NORMAL INCIDENCE FOR GRADED INDEX SURFACES

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1. Geometry

A plane wave is incident normally from vacuum ($n_0 = 1$) onto a smooth surface. The substrate has three layers; the top most layer has thickness d_1 and permittivity ϵ_1. The corresponding numbers for the next layer are d_2, ϵ_2, while the third layer which is semi-infinite has index n_3. The Hallikainen model [1] is used to relate volumetric soil moisture to the permittivity. Here, we consider the relation for the real part of the permittivity for a typical loam soil:

$$\epsilon^{'}(mv) = 2.8571 + 3.9678 * mv + 118.85 * mv^2$$

2. Solution

In each region, the fields are decomposed into upward (e^{+ikz}) and downward propagating (e^{-ikz}) plane waves, except in the the lower-most region (n_3), where fields only propagate downward. Applying the boundary conditions of continuity of tangential fields at the interfaces, we get:

$$E_0^- - E_0^+ = E_1^- - E_1^+ \quad \text{(for electric fields)} \tag{1}$$

$$E_0^- + E_0^+ = \frac{n_1}{n_0}(E_1^- + E_1^+) \quad \text{(for magnetic fields)} \tag{2}$$

where $E_i^{\pm}$ is the (upward + /downward -) field at the top most part within the ith layer ($i > 0$). The fields at the bottom of the ith layer are E_i^+/ϕ_i (upward) and $E_i^-\phi_i$ (downward), where $\phi_i = e^{ik_i d_i}$ is the phase difference. With all of this in place, the equations that need to be solved are given as:

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} E_0^- \\ E_0^+ \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ \frac{n_1}{n_0} & \frac{n_1}{n_0} \end{pmatrix} \begin{pmatrix} E_1^- \\ E_1^+ \end{pmatrix} \tag{3}$$

$$\begin{pmatrix} \phi_1 & -\phi_1^{-1} \\ \phi_1 & \phi_1^{-1} \end{pmatrix} \begin{pmatrix} E_1^- \\ E_1^+ \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ \frac{n_2}{n_1} & \frac{n_2}{n_1} \end{pmatrix} \begin{pmatrix} E_2^- \\ E_2^+ \end{pmatrix} \tag{4}$$

$$\begin{pmatrix} \phi_2 & -\phi_2^{-1} \\ \phi_2 & \phi_2^{-1} \end{pmatrix} \begin{pmatrix} E_2^- \\ E_2^+ \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ \frac{n_3}{n_2} & \frac{n_3}{n_2} \end{pmatrix} \begin{pmatrix} E_3^- \\ E_3^+ \end{pmatrix} \tag{5}$$

where $E_3^+ = 0$ and we are interested in $R = \|\frac{E_0^+}{E_0^-}\|^2$. This analysis can be extended to multiple layers by cascading more matrices between $\begin{pmatrix} E_0^- \\ E_0^+ \end{pmatrix}$ and $\begin{pmatrix} E_3^- \\ E_3^+ \end{pmatrix}$, analogous to equation (4). The equations are solved in Mathematica and the code is presented in the appendix.

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3. Validation

To validate the code, a few special cases are calculated. Firstly, the Fresnel coefficient is correctly retrieved from the three layer case by setting $n_1 = n_2 = n_3$, $d_1 = d_2 = 0$, which gives $R = \text{Abs}\left[\frac{n_3-1}{n_3+1}\right]^2$. Next, the thin film problem is recreated by setting $n_1 = n_2$, $d_1 = d_2 = d/2$ and $n_0 = n_3$, which gives the well known interference pattern seen as plotted in Figure (1). The analytical expression for the reflectivity in this case is calculated by summing an infinite series, and it is given by $R = (r\frac{1-\phi}{1-r^2\phi})^2$, where $r = \frac{n_1-n_0}{n_1+n_0}$, $\phi = e^{i2dk_0n_1}$. Here, $R_{\max} = (\frac{2r}{1+r^2})^2$ and $R_{\min} = 0$, which in the case plotted in Figure (1) is $R_{\max} = 0.85$ and $R_{\min} = 0$ for $n_0 = n_3 = 1$, $n_1 = n_2 = 5$.

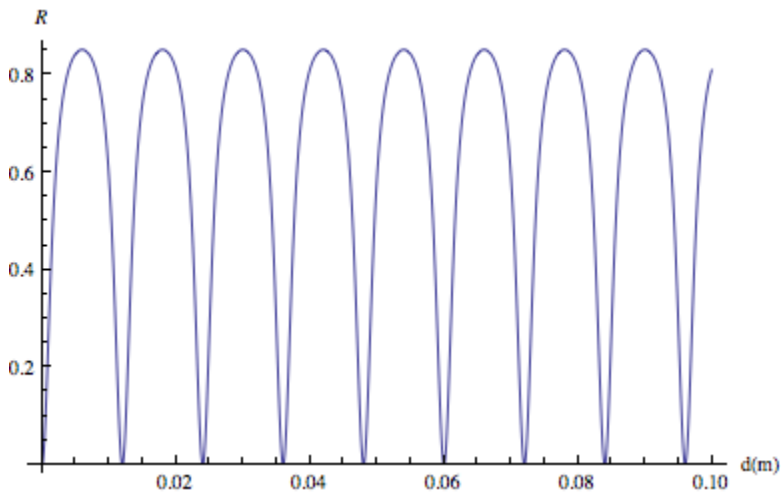

Figure 1. Reflectivity as a function of thin film thickness at $\lambda = 0.24$m and $n_0 = n_3 = 1$, $n_1 = n_2 = 5$.

4. Results

4.1. **Three layers.** We consider four cases and plot the reflectivity as a function of wavelength[1] in Figure (2). Table (1) summarizes the four cases and the value of R at $\lambda = 24\,\text{cm}$. Note that in all cases, $d_1 = d_2 = 2.5\,\text{cm}$.

From the figure it is evident that around $\lambda = 33\,\text{cm}$, the reflectivity difference between case #1 and #2 becomes largest. This is equivalent to the case where $\lambda = 24\,\text{cm}$ and $d_1 = d_2 = 2.5*24/33 = 1.82\,\text{cm}$. Table (2) summarizes the reflectivities with this new layer thickness.

4.2. **Eleven layers.** The same procedure is carried forward with a larger number of layers. The calculation is done via the transfer matrix approach and the relevant code is presented in the appendix. In particular, ten layers with the same thickness are simulated with a linearly varying moisture content. Figure (4) shows the reflectivity as a function of wavelength. From the graphs it is clear that even the choice of the constant profile for comparison is critical.

Acknowledgement

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[1]In practice, the layer thickness is the actual quantity that is varying, since the permittivity is a function of wavelength. dB value in tables is calculated as 10 $\text{Log}_{10}(1/R)$.

Layer	#1	#2	#3	#4
1	30%	25%	5 %	6%
2	20%	25%	7 %	6%
3	10%	10%	10 %	10%
R (sandy %	41.34	44.85	8.17	10.26
loam) dB	-3.84	-3.48	-10.88	-9.89
R (loam)%	41.32	44.83	8.17	10.26
dB	-3.84	-3.48	-10.88	-9.88
R (sandy %	41.30	44.8	8.16	10.24
silt) dB	-3.84	-3.48	-10.88	-9.90
R (silt %	41.27	44.74	8.15	10.22
loam) dB	-3.84	-3.49	-10.89	-9.91
R (silty %	41.22	44.66	8.12	10.18
clay) dB	-3.85	-3.5	-10.9	-9.92

TABLE 1. Reflectivities for a three layer substrate with given volumetric soil moisture values. $d_1 = d_2 = 2.5$ cm, $\lambda = 24\,\text{cm}$. The soil types are from [1].

Case	#1	#2	#3	#4
R (%)	32.55	12.78	6.35	6.50
(dB)	-4.87	-8.93	-11.97	-11.87

TABLE 2. Reflectivities for loam soil when $d_1 = d_2 = 1.82$ cm, $\lambda = 24\,\text{cm}$

REFERENCES

1. M. T. Hallikainen, F. T. Ulaby, M. C. Dobson, M. A. El-Rayes, and L. Wu, Microwave Dielectric Behavior of Wet Soil Part I: Empirical Models and Experimental Observations, IEEE Transactions on Geoscience and Remote Sensing, vol. 23, pp. 2534, 1985.

Appendix

Mathematica code:

```
Clear["Global`*"]
n0 = 1;
(*Defines the phase function*)
ph[n_, d_, lambda_] := E^(2 \[Pi]*I*n*d/lambda);

(*Matrices coming from imposing boundary conditions*)
Npq[np_, nq_] := {{1, -1}, {np/nq, np/nq}};
P0 = {{1, -1}, {1, 1}};
Pn[ph_, n_, d_, lambda_] := {{ph[n, d, lambda], -1/ph[n, d, lambda]},
{ph[n, d, lambda], 1/ph[n, d, lambda]}};

(*Final matrix translating fields in n3 to fields in n0*)
S[n1_, d1_, n2_, d2_, n3_, lambda_] :=
  Inverse[P0].Npq[n1, n0].Inverse[Pn[ph, n1, d1, lambda]].
  Npq[n2, n1].Inverse[Pn[ph, n2, d2, lambda]].Npq[n3, n2];

(*Reflection coefficient*)
R[n1_, d1_, n2_, d2_, n3_, lambda_] :=
  Simplify[Abs[ S[n1, d1, n2, d2, n3, lambda][[2, 1]]/
      S[n1, d1, n2, d2, n3, lambda][[1, 1]]]^2];

(*Verification of Fresnel coefficients for single interface*)
R[n1,0, n1, 0, n3, k]
```

$$\mathrm{Abs}\left[\frac{-1+n3}{1+n3}\right]^2$$

```
(*Insert Hallikainen's model for soil moisture to dielectric constant \
relation, here for loam soils*)
mvton[mv_] := Sqrt[2.8571 + 3.9678*mv + 118.85*mv^2]

Plot[{R[mvton[.3], 0.025, mvton[.2], 0.025, mvton[0.1], k],
  R[mvton[.25], 0.025, mvton[.25], 0.025, mvton[0.1], k],
  R[mvton[.05], 0.025, mvton[.07], 0.025, mvton[0.1], k],
  R[mvton[.06], 0.025, mvton[.06], 0.025, mvton[0.1], k]}, {k, 0.1,
  0.8}, AxesLabel -> {"\[Lambda](m)", R}, AxesOrigin -> {0.09, 0.0},
 PlotStyle -> {Orange, {Orange, Dashed}, Blue, {Blue, Dashed}}]

 (*Calculation for more layers.*)
S10[dtot_, lambda_, ar_] :=
  Inverse[P0].Npq[ar[[1]], 1].Inverse[Pn[ph, ar[[1]], dtot/10, lambda]].
  Npq[ar[[2]], ar[[1]]].Inverse[Pn[ph, ar[[2]], dtot/10, lambda]].
  Npq[ar[[3]], ar[[2]]].Inverse[Pn[ph, ar[[3]], dtot/10, lambda]].
  Npq[ar[[4]], ar[[3]]].Inverse[Pn[ph, ar[[4]], dtot/10, lambda]].
  Npq[ar[[5]], ar[[4]]].Inverse[Pn[ph, ar[[5]], dtot/10, lambda]].
  Npq[ar[[6]], ar[[5]]].Inverse[Pn[ph, ar[[6]], dtot/10, lambda]].
  Npq[ar[[7]], ar[[6]]].Inverse[Pn[ph, ar[[7]], dtot/10, lambda]].
  Npq[ar[[8]], ar[[7]]].Inverse[Pn[ph, ar[[8]], dtot/10, lambda]].
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  Npq[ar[[9]], ar[[8]]].Inverse[Pn[ph, ar[[9]], dtot/10, lambda]].
  Npq[ar[[10]], ar[[9]]].Inverse[Pn[ph, ar[[10]], dtot/10, lambda]].
  Npq[ar[[11]], ar[[10]]]

(*Reflection coefficient*)
R10[dtot_, lambda_, ar_] :=
 Simplify[Abs[
    S10[dtot, lambda, ar][[2, 1]]/S10[dtot, lambda, ar][[1, 1]]]^2]
dtot = 0.05; ar = Table[mvton[i], {i, .30, 0.10, -0.02}]; ar2 =
 Table[mvton[i], {i, .05, 0.10, 0.005}]

Plot[{10 Log[10, R10[dtot, k, ar]],
  10 Log[10,R[mvton[.30], dtot/2, mvton[.30], dtot/2, mvton[0.1], k]],
  10 Log[10,R[mvton[.20], dtot/2, mvton[.20], dtot/2, mvton[0.1], k]]},
  {k, 0.1, .7}, AxesLabel -> {"\[Lambda](m)", R (Log)},
 AxesOrigin -> {0.09, -2}, PlotStyle -> {Orange, Blue, Red}]
```

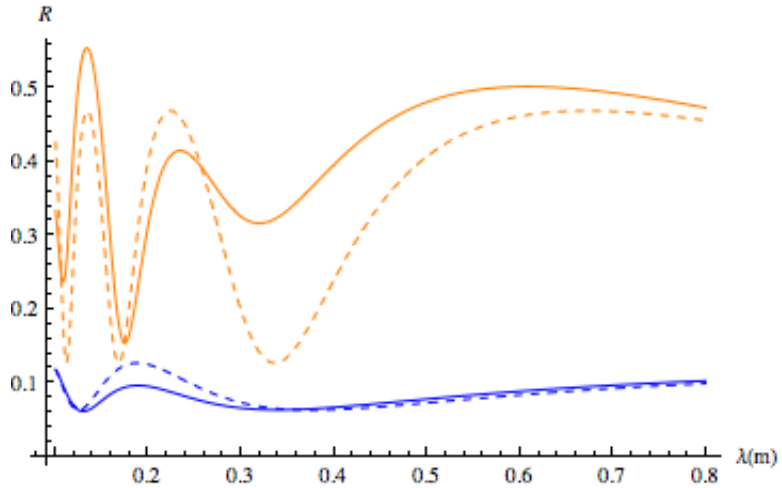

FIGURE 2. Reflectivity as a function of wavelength for $d = 5$cm. Orange corresponds to case #1(30-20-10), dashed-orange corresponds to case #2(25-25-10), blue corresponds to case #3(5-7-10), and dashed-blue corresponds to case #4(6-6-10).

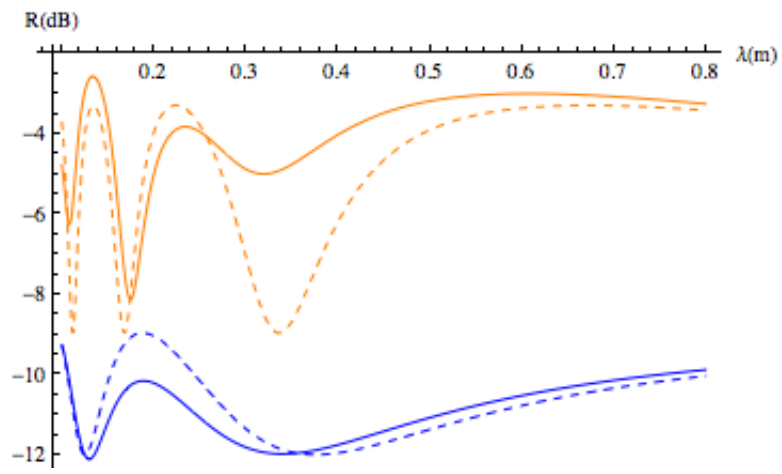

FIGURE 3. Same as Figure (2) in dB scale.

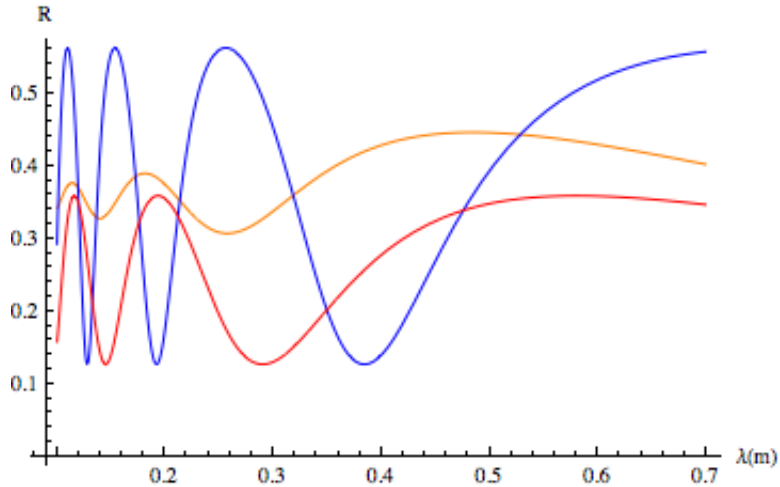

FIGURE 4. Reflectivity as a function of wavelength for ten layers and $d = 5$cm. Orange corresponds to case #1 (30% to 10% in steps of -2%, blue corresponds to case #2 (constant 30%), and red corresponds to case #3(constant 20%). All cases have 10% as the final mv.

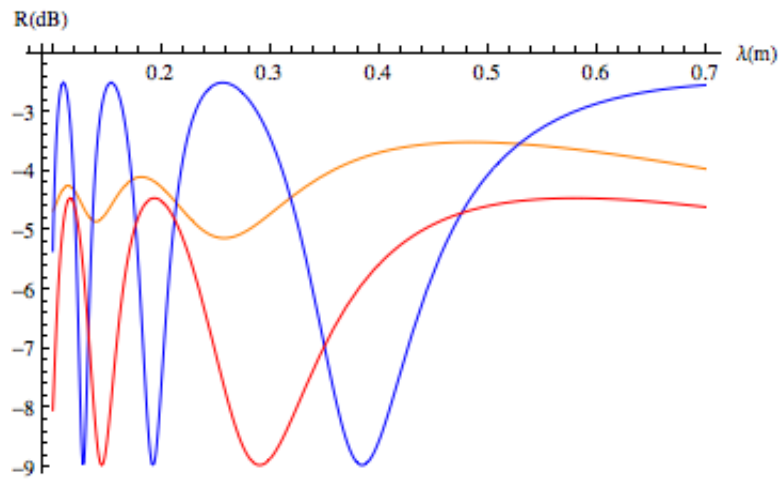

FIGURE 5. Same as Figure (4) in dB scale.