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The Dolinar receiver in an information theoretic framework

by

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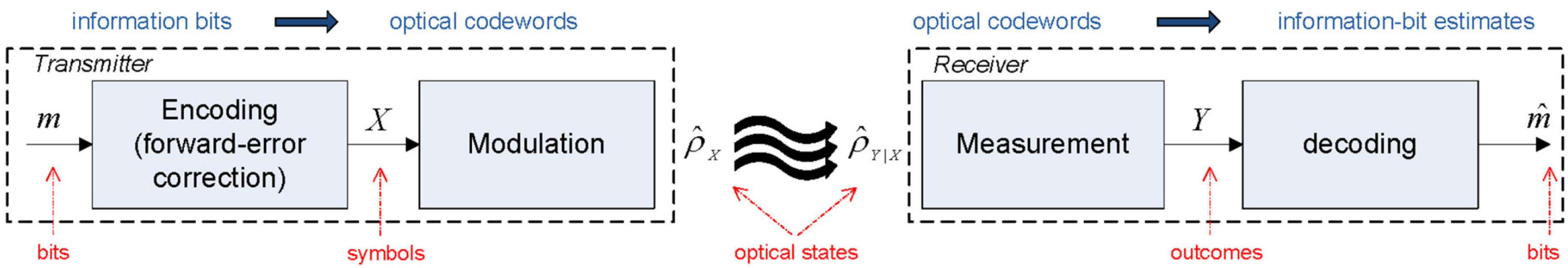
Adaptive-feedback receivers

- Preliminaries
  - Optical communication at the quantum limit
  - Dimensional and energy efficiency in optical communication
  - Binary coherent-state minimum probability of error measurement (Dolinar receiver)
- General formulation of adaptive-feedback receivers
- Optimal local oscillator for binary coherent-state constellation
- Optical local oscillator for ternary coherent-state constellation
- Conclusions

Optical communication at the quantum limit



- Light is fundamentally quantum mechanical
- Measurements inherently yield random observations
- Statistics of measurement outcomes determine channel capacity
- The ultimate capacity of optical communications must therefore optimize over…
  - …the alphabet of optical states
  - …the priors for the optical states
  - …the measurement at the receiver

Single-mode free-space capacity limits

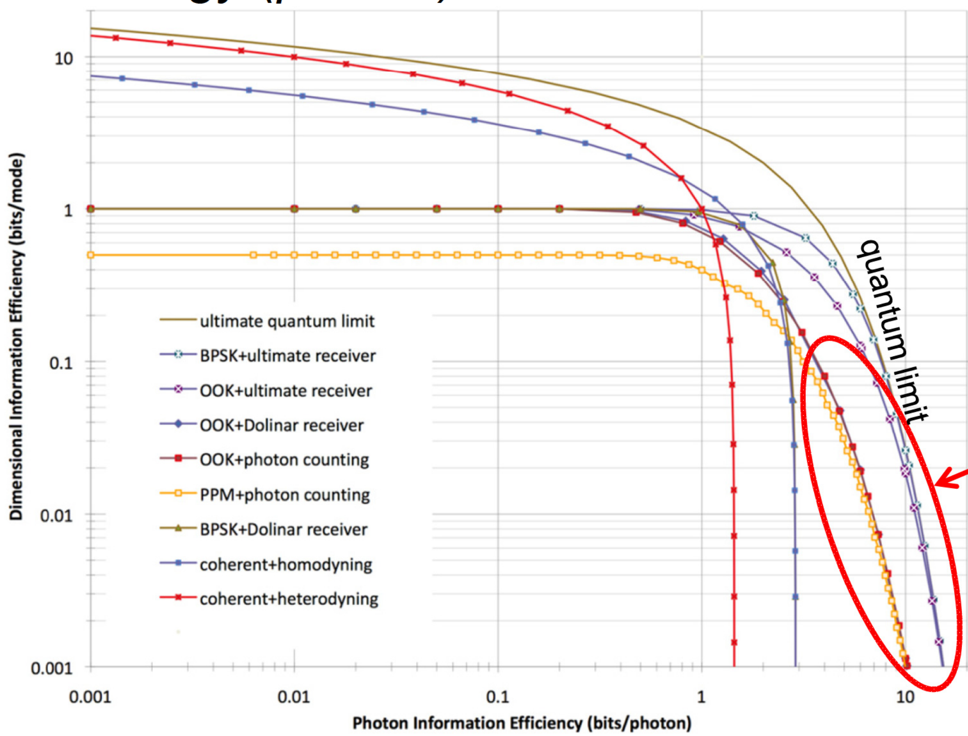
- Dimensional information efficiency: $c_d \equiv C/D$
  - units: *bits per dimension*

capacity in bits per channel use

dimensions: spatial, spectral, polarization etc.

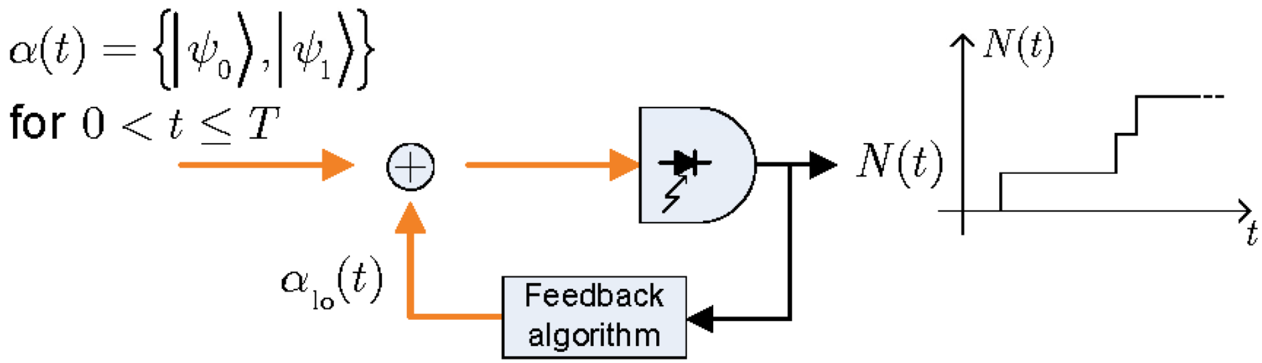
- Photon information efficiency: $c_p \equiv C/E$
  - units: *bits per unit-energy (photon)*

energy per channel use



it is desirable to close this gap

The Dolinar receiver

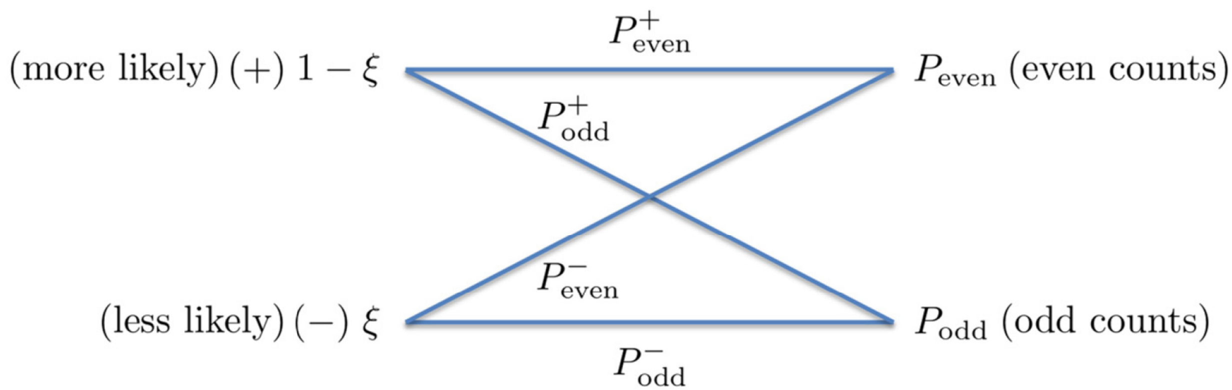


- Input state is one of two coherent states for $0 < t \leq T$
  - $|\psi_0\rangle$ with probability ξ , and $|\psi_1\rangle$ with probability $1 - \xi \quad (\xi < 1/2)$
- Local oscillator field is $|\alpha_{lo}(t)\rangle$
  - chosen to minimize the probability of error in distinguishing the two states in the next increment
- ML decision rule at the end of the observation interval:
  - if even number of photon-arrivals observed, then choose $|\psi_1\rangle$
  - if odd number of photon-arrivals observed, then choose $|\psi_0\rangle$

more likely hypothesis

Minimum probability of error measurement

- Binary asymmetric channel with cross-over probabilities being a function of the *apriori* probabilities

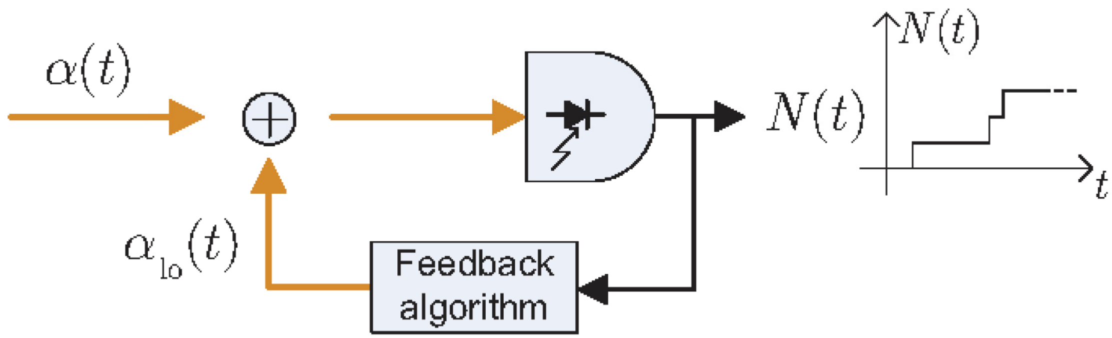


$$P^{+}_{\text{even}} = \frac{1}{2}\left(1 + \frac{1-2\xi s}{\sqrt{1-4\xi(1-\xi)s}}\right) \qquad P^{-}_{\text{even}} = \frac{1}{2}\left(1 - \frac{1-2(1-\xi)s}{\sqrt{1-4\xi(1-\xi)s}}\right) \qquad s \equiv |\langle\psi_0|\psi_1\rangle|^2$$

- The probability of error meets the Helstrom lower bound
  - lowest probability of error quantum mechanically permissible in distinguishing two coherent states

$$P_e = \xi P^{-}_{\text{even}} + (1-\xi)P^{+}_{\text{odd}} = \frac{1}{2}\left(1 - \sqrt{1-4\xi(1-\xi)s}\right)$$

General formulation



- Input signal $\alpha(t) = \alpha_k$ for $0 < t \leq T$; $\alpha_k \in \mathbb{C}$, $k = 1, \ldots, \mathcal{K}$
  - normalized such that $\alpha(t)$ has units $\sqrt{\text{photons/s}}$
  - α_k has apriori probability q_k
- Local oscillator $\alpha_{\text{lo}}(t)$ displaces incoming field
  - $\alpha_{\text{lo}}(t)$ is normalized to have units $\sqrt{\text{photons/s}}$
- Photodetector is ideal
  - ∞-bandwidth, unity quantum efficiency, and no dark counts
- $N(t)$ is a conditional inhomogenous Poisson counting process with rate $\lambda_k(t) = |\alpha_k + \alpha_{\text{lo}}(t)|^2$

Differential mutual information

- The maximum rate of reliable information transfer is

$$C \equiv \max_{\{q_k\},\alpha_{\text{lo}}(t)} I\big(K;\{N(t):t\in(0,T]\}\big)$$

- For given $\{q_k\}$, the chain rule for mutual information yields

$$\max_{\alpha_{\text{lo}}(t)} I\big(K;\{N(t):t\in(0,T]\}\big) = \int_0^T dt\, \mathrm{E}_N\left[\max_{\alpha_{\text{lo}}(t)} \underbrace{\lim_{\Delta T\to 0} \frac{i(K;N(t+\Delta T)-N(t)|\{n(\tau):\tau\in(0,t]\})}{\Delta T}}_{\text{differential mutual information: } i_d(t)}\right]$$

- The differential mutual information for a Poisson process is

$$i_d(t) = -\overline{\lambda}(t)\log\overline{\lambda}(t) + \sum_{k=1}^{\mathcal{K}} p_k(t)\lambda_k(t)\log\lambda_k(t)$$

$$\overline{\lambda}(t) \equiv \sum_k p_k(t)\lambda_k(t)$$

$$p_k(t) \equiv P(K=k|\{n(\tau):\tau\in(0,t]\})$$

$$\lambda_k(t) \equiv |\alpha_k + \alpha_{\text{lo}}(t)|^2$$

- $i_d(t) \geq 0$ because $-\lambda\log\lambda$ is a concave function

Critical points of solution

- Assume infinite modulation bandwidth for local oscillator
- Denote $\alpha_{\text{lo}}(t) \equiv \underbrace{a(t)}_{\geq 0} \exp(i\phi(t))$
- Differentiating $i_d(t)$ with respect to a, ϕ at each time t gives

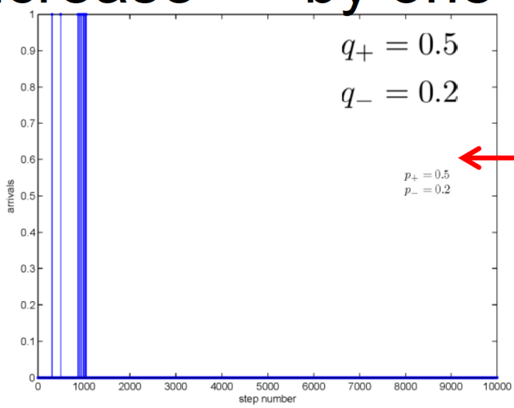
$$\begin{cases} \sum_k p_k \log\left(\lambda_k/\overline{\lambda}\right)\left(a + |\alpha_k| \cos(\phi - \phi_k)\right) = 0 \\ \sum_k p_k \log\left(\lambda_k/\overline{\lambda}\right) \sin(\phi - \phi_k) = 0 \end{cases}$$

$\alpha_k \equiv |\alpha_k| e^{i\phi_k}$

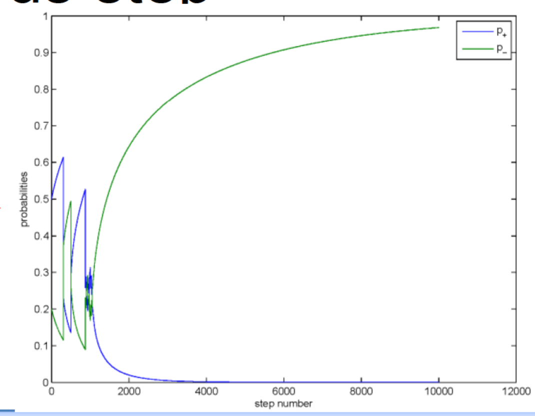
- *In principle* optimal local oscillator can be solved for arbitrary $\{\alpha_k\}$, but, *in practice* difficult to do analytically
  - Binary case has analytic solution
  - Higher-dimensional cases can be solved numerically
- If all $\{\alpha_k\}$ are aligned on a line, then $\alpha_{\text{lo}}(t)$ is on the same line

Algorithm for simulating optimal feedback

- Incremental form of the solution is suitable for simulation
- Initialize:
  - Choose true hypothesis using apriori probabilities $\{q_k\}$
  - Set $p_k(0) = q_k$; choose step size ΔT
- Repeat for $m = 1, \ldots, \lfloor T/\Delta T \rfloor$
  - Find optimal $\alpha_{\mathrm{lo}}(m\Delta T)$ using $p_k(m\Delta T)$
  - Simulate whether arrival occurs in next time increment $(m+1)\Delta T$
  - Update conditional probabilities, $p_k((m+1)\Delta T)$, using Bayes' rule
- Increase m by one and return to previous step



Sample run:
Arrivals
Probability evolution



Binary signaling constellation

- Assume BPSK input constellation $\alpha_k \in \{|\alpha\rangle, |-\alpha\rangle\}$
  - subscript '$+$' refers to positive amplitude state

with probability $p_-(0)$

with probability $p_+(0) < 0.5$

- The solution to the optimality equations yields

$$\alpha_{\mathrm{lo}}(t) = \frac{\alpha}{1 - 2p_+(t)}$$

- The probability product evolves deterministically, as

$$p_+(t)p_-(t) = p_+(0)p_-(0)e^{-4\alpha^2 t}$$

- The closed-form optimal local oscillator solution is

$$\alpha_{\mathrm{lo}}(t) = \frac{\alpha(-1)^{N(t)}}{\sqrt{1 - 4p_+(0)\underbrace{p_-(0)}_{= 1 - p_+(0)}e^{-4\alpha^2 t}}} \qquad 0 < t \leq T$$

- Local oscillator is identical that of Dolinar Receiver!

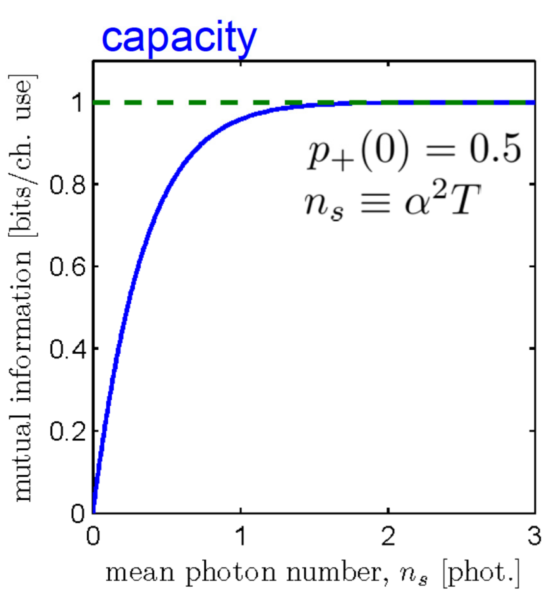
Capacity of BPSK

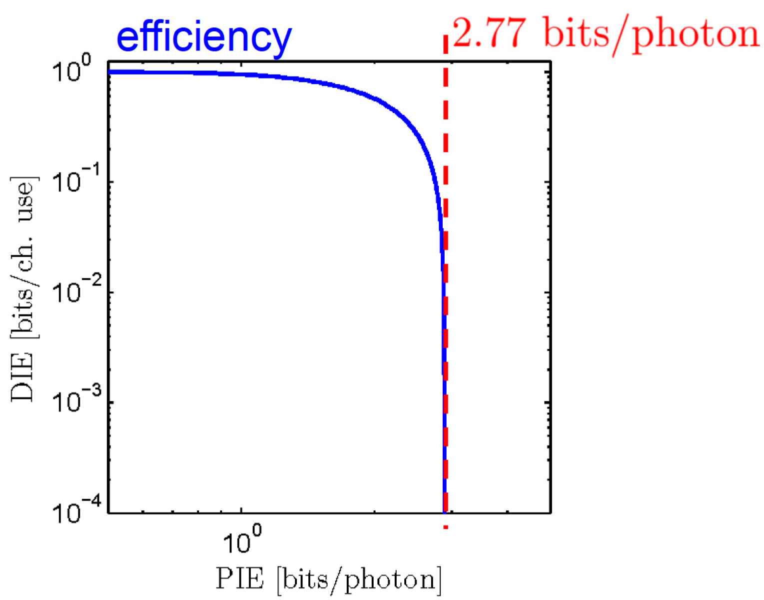
- First evaluating $i_d(t)$, then the mutual information, we obtain

$$\max_{\alpha_{\mathrm{lo}}(t)} I\big(K;\{N(t):t\in(0,T]\}\big)=h_2\big(p_+(0)\big)-h_2(P_{\mathrm{er}})$$

$$P_{\mathrm{er}}\equiv\frac{1}{2}\left(1-\sqrt{1-4p_+(0)\big(1-p_+(0)\big)e^{-4\alpha^2T}}\right)$$

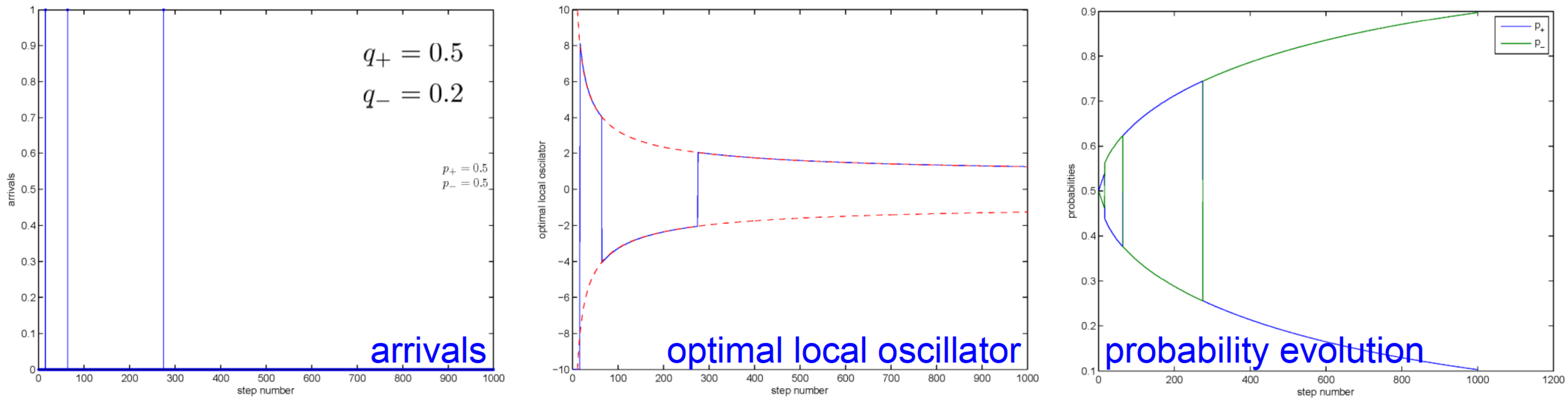
- Capacity and photon versus dimension efficiency plots



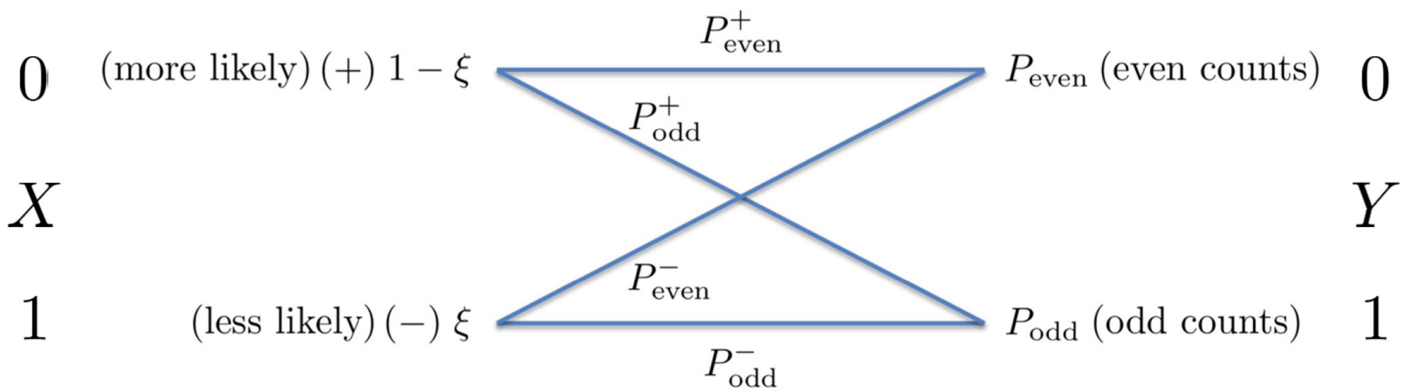


The binary asymmetric channel

- The Dolinar receiver is a *hard-decision* receiver
  - At the end of every symbol, one of two hypotheses are chosen
- The adaptive receiver is a *soft-decision* receiver
  - The receiver is maximizes the mutual information between input and the output waveform (photon-counting process)
- The Dolinar receiver and mutual-information maximizing adaptive receiver have the *same* capacity
  - No ‘soft information’ is obtained by resolving the count epochs



The Dolinar receiver: revisited



- The mutual information of the binary asymmetric channel is

$$I(X;Y) = H(X) - H(X|Y)$$

- The mutual information of the adaptive feedback receiver is

$$I(X;Y) = H(X) - h_2(P_{\mathrm{er}}) \longrightarrow H(X|Y) = h_2(P_{\mathrm{er}})$$

- This holds because P_{odd}^{+} and P_{even}^{-} are such that

$$H(X \oplus Y|Y) = H(X \oplus Y)$$

Arbitrary binary coherent-state constellation

- Earlier derivations for BPSK extend trivially to arbitrary binary coherent states
- Assume the states are $\{|\alpha_1\rangle, |\alpha_2\rangle\}$, $\alpha_1, \alpha_2 \in \mathbb{C}$, and have apriori probability distribution $\{q_1, 1-q_1\}$
- The optimal local oscillator must be

$\theta \equiv \angle(\alpha_1 - \alpha_2)$

$$\alpha_{\text{lo}} = -\frac{1}{2}(\alpha_1 + \alpha_2) + e^{j\theta}\alpha'_{\text{lo}}$$

optimal local oscillator for the real-valued, and antipodal constellation $\{|-|\alpha_1 - \alpha_2|/2\rangle, ||\alpha_1 - \alpha_2|/2\rangle\}$, and apriori distribution $\{q_1, 1-q_1\}$

- Proof is by contradiction

A ternary signaling constellation

- Assume ternary input constellation $\alpha_k \in \{|-\alpha\rangle, |0\rangle, |\alpha\rangle\}$
  - subscript '$+$' refers to positive amplitude state
  - subscript '$-$' refers to negative amplitude state

with probability $p_-(0)$
with probability $p_0(0)$
with probability $p_+(0)$

- Analytical solution to the optimality equations is intractable, yet, it can be solved numerically

- As an alternative we propose the following local oscillator function, whose performance is very close to optimal

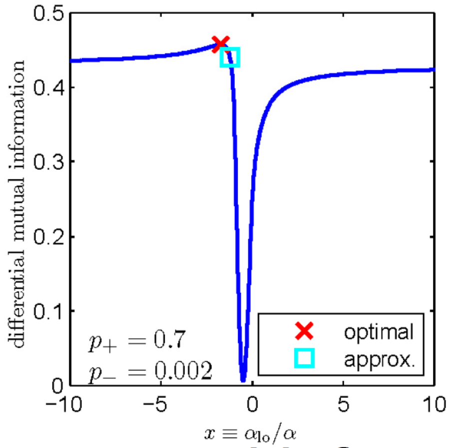
$$\alpha_{\mathrm{lo}}(t) = \begin{cases} \alpha \frac{1+p_-(t)+p_+(t)}{2(p_-(t)-p_+(t))} & \text{for } f\left(p_+(t), p_-(t)\right) > 0 \\ 0 & \text{otherwise} \end{cases}$$

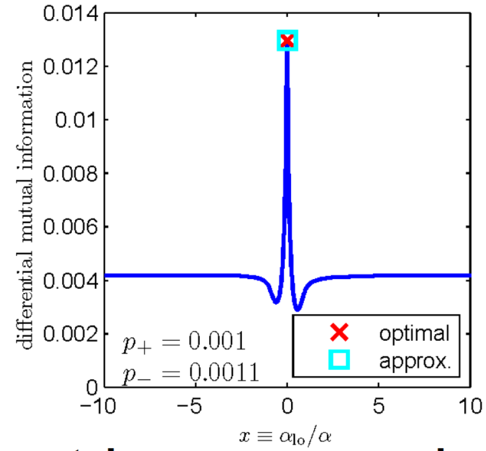
difference of mutual information attained
with the two local-oscillator possibilities

- The first case solution satisfies $\frac{\mathrm{d}}{\mathrm{d}t} \prod_{k=\{-,0,+\}} p_k(t) = -5\alpha^2 \prod_{k=\{-,0,+\}} p_k(t)$ between detection epochs

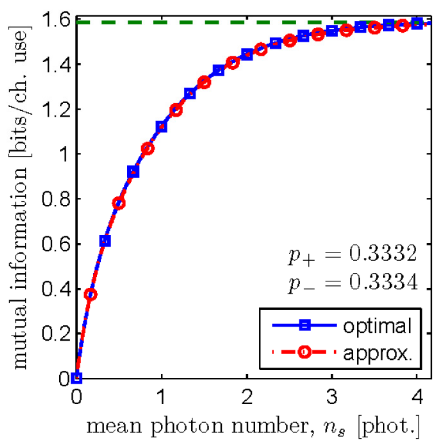
Optimal versus approximate local oscillator

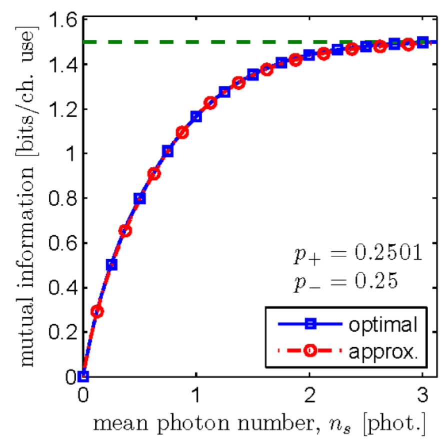
- Difference in $i_d(t)$ attained with the optimal and approximate local oscillators is small, and…





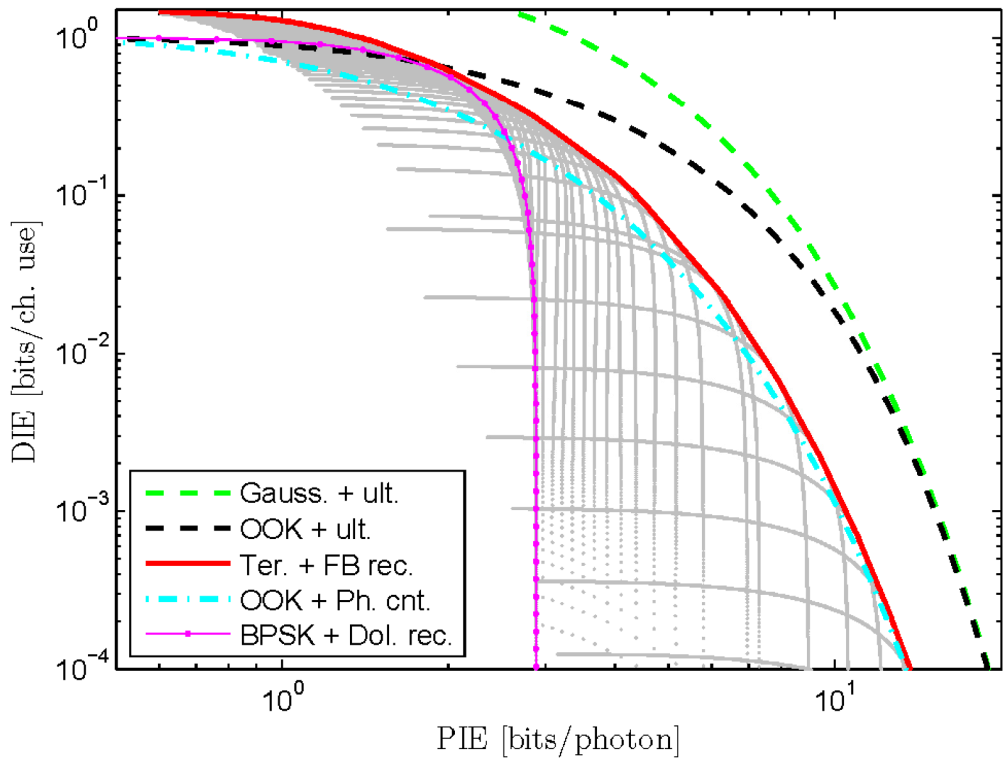
- …the mutual information agreement is very good





Capacity of ternary constellation

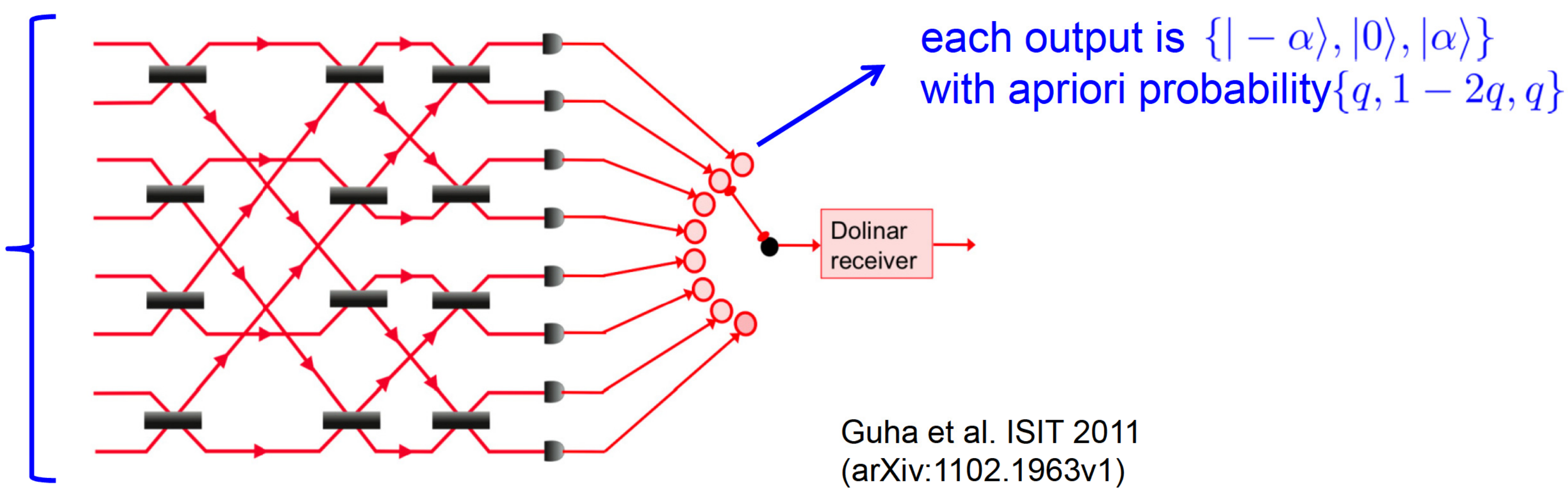
- Photon information efficiency (PIE) versus dimensional information efficiency (DIE) tradeoff
  - Adding the vacuum state overcomes finite PIE asymptote of BPSK
  - Better than OOK + photon-counting but gains diminish at high PIE



An adaptive-feedback receiver

- Guha *et al.* (ISIT 2011) use an adaptive-feedback receiver with for this ternary constellation
  - local oscillator is set to zero until a click is registered
  - once a photon is detected, the Dolinar receiver is applied in the remaining duration

inputs coherent states



Guha et al. ISIT 2011 (arXiv:1102.1963v1)

Optimality of adaptive-feedback receiver

- Setting $p_+(0) = p_-(0)$ in our approximation to the optimal local oscillator, we find that $\alpha_{\text{lo}}(0) = 0$ is *optimal* when

$$q < 0.5e^{-2} \approx 0.068$$

- $\alpha_{\text{lo}}(t) = 0$ remains optimal as long as $N(t) = 0$
- If an arrival is registered at t_1, $p_0(t_1^+) = 0,\ p_+(t_1^+) = \frac{1}{2},\ p_-(t_1^+) = \frac{1}{2}$, so the constellation reduces to binary and the Dolinar receiver becomes optimal adaptive-feedback receiver
- This strategy is optimal for $\text{PIE} \geq 2.5\,\text{bits/photon}$
- Hybrid BPSK + Pulse-Position Modulation (PPM), with PPM order ≥ 8 satisfies $q < 0.068$

Conclusions

- Adaptive-feedback based optical communication receivers for coherent states are studied in an information-theoretic setting
  - local oscillator is incrementally optimized
- Dolinar receiver is the optimal adaptive-feedback receiver in the binary case
  - no soft information
- BPSK+vacuum-state local oscillator can be approximated in partially-closed form
  - "wait for a photon, then apply the Dolinar receiver" is an optimal feedback strategy when non-vacuum states satisfy $q < 0.068$
- Analysis framework can be applied, in principle, to arbitrary coherent-state constellations