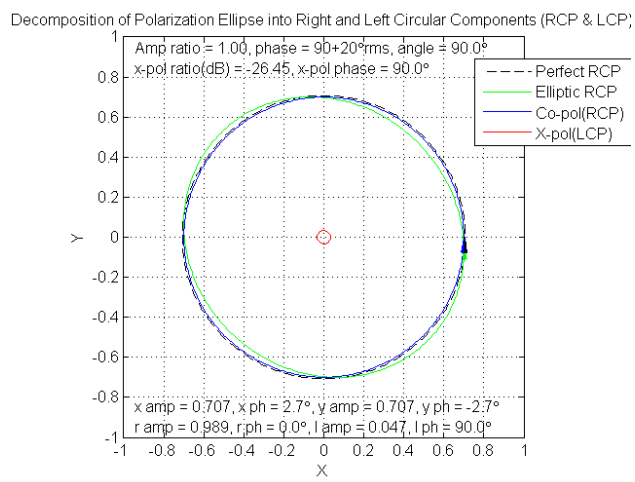


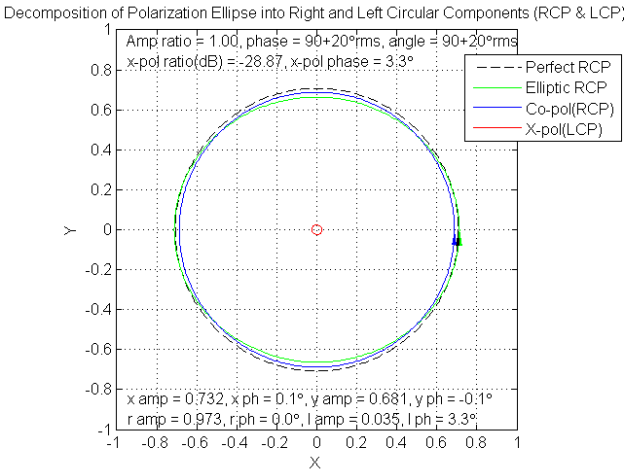
Fig. **4c – Polarization for the array of the 2 elements**



Phase = 90 + 20°rms: 74.61, 97.43, 85.49, 112.35, 68.22, 90.65, 101.05, 112.01, 120.88, 91.72

Fig. 5 – Polarization of a 10-element array with elements

of random phase errors with 20° rms



Phase = 90 + 20°rms = 60.17, 75.15, 68.77, 137.01, 77.69, 104.96, 86.15, 107.77, 74.70, 61.95
Angle = 90 + 20°rms = 74.61, 97.43, 85.49, 112.35, 68.22, 90.65, 101.05, 112.01, 120.88, 91.72

Fig. 6 – Polarization of a 10-element array with elements of random phase and angle errors, both with 20° rms.

7. Summary And Conclusions

In this paper, we have presented a systematic formulation of the cross polarization problem for the individual antennas as well as arrays of such antennas. We have established bounds for the errors of the array cross polarization in terms of the individual element errors. Have identified and formulated the source of some the errors and finally have provided specific numerical examples for single antennas as well as arrays, which help in establishing bounds on the required errors for the elements of the arrays.

8. Acknowledgements

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BIOGRAPHY

Vahraz Jamnejad *is a principal scientist at the Jet Propulsion Laboratory, California Institute of Technology. He received his M.S. and Ph.D. in electrical engineering from the University of Illinois at Urbana-Champaign, specializing in electromagnetics and antennas. At JPL, he has been engaged in research and software and hardware development in various areas of spacecraft antenna technology and satellite communication systems. Among other things, he has been involved in the study, design, and development of ground and spacecraft antennas for future generations of Land Mobile Satellite Systems at L band, Personal Access Satellite Systems at K/Ka band, as well as feed arrays and reflectors for future planetary missions. His latest work on communication satellite systems involved the development of ground mobile antennas for K/Ka band mobile terminal, for use with ACTS satellite system. In the past few years, he has been active in research in parallel computational electromagnetics as well as in developing antennas for MARS sample return orbiter. More recently he has studied the applicability of large arrays of small aperture reflector antennas for the NASA Deep Space Network (DSN). He is also involved in the detailed study and analysis of the near field of large DSN antennas. Over the years, he has received many US patents and NASA certificates of recognition.*

APPENDIX I. ANTENNA TRANSMISSION AND RECEPTION OF ARBITRARILY POLARIZED PLANE WAVES

Here we present a useful summary of the reception of an arbitrarily polarized plane wave by an antenna of a given polarization. A detailed account of the various aspects of this subject may be found in references [8-13].

An antenna can be essentially and completely characterized by three parameters:

1- Impedance, Z_a

2- Gain function $G(\theta,\phi)$

3- Polarization Vector function $\hat{p}(\theta,\phi)$

We briefly describe each of these three parameters.

Antenna impedance basically characterizes its matching compatibility with the propagation medium on the one hand and with the receiver or transmitter circuitry on the other hand. It is given as

$$Z_a = R_a + jX_a$$

In which R_a and X_a are the resistive and reactive components of the impedance. The resistance can also be decomposed as

$$R_a = R_{rad} + jX_{loss}$$

In which R_{rad} is the radiation resistance, and R_{loss} represents the ohmic and other internal antenna losses.

The power gain function, $G(\theta,\phi)$ [or voltage gain function $g(\theta,\phi)=\sqrt{G(\theta,\phi)}$], represents the antenna's ability to transmit or receive in different directions and is a real positive function, such that

$$\int_0^{2\pi}\int_0^{\pi} G(\theta,\phi)\sin(\theta)d\theta d\phi = 4\pi$$

Finally, polarization vector is a complex unit vector function which describes both time (phase) and space (polarization) properties of the antenna in different directions. A useful description of this important vector is given in Appendix II.

The most useful parameter of the antenna in transmit mode is the electric field vector, which in terms of the above three parameters and the total transmitted power, P_t , is given as:

$$\overline{E}(\theta,\phi) = \sqrt{\frac{P_t Z_0}{4\pi r^2}}\, g(\theta,\phi)\hat{p}(\theta,\phi) = E(\theta,\phi)\hat{p}(\theta,\phi) \ ,$$

and $$E(\theta,\phi) = \sqrt{\frac{P_t Z_0}{4\pi r^2}}\, g(\theta,\phi) \ ,$$

in which $Z_0 \cong 377 \text{ ohm}$, is the free space wave impedance, and r is the distance from the antenna to the point at which the field is observed.

The most useful parameter of the antenna in the receive mode is the "length" vector which is given as

$$\overline{L}(\theta,\phi) = \lambda\sqrt{\frac{R_a}{\pi Z_0}}\, g(\theta,\phi)\hat{p}(\theta,\phi) = L(\theta,\phi)\hat{p}(\theta,\phi)$$

and $$L(\theta,\phi) = \lambda\sqrt{\frac{R_a}{\pi Z_0}}\, g(\theta,\phi) \ ,$$

in which $\lambda = c/f$ (with c the speed of light and f the frequency), is the wavelength, and R_a is the antenna resistance.

Then the open circuit voltage at the receiving antenna due to an incoming field $\overline{E}_t$ is given very appropriately by

$$V = \overline{E}_t \cdot \overline{L}_r^* = E_t L_r (\hat{p}_t \cdot \hat{p}_r^*)$$

The power delivered to the receiver can then be simply calculated. For example, the power delivered to a matched load $Z_l = R_a - jX_a$ is given by

$$P_r = \frac{|V|^2}{4R_a} = \frac{(E_t L_r)^2}{4R_a}\eta_p = \frac{P_t G_t G_r}{(4\pi r/\lambda)^2}\eta_p$$

In which η_p is the polarization efficiency defined as

$$\eta_p = \left|(\hat{p}_t \cdot \hat{p}_r^*)\right|^2$$

The power equation above is the familiar Friis transmission formula in which the polarization properties of the transmit and receive antennas have also been incorporated.

For completeness, it should be mentioned that, in general, under different matching conditions, the received power can be written as

$$P_r = \frac{k_m^2}{4R_a}|V|^2 = \frac{(k_m E_t L_r)^2}{4R_a}\eta_p = k_m^2 \frac{P_t G_t G_r}{(4\pi r/\lambda)^2}\eta_p$$

In which k_m is a matching constant less than unity.

APPENDIX II. DEFINITION OF POLARIZATION VECTOR

Polarization vector is a complex unit vector function which describes both time (phase) and space (polarization) properties of the electromagnetic field in different directions.

Referring to Figure A.1, an incident plane wave with a coherent polarization is, in general, elliptically polarized and can be considered as a combination of two orthogonal linear wave components in the plane normal to the direction of propagation (e.g., x and y directed or horizontal, h and vertical, v) wave components given as

$$\hat{p} = p_h \hat{h} + p_v \hat{v}$$

In which $\hat{h}$ and $\hat{v}$ are orthogonal unit vectors in the horizontal (parallel to the ground) or "vertical" (in the plane of incidence) directions. The incident wave can be equally decomposed into a pair of right-hand circularly polarized (RCP) and left-hand circularly polarized (RCP) components with respect to the direction of the propagation and given as

$$\hat{p} = p_r \hat{r} + p_l \hat{l}$$

In which $\hat{r}$ and $\hat{l}$ are complex unit vectors which are related to the horizontal and vertical real unit vectors by

$$\hat{r} = \frac{1}{\sqrt{2}}(\hat{h} - j\hat{v}) \ , \qquad \hat{l} = \frac{1}{\sqrt{2}}(\hat{h} + j\hat{v})$$

The inverse relations are also given as

$$\hat{h} = \frac{1}{\sqrt{2}}(\hat{r} + \hat{l}) \ , \qquad \hat{v} = \frac{j}{\sqrt{2}}(\hat{r} - \hat{l})$$

in which $j = \sqrt{-1}$.

Notice that complex vectors which have a hybrid nature (being both time phase vectors and space vectors), must be dealt with proper care. In particular, the inner product of the complex vectors $\overline{u}$ and $\overline{v}$ is defined as $\overline{u}.\overline{v}^*$ in which * designates complex conjugation and affects the complex scalar components of the vector $\overline{v}$. Thus we have

$$\hat{r} \cdot \hat{r}^* = \hat{l} \cdot \hat{l}^* = 1$$

and $$\hat{r} \cdot \hat{l}^* = \hat{l} \cdot \hat{r}^* = 0$$

(For a complete and comprehensive discussion of circularly polarized waves and complex vectors, Reference [13] is suggested.)

The linear x or horizontal unit vector $\hat{h}$ represents an oscillating field along the horizontal (x) axis, while the linear y or vertical unit vector $\hat{v}$ represents an oscillating field along the vertical axis. The RCP unit vector $\hat{r}$ represents a unit vector rotating in the positive (counterclockwise) direction, while the LCP unit vector $\hat{l}$ represents a unit vector rotating in the negative (clockwise) direction.

Now, in view of the above definitions, the circular right- and left-polarized components of the plane wave can be related to its linear (horizontal and vertical) components by

$$p_r = \frac{1}{\sqrt{2}}(p_h + jp_v) \ , \quad p_l = \frac{1}{\sqrt{2}}(p_h - jp_v)$$

And conversely,

$$p_h = \frac{1}{\sqrt{2}}(p_r + p_l) \ , \quad p_v = -\frac{j}{\sqrt{2}}(p_r - p_l)$$

In time domain we have

$$p_x(t) = \mathrm{Re}(p_x \mathrm{e}^{j2\pi t}) \ , \quad p_y(t) = \mathrm{Re}(p_y \mathrm{e}^{j2\pi t})$$

These last two can be used for the actual calculation of the field values in time. For clarity, graphic representations of the RCP and LCP field behavior in space-time coordinates are given in Figures A.2 and A.3, respectively.

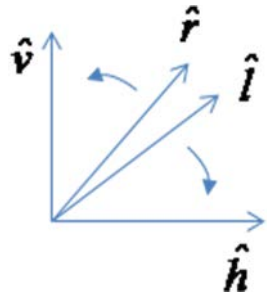

Figure A.1 – Graphic representation of the polarization vectors

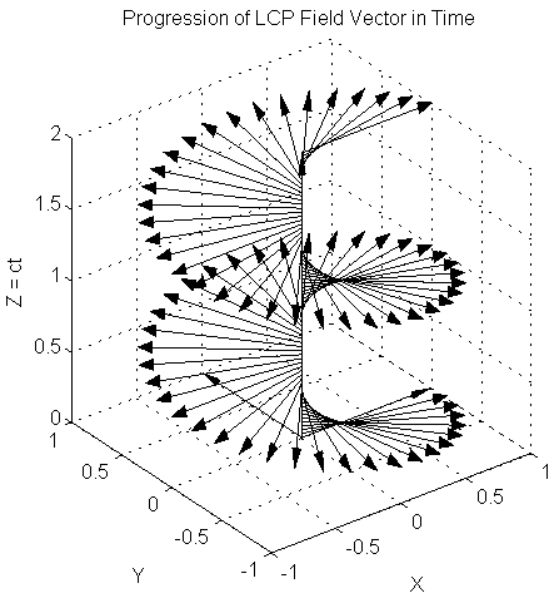

Fig. A.2- Progression of LCP field vector in time - 3D representation

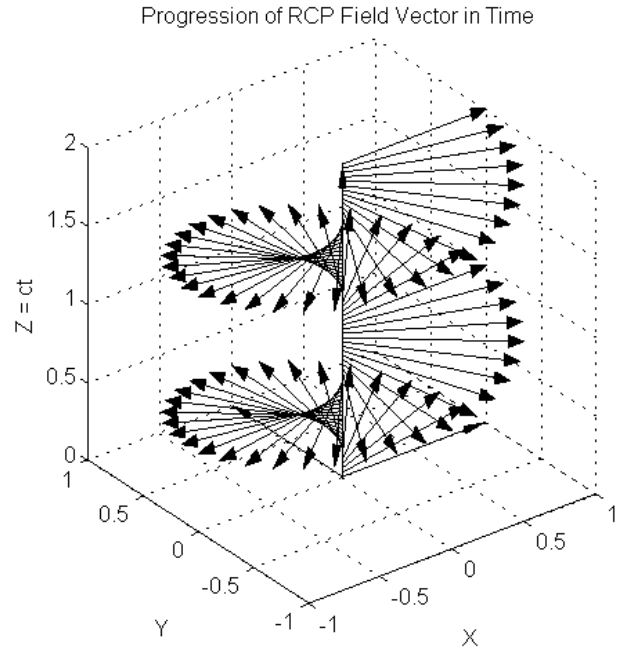

Fig. A.3- Progression of RCP field vector in time - 3D representation