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TRANSFORMING MEAN AND OSCULATING ELEMENTS USING NUMERICAL METHODS

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Mean element propagation of perturbed two body orbits has as its mathematical basis averaging theory of nonlinear dynamical systems. Averaged mean elements define the long-term evolution characteristics of an orbit. Using averaging theory, a near identity transformation can be found that transforms the mean elements back to the osculating elements that contain short period terms in addition to the secular and long period mean elements. The ability to perform the conversion is necessary so that orbit design conducted in mean elements can be converted back into osculating results. In the present work, this near identity transformation is found using the Fast Fourier Transform. An efficient method is found that is capable of recovering the osculating elements to first order.

INTRODUCTION

Mean element theories have proven useful for efficiently determining the long term characteristics of an orbit, however to initialize an appropriate mean orbit requires a transformation from osculating-to-mean or to find a mean orbit's osculating counterpart requires the inverse transformation. Fortunately the mean element theory provides an available mechanism for finding a transformation that is consistent with the theory. Indeed, the process to obtain the mean elements using averaging produces transformation equations directly. This is a well known process and is documented thoroughly by Sanders [1], and will be reviewed later in this paper. Averaging theory's application to satellite astrodynamics has a long tradition that has focused primarily on deriving analytic or semi-analytic theories for finding satellite mean orbital elements. Application of the theory for finding explicit mean-to-osculating or osculating-to-mean theories and its associated tools hasn't received as much attention; however numerous authors have developed rich techniques and powerful tools for finding these transformations. Examples include JPL's OSMEAN tool [2] or Draper's Semi-analytic Satellite Theory (DSST) which has been documented extensively by Danielson [3] and McClain [4] and [5]. These prior works obtain the transformations by using analytical techniques to compute explicit formulae to the Fourier series that correspond to the short period terms of the orbit being examined. In the present study, numerical techniques based on the Fast Fourier Transform (FFT) are utilized for deriving the Fourier series' coefficients. This has the advantage of being able to find the short period terms due to any perturbing acceleration without the need for an explicit derivation of the associated coefficients. Nor are there any approximations present due to truncating eccentricity expansions. Prior research by Lutzky [6] utilized a numerical integration technique to find the Fourier Series coeffi-

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cients however their work focused on directly integrating the integrals defining the Fourier coefficients. The present study takes advantage of progress in FFT's to more efficiently find the coefficients. Furthermore, unlike Lutzky's [6] work, specific issues with regard to mean motion resonances are examined.

BASIC THEORY OF THE NEAR IDENTITY TRANSFORMATION

The trajectory of a satellite in two-body perturbed motion can be modeled using a variation of parameter formulation of 6 1^{st}-order differential equations of motion (EOMs). In a basic form with only one fast angle, perhaps a slow time dependence, and no resonances (the case of mean motion resonances will be examined later) the EOMs can be represented as,

$$\begin{bmatrix} \dot{\boldsymbol{\alpha}} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ n(a) \end{bmatrix} + \varepsilon \begin{bmatrix} \mathbf{f}^{\alpha}(\boldsymbol{\alpha}, \boldsymbol{\theta}, \lambda) \\ f^{\lambda}(\boldsymbol{\alpha}, \boldsymbol{\theta}, \lambda) \end{bmatrix}, \tag{1}$$

where the osculating elements have been selected to be an equinoctial set with $\mathbf{x} \equiv \{\boldsymbol{\alpha}, \lambda\}$, $\boldsymbol{\alpha} \equiv \{a, h, p, q, r\}$, $n(a) = \sqrt{\mu/a^3}$ is the Keplerian mean motion. The other angular variables $\boldsymbol{\theta}$ in Eq. (1) are secular and slowly varying with rates that are themselves of first order $\dot{\boldsymbol{\theta}} = O(\varepsilon)$ relative to the mean motion. Typical examples, of this include, other body orbital periods, Venus' rotation period, slowly varying atmospheres, etc. The mean longitude λ has been singled out because it is the short periodic ('fast') angle in this problem. Since this problem is in a standard variation of parameters formulation it follows (c.f. Sanders [1] or McClain[5]) that there exists a near-identity transformation which eliminates the mean longitude by averaging and produces a system of equations that approximate the original system in Eq. (1) to first order $O(\varepsilon)$ on time scales of order $O(1/\varepsilon)$. The set of averaged equations of motion take the form,

$$\begin{bmatrix} \dot{\bar{\boldsymbol{\alpha}}} \\ \dot{\bar{\lambda}} \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{n}(\bar{a}) \end{bmatrix} + \varepsilon \begin{bmatrix} \mathbf{g}^{\alpha}(\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}; \bar{\lambda}) \\ g^{\lambda}(\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}; \bar{\lambda}) \end{bmatrix} + O(\varepsilon^2). \tag{2}$$

The overbar is used to represent the mean elements $\{\bar{\boldsymbol{\alpha}}, \bar{\lambda}\}$ and the perturbing functions $\mathbf{g}^{\alpha}$ and g^{λ} are to be determined. The presence of the *mean* mean longitude $\bar{\lambda}$ as a parameter for finding $\mathbf{g}^{\alpha}$ and g^{λ} will become evident later when the averaging operation is defined. Note that since the slow angles $\boldsymbol{\theta}$ vary secularly the mean and osculating versions are equivalent, that is $\bar{\boldsymbol{\theta}} = \boldsymbol{\theta}$.

The form of the first order near identity transformation is represented as,

$$\begin{aligned} \boldsymbol{\alpha} &= \bar{\boldsymbol{\alpha}} + \varepsilon \mathbf{u}^1(\bar{\boldsymbol{\alpha}}, \bar{\lambda}, \boldsymbol{\theta}) + O(\varepsilon^2), \\ \lambda &= \bar{\lambda} + \varepsilon v^1(\bar{\boldsymbol{\alpha}}, \bar{\lambda}, \boldsymbol{\theta}) + O(\varepsilon^2). \end{aligned} \tag{3}$$

The procedure for finding the solution to the first-order transformation function $\mathbf{u}^1$ is now outlined; a similar procedure for finding v^1 also applies. Substitute Eq. (3) into Eq. (1) and expand the $\mathbf{u}^1$ equation,

$$\dot{\bar{\boldsymbol{\alpha}}} + \varepsilon \frac{d\mathbf{u}^1}{dt} = \varepsilon \mathbf{f}^{\alpha}\left(\bar{\boldsymbol{\alpha}} + \varepsilon \mathbf{u}^1, \boldsymbol{\theta}, \bar{\lambda} + \varepsilon v\right) \tag{4}$$

with,

$$\frac{d\mathbf{u}^1}{dt} = \frac{\partial \mathbf{u}^1}{\partial \lambda}\dot{\bar{\lambda}} + \frac{\partial \mathbf{u}^1}{\partial \boldsymbol{\alpha}}\dot{\bar{\boldsymbol{\alpha}}} + \frac{\partial \mathbf{u}^1}{\partial \boldsymbol{\theta}}\dot{\boldsymbol{\theta}} = \bar{n}\frac{\partial \mathbf{u}^1}{\partial \lambda} + O(\varepsilon) \tag{5}$$

Using $\dot{\bar{\boldsymbol{\alpha}}} = \varepsilon \mathbf{g}^{\alpha}$ and collecting the first order terms leads to,

$$O(\varepsilon): \quad \mathbf{g}^{\alpha}\left(\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}; \bar{\lambda}\right) + \bar{n}\frac{\partial \mathbf{u}^1}{\partial \lambda} = \mathbf{f}^{\alpha}\left(\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}, \bar{\lambda}\right) \tag{6}$$

Now define the unknown function $\mathbf{g}^{\alpha}$ to be the average of $\mathbf{f}^{\alpha}$, that is,

$$\mathbf{g}^{\alpha}\left(\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}; \bar{\lambda}\right) \equiv \bar{\mathbf{f}}^{\alpha}\left(\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}; \bar{\lambda}\right) = \frac{1}{2\pi}\int_{\bar{\lambda}-\pi}^{\bar{\lambda}+\pi} \mathbf{f}^{\alpha}\left(\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}, \lambda\right) d\lambda, \tag{7}$$

where the function $\mathbf{f}^{\alpha}$ is evaluated at the current values for the mean elements that includes not only $\{\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}\}$ but the current *mean* mean longitude $\bar{\lambda}$ as well. Making this definition for $\mathbf{g}^{\alpha}$ leads to the following partial differential equation for $\mathbf{u}^1$,

$$\bar{n}\frac{\partial \mathbf{u}^1}{\partial \lambda} = \mathbf{f}^{\alpha}\left(\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}, \bar{\lambda}, \Delta\lambda\right) - \bar{\mathbf{f}}^{\alpha}\left(\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}; \bar{\lambda}\right) \tag{8}$$

Note that the short periodic variable is now $\Delta\lambda$ and is evaluated on the interval $-\pi \le \Delta\lambda < \pi$ around the mean value $\bar{\lambda}$; also $\partial\left(\bar{\lambda} + \Delta\lambda\right) = \partial\Delta\lambda$. The mean element trajectory is provided by,

$$\begin{aligned} \bar{\boldsymbol{\alpha}} &= \bar{\boldsymbol{\alpha}}(t), \\ \bar{\lambda} &= \bar{\lambda}(t), \\ \boldsymbol{\theta} &= \dot{\boldsymbol{\theta}}t + \boldsymbol{\theta}_o, \end{aligned} \tag{9}$$

which is obtained from an appropriately formed mean element trajectory. An example technique for finding such a trajectory via numerical averaging was provided in an earlier paper by Ely [7]. Doing the same for the v^1 equation yields,

$$\bar{n}\frac{\partial v^1}{\partial \Delta\lambda} = f^{\lambda}\left(\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}, \bar{\lambda}, \Delta\lambda\right) - \bar{f}^{\lambda}\left(\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}; \bar{\lambda}\right) - \frac{3\bar{n}}{2\bar{a}}u_1^1 \tag{10}$$

Equations (8) and (10) represent the *homological* partial differential equations for finding the near identity transformations of $\mathbf{u}^1$ and v^1. Solutions to these equations can be formally obtained by assuming that the functions $\mathbf{f}$ can be expanded in a Fourier series as functions of the mean longitude variation $\Delta\lambda$,

$$\mathbf{f}\left(\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}, \bar{\lambda}, \Delta\lambda\right) - \bar{\mathbf{f}}\left(\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}; \bar{\lambda}\right) = \sum_{m=-\infty, m\neq 0}^{\infty} \mathbf{f}_m\left(\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}, \bar{\lambda}\right) e^{jm\Delta\lambda}. \tag{11}$$

Note that the zero-frequency term is simply the average value $\overline{\mathbf{f}}\left(\overline{\boldsymbol{\alpha}},\boldsymbol{\theta};\overline{\lambda}\right)$, hence the Fourier Series in Eq. (11) has a zero mean value. Substituting Eq. (11) into Eq. (8) and integrating yields the following solution for $\mathbf{u}^1$,

$$\mathbf{u}^1\left(\overline{\boldsymbol{\alpha}},\boldsymbol{\theta},\overline{\lambda},\Delta\lambda\right)=\frac{1}{\overline{n}}\sum_{m=-\infty,m\neq 0}^{\infty}\frac{\mathbf{f}_m^{\boldsymbol{\alpha}}\left(\overline{\boldsymbol{\alpha}},\boldsymbol{\theta},\overline{\lambda}\right)}{jm}e^{jm\Delta\lambda}, \tag{12}$$

Where the constant of integration is also zero by virtue of defining the average of the transformation to be zero, that is $\overline{\mathbf{u}}^1\left(\overline{\boldsymbol{\alpha}},\boldsymbol{\theta},\overline{\lambda},\Delta\lambda\right)\equiv\mathbf{0}$. Likewise, substituting Eq. (11) into Eq. (10) leads to,

$$\begin{aligned} v^1\left(\overline{\boldsymbol{\alpha}},\boldsymbol{\theta},\overline{\lambda},\Delta\lambda\right)&=\frac{1}{\overline{n}}\int_{\Delta\lambda}\sum_{m=-\infty,m\neq 0}^{\infty}f_m^{\lambda}\left(\overline{\boldsymbol{\alpha}},\boldsymbol{\theta},\overline{\lambda}\right)e^{jm\Delta\lambda}-\frac{3}{2\overline{a}}\sum_{m=-\infty,m\neq 0}^{\infty}\frac{f_m^{a}\left(\overline{\boldsymbol{\alpha}},\boldsymbol{\theta},\overline{\lambda}\right)}{jm}e^{jm\Delta\lambda}d\Delta\lambda \\ &=\frac{1}{\overline{n}}\sum_{m=-\infty,m\neq 0}^{\infty}\left(\frac{f_m^{\lambda}\left(\overline{\boldsymbol{\alpha}},\boldsymbol{\theta},\overline{\lambda}\right)}{jm}+\frac{3}{2\overline{a}}\frac{f_m^{a}\left(\overline{\boldsymbol{\alpha}},\boldsymbol{\theta},\overline{\lambda}\right)}{m^2}\right)e^{jm\Delta\lambda}. \end{aligned} \tag{13}$$

Now the near identity transformation can be found by evaluating the transformation at the current mean mean longitude $\overline{\lambda}$ by setting $\Delta\lambda$ to zero with the result,

$$\begin{aligned} \boldsymbol{\alpha}&=\overline{\boldsymbol{\alpha}}+\frac{1}{\overline{n}}\sum_{m=-\infty,m\neq 0}^{\infty}\frac{\mathbf{f}_m^{\boldsymbol{\alpha}}\left(\overline{\boldsymbol{\alpha}},\boldsymbol{\theta},\overline{\lambda}\right)}{jm}+O\left(\varepsilon^2\right), \\ \lambda&=\overline{\lambda}+\frac{1}{\overline{n}}\sum_{m=-\infty,m\neq 0}^{\infty}\frac{f_m^{\lambda}\left(\overline{\boldsymbol{\alpha}},\boldsymbol{\theta},\overline{\lambda}\right)}{jm}+\frac{3}{2\overline{a}}\frac{f_m^{a}\left(\overline{\boldsymbol{\alpha}},\boldsymbol{\theta},\overline{\lambda}\right)}{m^2}+O\left(\varepsilon^2\right). \end{aligned} \tag{14}$$

To first order the transformation from osculating-to-mean elements can be found by transposing Eq. (14) with the following result,

$$\begin{aligned} \overline{\boldsymbol{\alpha}}&=\boldsymbol{\alpha}-\frac{1}{n}\sum_{m=-\infty,m\neq 0}^{\infty}\frac{\mathbf{f}_m^{\boldsymbol{\alpha}}\left(\boldsymbol{\alpha},\boldsymbol{\theta},\lambda\right)}{jm}+O\left(\varepsilon^2\right), \\ \overline{\lambda}&=\lambda-\frac{1}{n}\sum_{m=-\infty,m\neq 0}^{\infty}\frac{f_m^{\lambda}\left(\boldsymbol{\alpha},\boldsymbol{\theta},\lambda\right)}{jm}-\frac{3}{2a}\frac{f_m^{a}\left(\boldsymbol{\alpha},\boldsymbol{\theta},\lambda\right)}{m^2}+O\left(\varepsilon^2\right), \end{aligned} \tag{15}$$

where now the terms in Eq. (15) are evaluated with osculating elements (versus the mean elements). Typically osculating elements are known for an orbit's initial conditions, Eq. (15) enables an analyst to find the equivalent mean element initial conditions for conducting a mean element study that is consistent with the selected osculating initial conditions.

COMPUTING THE FOURIER COEFFICIENTS

In a departure with other techniques that find the Fourier coefficients analytically, such as JPL's OSMEAN [2] or Draper's Semi-Analytic Theory [3] software, in this research the coefficients are found numerically using the Discrete Fourier Transform (DFT). As developed in Ref. [8], the definition of the DFT coefficients $\mathbf{F}_m$ for a vector function $\mathbf{f}\left(\lambda\right)$ that is 2π -periodic is defined as,

Table 1: Initial Conditions and Parameters for Venus Orbiter Example

$\{a, e, i, \Omega, \omega, M\}$	{ 10082.179 km, 0.375, 85°, 51.831°, 10.036°, 0.0°}
Epoch	July 26, 1988 00:00:00 ET
Duration	1 day
$\{C_2 \cdots C_{10}, S_2 \cdots S_{10}\}$	MGNP180U gravity field
Sun Perturbations	Off in 1st case and on in 2nd case and using the DE421 ephemeris
μ	324858.77 km^3/sec^2
Frames	IAU 2000 Venus Pole and Fixed
FFT Order	N = 32, 64, 128, 256

$$\mathbf{F}_m = \frac{1}{N}\sum_{n=0}^{N-1}\mathbf{f}\left(\lambda_n\right)e^{-jm\lambda_n}, \quad \lambda_n = 2\pi\frac{n}{N}, \tag{16}$$

where N is the number of samples taken of the original equations $\mathbf{f}$, which for the algorithm presented on the prior page would be Eq. (11). The Fourier series coefficients $\mathbf{f}_m$ are related to the DFT coefficients via,

$$\mathbf{f}_m = \frac{\mathbf{F}_m}{N}, \quad m = 0, 1, \ldots, \frac{N}{2} - 1. \tag{17}$$

Note that the size of N dictates the number of frequency multiples of the mean longitude $m\lambda$ that will be reconstructed in forming the DFT. In particular, as can be seen in Eq. (17), N samples will recover $N/2$ frequency multiples of $m\lambda$. Thus if there are a large number of short periodic terms in the expansion with significant amplitudes then a large N will be required to recover them satisfactorily. The DFT coefficients are found efficiently using a Fast Fourier Transform (FFT) algorithm that has been implemented in most mathematical processing packages. In the present case, the FFT is computed using the Numerical Python (NumPy) package [9].

Example: Consider an orbiter at Venus with conditions similar to the Magellan orbiter given in Table 1. This problem includes perturbations from the zonal and tesseral harmonics and the Sun. Since Venus' rotational rate is on the order of 243 days there is no possibility of a significant mean motion resonance and hence sidereal angle can be treated as a secular slow variable. Likewise since Venus' orbital period is 0.6 years the apparent mean longitude of the Sun's perturbation can also be treated as a slow variable. The numerical averaging technique from Ely [7] is used to find the mean elements as a function of time which are then converted to osculating elements using Eqs. (14) and (17). The 1 day propagation is done in two ways, 1) standard propagation of the osculating equations of motion; 2) the transformation technique just described to convert a mean orbit to osculating.

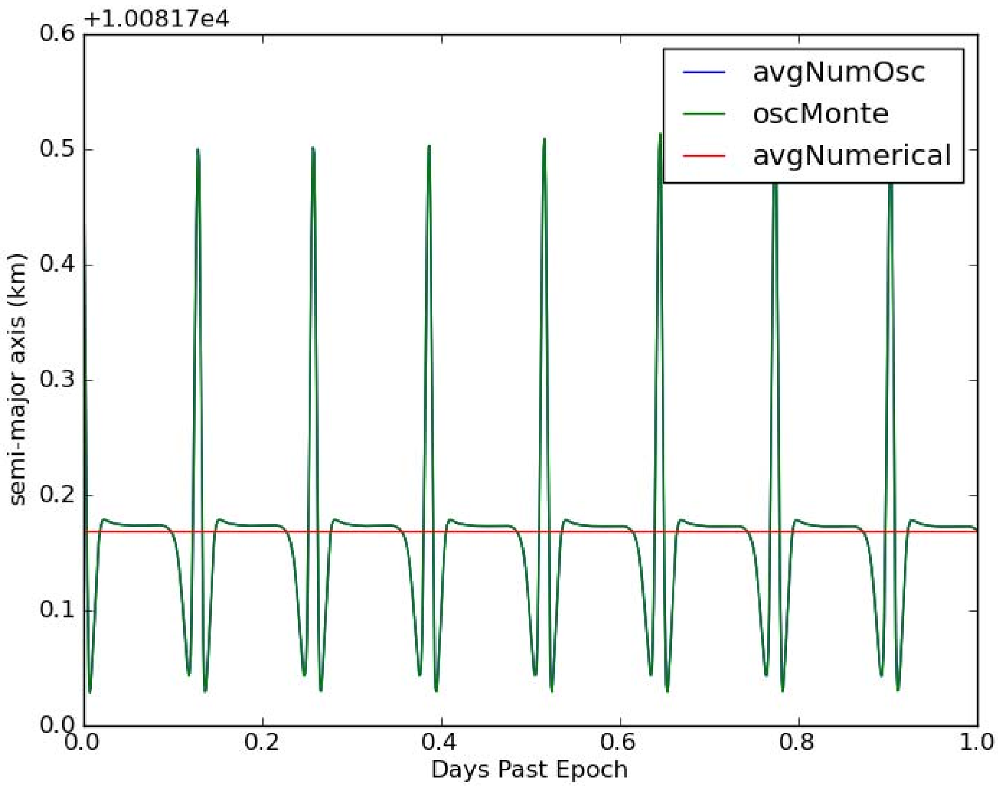
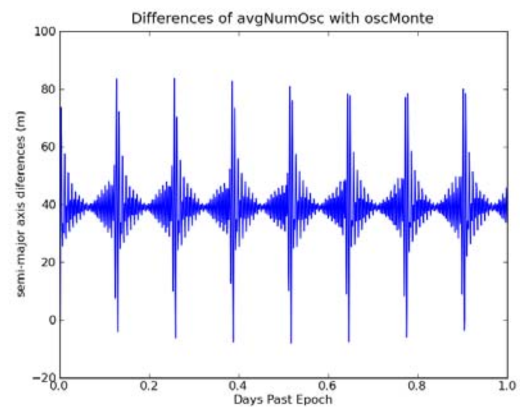


Figure 1: Semi-major axis of Venus orbiter propagated via numerical integration of mean elements (red line), 1st order numerical conversion of mean-to-osculating (green line), and direct numerical integration of the equations of motion (red line).

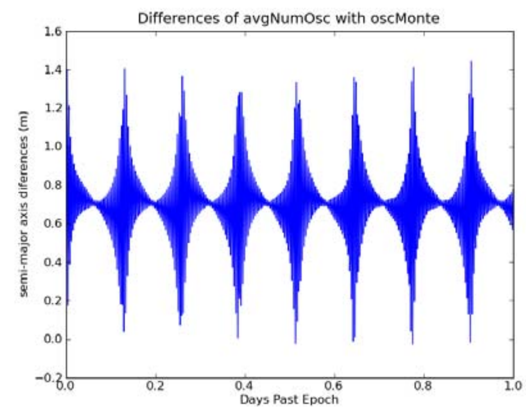
In a first case the Sun's perturbations are turned off, hence the only active perturbations are from the zonal and tesseral harmonics. An illustration of the recovery of the semi-major axis is shown in Figure 1 where the numerically averaged and integrated mean elements are shown in red ('avgNumerical'), the 1st-order numerical conversion of the mean elements to osculating is shown in green ('avgNumOsc'), and the direct numerical integration of the equations of motion are shown in blue ('oscMonte'). The order of the FFT used is N = 256. The recovery of the osculating elements from the mean is quite good, with no perceptible differences seen between 'avgNumOsc' and 'oscMonte' (i.e., they overlay each other). To see this more carefully and to examine the effect that the FFT order has differences of semi-major axis for orders N = 32, 64, 128, 256 are shown in Figure 2. The results illustrate the importance of selecting a proper order to ensure a sufficient spectrum is recovered with the FFT. In this case the N = 32 yields semi-major axis differences on the order of 80 m or less while N = 128 produces differences that are significantly less at 6 cm or less. However, increasing the order to N = 256 yields no substantial improvement.

In the next case the Sun's perturbations are activated. The inclusion of these terms yields a more pronounced difference than with the prior case with zonals/tesserals only, and is illustrated in Figure 3. The semi-major axis difference plot shows that the differences have increased from 6 cm to around 5 m. This change is attributable to the fact that the orbit has a high apoapsis where the solar perturbations dominate. Indeed a simulation (not shown) with the solar perturbations on

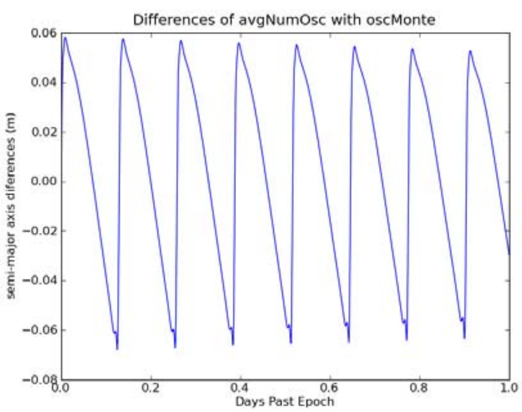
N = 32:



N = 64:



N = 128:



N = 256:

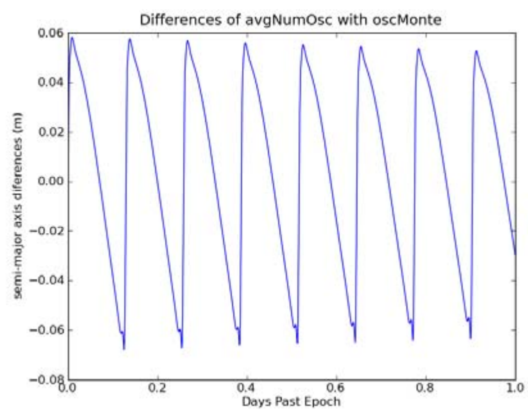


Figure 2: Differences of semi-major axis from the osculating trajectory obtained from the 1st-order numerical mean-to-osculating transformation and the directly integrated numerical trajectory using different orders for the FFT calculation.

and the eccentricity reduced to 0.001 reduce the semi-major axis differences from the 5 m level to about 0.1 m. To reduce the error further for highly eccentric case would require use of a higher order theory. This is the subject of a future study.

TESSERAL RESONANCE

The preceding basic theory applies to perturbations from zonal harmonics, drag, solar radiation pressure, other bodies, and tesseral harmonics under the conditions:

- The sidereal and the other body orbital rates are small and there are no significant resonances,
- The average is only over the mean longitude.

In the typical case where the mean motion of the satellite n and the central body's sidereal rotation rate ω can become commensurate, then there may exist either deep or shallow resonances that are dependent on the selected orbital initial conditions. In this case, Ely [7] showed that a transformation to new variables can be utilized to separate the slow resonant motions from the

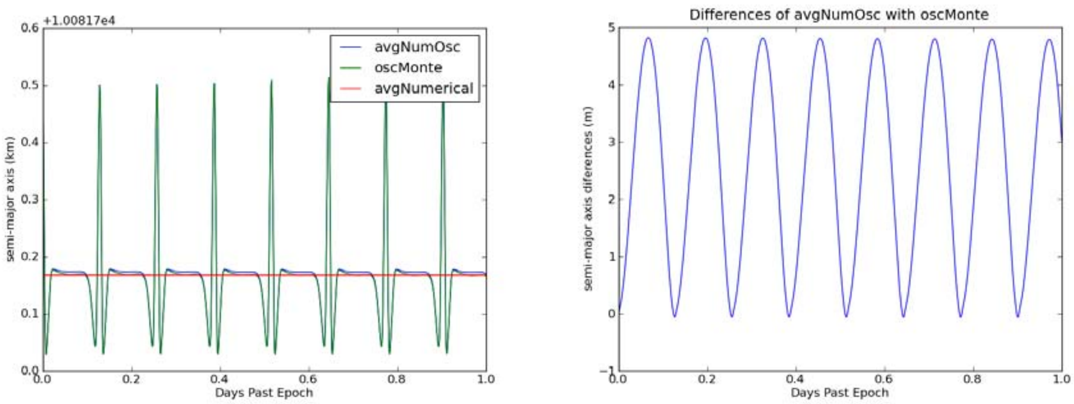


Figure 3: Inclusion of Sun perturbations in the Venus orbiter case at N = 256.

short periodic ones. Prior to this transformation the equations of motion have the following structure,

$$\begin{bmatrix} \dot{\boldsymbol{\alpha}} \\ \dot{\lambda} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ n(a) \\ \omega \end{bmatrix} + \varepsilon \begin{bmatrix} \mathbf{f}^{\boldsymbol{\alpha}}(\boldsymbol{\alpha}, \boldsymbol{\theta}, \lambda, \phi) \\ f^{\lambda}(\boldsymbol{\alpha}, \boldsymbol{\theta}, \lambda, \phi) \\ 0 \end{bmatrix}, \tag{18}$$

where the sidereal rate equation $\dot{\phi} = \omega$ has added a dimension to the problem. There are two cases to examine: one with no resonance and the other with deep/shallow resonance,

1. In the no resonance case, to *first* order the teseseral harmonics include only m-daily and short period terms that do not contribute to the mean element trajectories. There are two approaches,
    a. Average over both the mean longitude λ and the sidereal angle ϕ, and then find the tesseral short period terms and the m-'daily' terms (i.e., those terms with a period on the order of the central bodies sidereal rate) using a multiply frequency DFT.
    b. Average over only the zonal harmonics and do not include the teseral harmonics in the mean element equations of motion, then find the tesseral short period/m-'daily' terms using a multiple frequency DFT on the osculating element rate functions that *now* includes the tesserals. This approach is the more efficient than the prior one and yields equivalent answers.

   Finding multiply frequency DFTs will be the subject of a future effort.

2. For deep resonance or shallow resonance, the commensurability of the mean motion with the planet rotation rate introduces a new slow variable that can be revealed by introducing the stroboscopic node ψ. It is defined as,

$$\psi \equiv P\lambda - Q\phi, \tag{19}$$

Case 2 will be examined, Eq. (18) now becomes,

$$\begin{bmatrix} \dot{\boldsymbol{\alpha}} \\ \dot{\psi} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ n(a) \end{bmatrix} + \varepsilon \begin{bmatrix} \mathbf{f}^{\boldsymbol{\alpha}}(\boldsymbol{\alpha}, \boldsymbol{\theta}, \psi, \lambda) \\ Pn(a) - Q\omega + Pf^{\lambda}(\boldsymbol{\alpha}, \boldsymbol{\theta}, \psi, \lambda) \\ f^{\lambda}(\boldsymbol{\alpha}, \boldsymbol{\theta}, \psi, \lambda) \end{bmatrix}. \tag{20}$$

where the nodal rate $\dot{\psi}$ is of $O(\varepsilon)$. The near identity transformation for this system takes the form,

$$\begin{aligned}
\boldsymbol{\alpha} &= \bar{\boldsymbol{\alpha}} + \varepsilon \mathbf{u}^1\left(\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}, \bar{\psi}, \bar{\lambda}\right) + O\left(\varepsilon^2\right), \\
\psi &= \bar{\psi} + \varepsilon w^1\left(\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}, \bar{\psi}, \bar{\lambda}\right) + O\left(\varepsilon^2\right), \\
\lambda &= \bar{\lambda} + \varepsilon v^1\left(\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}, \bar{\psi}, \bar{\lambda}\right) + O\left(\varepsilon^2\right).
\end{aligned} \tag{21}$$

Now Eq. (20) has the same structural form as Eq. (1), thus the same procedure as before for finding the homological equations can be used with the following result for the $\boldsymbol{\alpha}$ elements,

$$\bar{n}\frac{\partial \mathbf{u}^1}{\partial \lambda} = \mathbf{f}^{\boldsymbol{\alpha}}\left(\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}, \bar{\psi}, \bar{\lambda}, \Delta\lambda\right) - \bar{\mathbf{f}}^{\boldsymbol{\alpha}}\left(\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}, \psi; \bar{\lambda}\right), \tag{22}$$

and for the mean longitude,

$$\bar{n}\frac{\partial v^1}{\partial \lambda} = f^{\lambda}\left(\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}, \bar{\psi}, \bar{\lambda}, \Delta\lambda\right) - \bar{f}^{\lambda}\left(\bar{\boldsymbol{\alpha}}, \boldsymbol{\theta}, \bar{\psi}; \bar{\lambda}\right) - \frac{3\bar{n}}{2\bar{a}}u_1^1 . \tag{23}$$

The expansion for the node reveals that,

$$\frac{\partial w^1}{\partial \lambda} = P\frac{\partial v^1}{\partial \lambda}, \tag{24}$$

hence only the mean longitude homological equation v^1 needs to be solved and then scaled by P to get w^1 for the node.

In Ely [7] it has shown that the formal Fourier series expansions for the perturbing rate equations due to tesseral harmonics with the change of variable to the stroboscopic node take the form,

$$\begin{bmatrix} \mathbf{f}^{\boldsymbol{\alpha},T} \\ f^{\lambda,T} \end{bmatrix} = \sum_{q=-\infty}^{q=\infty}\sum_{m=0}^{l} \mathbf{f}_{qm}^{T}(\boldsymbol{\alpha}) \exp j\left[-\frac{m}{Q}\psi\right] \exp j\left[\left(qQ - mP\right)\frac{\lambda}{Q}\right] \tag{25}$$

Furthermore, the averaging procedure was facilitated by limiting the tesseral field to $m = rQ,\ r = 0,\ldots,l/Q$. Dong so reduces Eq. (25) to,

$$\begin{bmatrix} \mathbf{f}^{\boldsymbol{\alpha},T} \\ f^{\lambda,T} \end{bmatrix}_{reduced} = \sum_{q=-\infty}^{q=\infty}\sum_{r=0}^{l/Q} \mathbf{f}_{q,rQ}^{T}(\boldsymbol{\alpha}) \exp j\left[-r\psi\right] \exp j\left[\left(q - rP\right)\lambda\right]. \tag{26}$$

The terms that survive the averaging procedure have indices that conform to $q = rP,\ r = 0,\ldots,l/Q$ which yields the following result,

$$\begin{bmatrix} \bar{\mathbf{f}}^{\boldsymbol{\alpha},T} \\ \bar{f}^{\lambda,T} \end{bmatrix} = \sum_{r=0}^{l/Q} \mathbf{f}_{rP,rQ}^{T}(\boldsymbol{\alpha}) \exp j\left[-r\psi\right]. \tag{27}$$

Hence, subtracting Eq. (27) from Eq. (25) produces,

$$\begin{bmatrix} \mathbf{f}^{\boldsymbol{\alpha},T} - \overline{\mathbf{f}}^{\boldsymbol{\alpha},T} \\ f^{\lambda,T} - \overline{f}^{\lambda,T} \end{bmatrix} = \sum_{\substack{q=-\infty \\ q\neq rP}}^{q=\infty} \sum_{\substack{m=0 \\ m\neq rQ,\, r=0,\ldots,l/Q}}^{l} \mathbf{f}^{T}_{q,m}(\boldsymbol{\alpha}) \exp j\left[-\frac{m}{Q}\psi\right] \exp j\left[\left(qQ-mP\right)\frac{\lambda}{Q}\right] \tag{28}$$

Using Eq. (28) in the homological equations,

$$\overline{n}\frac{\partial \mathbf{u}^1}{\partial \Delta\lambda} = \sum_{\substack{q=-\infty \\ q\neq rP}}^{q=\infty} \sum_{\substack{m=0 \\ m\neq rQ,\, r=0,\ldots,l/Q}}^{l} \mathbf{f}^{\boldsymbol{\alpha},T}_{q,m}(\overline{\boldsymbol{\alpha}}) \exp j\left[-\frac{m}{Q}\overline{\psi}\right] \exp j\left[\left(qQ-mP\right)\frac{\overline{\lambda}}{Q}\right] \exp j\left[\left(qQ-mP\right)\frac{\Delta\lambda}{Q}\right] \tag{29}$$

Integrating yields,

$$\mathbf{u}^1 = \frac{Q}{\overline{n}} \sum_{\substack{q=-\infty \\ q\neq rP}}^{q=\infty} \sum_{\substack{m=0 \\ m\neq rQ,\, r=0,\ldots,l/Q}}^{l} \frac{\mathbf{f}^{\boldsymbol{\alpha},T}_{q,m}(\overline{\boldsymbol{\alpha}})}{j\left(qQ-mP\right)} \exp j\left[-\frac{m}{Q}\overline{\psi}\right] \exp j\left[\left(qQ-mP\right)\frac{\overline{\lambda}}{Q}\right] \exp j\left[\left(qQ-mP\right)\frac{\Delta\lambda}{Q}\right] \tag{30}$$

Setting the delta mean longitude $\Delta\lambda$ to zero leads to the final form for the transformation,

$$\boldsymbol{\alpha} = \overline{\boldsymbol{\alpha}} + \frac{Q}{\overline{n}} \sum_{\substack{q=-\infty \\ q\neq rP}}^{q=\infty} \sum_{\substack{m=0 \\ m\neq rQ,\, r=0,\ldots,l/Q}}^{l} \frac{\mathbf{f}^{\boldsymbol{\alpha},T}_{q,m}(\overline{\boldsymbol{\alpha}})}{j\left(qQ-mP\right)} \exp j\left[-\frac{m}{Q}\overline{\psi}\right] \exp j\left[\left(qQ-mP\right)\frac{\overline{\lambda}}{Q}\right]. \tag{31}$$

Performing a similar operation for v^1 yields the following expression for the mean longitude,

$$\lambda = \overline{\lambda} + \frac{Q}{\overline{n}} \sum_{\substack{q=-\infty \\ q\neq rP}}^{q=\infty} \sum_{\substack{m=0 \\ m\neq rQ,\, r=0,\ldots,l/Q}}^{l} \left[\frac{f^{\lambda,T}_{q,m}(\overline{\boldsymbol{\alpha}})}{j\left(qQ-mP\right)} + \frac{3Q}{2\overline{a}}\frac{f^{a,T}_{m}\left(\overline{\boldsymbol{\alpha}},\boldsymbol{\theta},\overline{\lambda}\right)}{\left(qQ-mP\right)^2}\right] \exp j\left[-\frac{m}{Q}\overline{\psi}\right] \exp j\left[\left(qQ-mP\right)\frac{\overline{\lambda}}{Q}\right] \tag{32}$$

A few observations about Eqs. (28) and Eq. (31) and (32) are in order:

1. The FFTs of Eq. (28) are required for evaluating Eq. (31). The FFT is performed via evaluating the rate equations as functions of the mean longitude. Note that the minimal period of the function is actually $2\pi Q$ (not just 2π), hence sampling the function should span $2\pi Q$ which is in fact a sufficient period to recover the m-'daily' oscillatory terms.
2. Because the FFT needs to capture frequencies that include not only multiples of the mean longitude but those that are multiples of the m-'dailies' as well the order of the FFT needed to capture all frequencies increases as compared to the zonal-only case. This will be illustrated in an example.
3. Operationally, the process of subtracting the mean rate terms from the osculating rate terms in Eq. (28) ensures that the conditions specified on the excluded indices in the right hand side of the equation are satisfied when performing the FFTs. That is, when Eqs.(31) and (32) are evaluated no zero divisor's occur (as would be the case if the excluded indices were present).

The process for finding the osculating-to-mean transformation is the same as before transpose Eqs. (31) and (32) and replace mean elements with their osculating counterparts.

Example: Consider a Mars orbiter that is in a $Q:P=12:1$ resonance with Mars' rotation rate with initial conditions and other parameters as indicated below in Table 2.

Table 2: Initial Conditions and Parameters for 12:1 Mars Resonant Orbiter Example

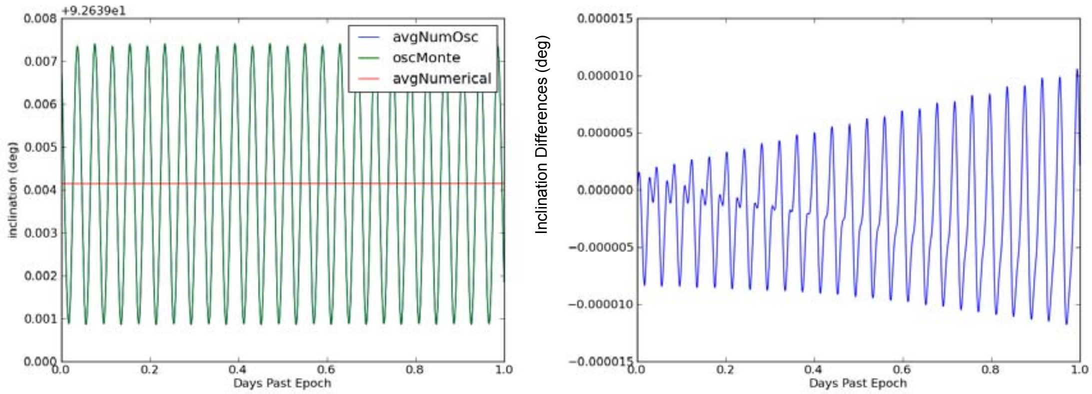
$\{a,e,i,\Omega,\omega,M\}$	{ 3694.794 km, 0.0123, 92.643°, 41.098°, -77.868°, -176.200°}
Epoch	January 1, 2000 12:00:00 ET
Duration	1 day
$\{C_2 \cdots C_{20}, S_2 \cdots S_{20}\}$	MGS75E gravity field – a case with zonals only and another tesserals/zonals.
Sun Perturbations	DE421 ephemeris
Drag Perturbations	Mass = 912 kg, drag radius = 1.5 m, $C_d = 2.1$, Exponential density model with $\rho_o = 1.25\text{E-}5\ \frac{\text{kg}}{\text{km}^3}$ and $H = 46.0$ km
Solar Radiation Perturbations	Solar pressure radius = 5 m, Solar Flux@1 A.U. = $0.10205\text{E}+09\ \frac{\text{kg km}^3}{\text{m}^2\ \text{s}^2}$.
μ	42828.377043 km^3/sec^2
Frames	IAU 2000 Mars Pole and Fixed
FFT Order	N = 64, 128, 256, 512, 1024

These elements are similar to those for MRO however the semi-major axis has been adjusted to put it in a deep resonance. Two cases will be examined. In both cases, perturbations from the Sun, drag, and solar radiation pressure are included. In the first case only the zonal harmonics are active, and in the second the tesserals are active along with the zonals. The inclination recovery for both of these cases is shown in Figure 4 where the highest FFT order N necessary to sufficiently recover the short periodics is shown. In the case with only the zonal harmonics a FFT of order 64 proves sufficient, however the inclusion of the tesserals increases this the necessary order to 512. The explanation for this is seen with the presence of a dramatic m-'daily' signature in the case with the active tesseral harmonics. That is, there is a daily period for the short periodic inclination that is not present in the zonal harmonic case. To adequately capture periods on the order of a day or less requires the FFT to carry more terms, in this case 512. It turns out that this is a function of the resonance. Indeed simulations for a 2:1 resonance case (not shown) indicate that the order of the FFT can be reduced back to 64 for that particular example.

CONCLUSIONS

The preceding work has shown the utility of a numerical FFT based approach at converting mean element orbits back to their osculating counterparts, and vice-versa. The method is efficient and does not require the explicit computation of an analytic Fourier series; rather the coefficients are obtained via direct calculations of the osculating element rate functions. This generality allows the method to be applied to a large set of perturbing accelerations, which for this study in-

Case 1: Zonals-Only, Sun, Drag, and Solar Radiation Pressure Active, FFT highest N necessary = 64



Case 2: Zonals and Tesserals, Sun, Drag, and Solar Radiation Pressure Active, FFT highest N neccesary = 512

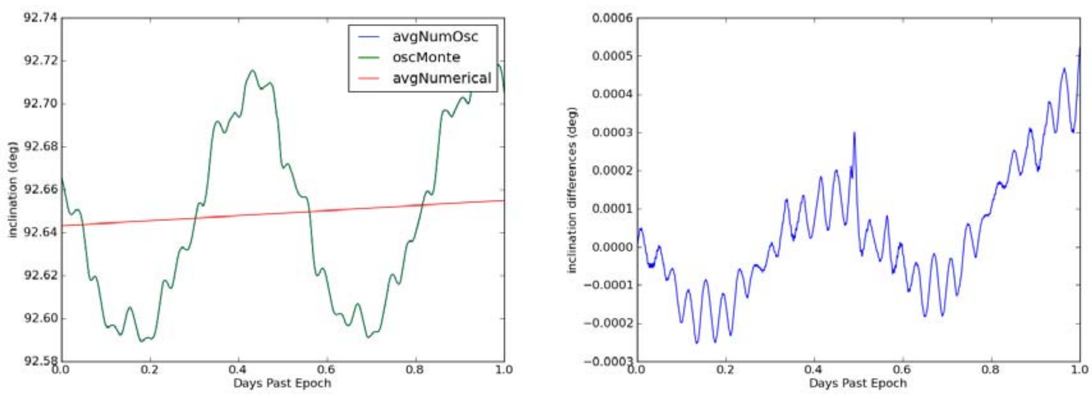


Figure 4: Comparison of inclination recovery of the mean-to-osculating transformation with zonal harmonics (Case 1) and zonal plus tesseral harmonics (Case 2) and the highest order of N necessary for the FFT to sufficiently recover the short periodic terms.

clude the zonal and tesseral harmonics, other bodies, drag, and solar radiation pressure. The following cases have been examined,

1. Purely autonomous resulting from zonal-only harmonics, drag, and solar radiation pressure,
2. Slow time variations from other bodies with slow orbital revolution periods and/or slow sidereal rates,
3. Mean motion resonances both deep and shallow.

These preliminary results indicate the method is efficient and accurate (at least to first order). Future studies will examine the non-resonant case with the tesseral harmonics and the cases where the other body perturbations are faster (i.e., Earth perturbations on Moon orbits). Both of these situations will require a multiply Fourier series expansion and thus a multiple frequency FFT. Additionally, it is anticipated that improvements in accuracy can be obtained by including appropriate 2^{nd}-order effects from the larger zonal harmonics, and zonal harmonic/drag coupling.

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