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# Athena's Computable General Equilibrium Model

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## Abstract

Athena is a single-user simulation to help U.S. Army intelligence analysts and others anticipate the consequences, intended and otherwise, of potential long-term stability and support operations in complex contemporary environments. Athena encompasses models of physical effects, politics, information, intelligence, civilian attitudes, demographics, and economics.

Computable general equilibrium models (CGEs) have been under development by economists for fifty years and contributions to their development have led to several Nobel Prizes. A CGE is at the core of Athena's economics model.

This paper presents the derivation of Athena's CGE and tells modelers how to expand the model to provide higher resolution.

To use a CGE dynamically, we extracted all slowly changing and delayed economic phenomena and modeled them and all non-economic phenomena in other components of the simulation.

One of the most important features of CGEs of stable regions is that their parameters can be calibrated by routinely available social accounting matrix data. In unstable regions, however, it would be naïve to use historical data blindly. By recognizing a distinction between shape and size parameters in the CGE model, data from nearby or similar regions and from recent times may be useful.

A typical CGE computes a long-term equilibrium. In a simulation, long-term solutions are needed to estimate latent demands, but the simulation is driven by current events and by responses to medium- and short-term economic equilibria, in which only prices, production quantities, and perhaps jobs are assumed to respond quickly enough to reach equilibrium. As a result, three passes through the CGE are used at each simulation time step.

CGEs typically assume that all markets are competitive and free. This can be a poor assumption in the regions of interest. In Athena applications, for example, international black markets, with exogenous prices determined by international competition, are often not only a major part of the economy, but a source of funding for insurgents.

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## Preface

Athena is a single-user simulation system to provide decision support to U.S. Army intelligence analysts and others who are addressing long-term stability and support operations in complex contemporary environments of regional and national scope in which the key to success is gaining the support (or at least the acquiescence) of the population. Courses of action being studied may involve the use of any or all of the elements of national power, which include diplomacy, information and intelligence operations, military action, and economic activities. Actions—and failures to act—have consequences of many kinds. The Athena simulation is intended to help the analysts anticipate those consequences, whether intended or not.

Athena has several modeling foci: physical effects (aka “the ground model”)<sup>1</sup>, politics<sup>2</sup>, information and intelligence<sup>3</sup>, civilian attitudes<sup>4</sup>, demographics<sup>5</sup>, and economics. All but the last are primarily discussed in other documents.

We describe Athena's models as “foci” because they are not independent of each other. Quite the contrary: They depend intimately on each other. The economics model, for example, relies upon the demographics model for the size of the available workforce. Thus, the demographics model contains the submodels of how civilians respond to market wages, price levels, insurgency recruitment campaigns, subsistence agricultural issues, and the tendency toward urban drift. The attitudes model assesses whether neighborhood security is so low that people are afraid to go to work or to shop. The politics model decides how the political actors' strategies cause them to spend their money. The ground model is responsible for keeping track of production capacities and inventories. And so on.

The purpose of this paper is to explain how to build and expand this *kind* of model of the economy of a region for use in Athena or an Athena-like model. Consequently, the six sectors of the current model design are referred to now and again, but discussion of those six sectors does not get very explicit until Section 4.5, which starts on page 16. That section might answer some of the questions readers have accumulated by that point in the text.

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<sup>1</sup> The physical effects model addresses the events and situations in the objective world, such as the actors' use of their assets, construction and destruction of infrastructure, armed conflict between opposing forces, collateral damage, essential services shortages, natural disasters, and so on.

<sup>2</sup> The politics model deals with the actors who can affect the state of affairs in the region. It allows them to pursue their goals and strategies as constrained by their assets and other actors' tactics. The bottom line is who is in control of each neighborhood.

<sup>3</sup> The role of the information and intelligence model is to allow the actors to use their information assets to affect the civilians' perception of reality.

<sup>4</sup> The attitudes model tracks the impact on civilian cooperation with force groups and satisfaction with regard to the four fundamental concerns of autonomy, safety, culture, and quality of life. Like the other models, it is resolved to neighborhood groups.

<sup>5</sup> The demographics model keeps track of the population: consumers, workers, subsistence farmers, refugees, etc.

At the focal point of the economics model, there is a computable general equilibrium model, which is responsible for assessing how the relevant aspects of the economy—wages and prices and unemployment, shortages and idle capacity, the consumer price index, and the gross domestic product—respond to things that happen in the simulation.

Although we believe there are novel aspects to this work, this is an application paper, and novelty is not essential to fulfilling its purpose.

# 1 Computable General Equilibrium Models

## 1.1 Credibility

*Computable general equilibrium models—CGEs—*of national economies have been a major research topic among economists for half a century and have led to several Nobel prizes.<sup>6, 7</sup> As recently as ten years ago, representatives of The World Bank and the International Food Policy Research Institute said, “CGE models today are routinely used by governments in policy formulation and debate.”<sup>8</sup>

A computable general equilibrium model is a set of simultaneous non-linear equations that describes the flows of value between the sectors of an economy. These flows are usually displayed in a *social accounting matrix*, or *SAM*, tableau as shown in Figures 1 and 2 on the next two pages. Generally speaking, streams of money flow in one direction while goods and services flow in the other. Parameters for the CGE equations are typically calibrated by data from a SAM that contains historical data.<sup>9, 10</sup> Inputs to the model typically describe changes in policy variables, such as tax rates, or in demographic variables, such as the size of the labor force. In addition to detailed cash flows, outputs of interest include prices and quantities of goods and services. Tax revenues, unemployment rates, shortages, cost of living indices such as the consumer price index, and the gross domestic product are among the derivative results.

Modern algorithms and modern computers make solution a cinch—given a close-enough approximate solution to start. Calibration to a base-case SAM gives a perfect starting solution!

*Dynamic stochastic general equilibrium models—DSGEs—*are currently a hot topic among theoretical economists. The basic idea is that the participants in the economy make their decisions not on past data, but on their uncertain expectations about the future. To implement this idea, DSGEs use immense amounts of data—several years of social accounting matrices—to make econometric projections of the uncertain future. This works well in a stable society, so long as there are no unmodeled changes in policy variables.<sup>11</sup>

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<sup>6</sup> The seminal work is [Johansen 1960].

<sup>7</sup> Cursory examination of the award citations published on the Nobel Foundation website [Nobel Prizes 2011] yielded the following list: John Hicks/Kenneth Arrow 1972, Wassily Leontief 1973, Gerard Debreu 1983, Richard Stone 1984, and Robert E. Lucas, Jr. 1995.

<sup>8</sup> See [Devarajan & Robinson 2002], page 1.

<sup>9</sup> Two of the standard references are [Stone 1962] and [Pyatt 1985].

<sup>10</sup> The data needed for the SAM is some of the national economic statistics for the country or region being modeled and may be available from United Nations or U.S. Central Intelligence Agency sources. Since the SAM is particularized to the specific CGE, the data will have to be processed to express it in the appropriate format.

<sup>11</sup> Robert E. Lucas, Jr., earned his Nobel Prize partially by virtue of what is now known as the Lucas' Critique [Lucas 1976]. In that critique, he identified “policy variables” as those parameters that describe the feedback of output data back into choices in the model. He pointed out that econometric data based on unchanging policies cannot be used to predict the effects of changes in those policy variables.

ATHENA'S COMPUTABLE GENERAL EQUILIBRIUM MODEL

Sectors	<i>goods</i>	<i>black</i>	<i>pop</i>	<i>actors</i>	<i>region</i>	<i>world</i>	Row Sums (Revenues)	Base Case Quantities
<i>goods</i>	Payments for intermediate goods during production	Black market payments for other goods	Consumer payments for goods	Sum of actors' payments for goods	Payments for goods from the rest of the region	\$ value of exported goods	Goods sector revenue from sales	Number of GoodsBKT sold per year
<i>black</i>	(zero) Black market goods are not used in production	Value of black market product consumed within the sector	Consumer purchases of black market goods	Sum of actors' payments for black market goods	Other purchases of black market goods within the region	\$ value of exported black market goods	Black market revenues from sales	Tonnes of black market product (opium) sold per year
<i>pop</i>	Payments for labor used in production of goods	Payments for labor used in production of black goods	Consumer payments for household help, gardeners, etc.	Sum of actors' payments for labor	Other payments for labor within the region	\$ value of exported labor	Populace revenue from wages	Work-yr/yr of labor supplied (the number of jobs in the region)
<i>actors</i>	Taxes & tax-like goods-sector payments to actors	Tax-like payments (CGE also has black's net revenue)	income taxes, "protection", and other tax-like payments	Sum of actors' payments to other actors	Other payments to actors from within the region	Foreign aid directly to actors, FAA (CGE also has graft * FAR)	Actors' revenue (not including net revenue of black sector)	REM is remittances and is not included in the SAM.
<i>region</i>	Tax-like payments for use of production capacity	Tax-like payments for use of production capacity	All other consumer payments that stay in the region	All other actors' payments that stay in the region	Other payments in the region from elsewhere in the region	Foreign aid directly to the region, FAR (CGE loses graft fraction)	Revenues in the rest of the region	graft is the fraction of FAR that will go to the actors.
<i>world</i>	Goods-sector payments for imported goods	Black payments for imports other than feedstock (which is entered separately)	Consumer payments for imported goods	Sum of actors' payments for imported goods	Other imports into the region	(blank)	Total import payments (other than black feedstocks)	AFwb is the amount of black feedstock per unit of black output.
Column Sums (Expenditures)	Total spending by goods sector	Total spending by black sector	Total consumer spending	Total actors' sector spending	Total rest-of-the-region spending	Total exports plus foreign aid		PFwb is the black feedstock price.
Base Case Prices	(one) \$/ GoodsBKT	\$/ tonne (of opium)	\$/ work-yr	Where appropriate, values are per-year rates.				

**Figure 1. Narrative Tableau.** A Social Accounting Matrix (SAM) shows the annual value of transactions between the economic sectors of a region as shown in this illustration, which has words where numbers go. The SAM for a base case is used to calibrate the parameters in Athena's Computable General Equilibrium (CGE) model. A few of the cells in the CGE have different definitions than in the SAM due to transfers to actors and exogenous revenues by the populace (REM). The Revenues column and the Expenditures row are routinely attached to the tableau. Base case prices and production quantities have been attached, but there are still a few other parameters needed to complete the description of the economy. *A blue font identifies cells containing required input data.*

Many of Athena's inputs are "policy variables". Hence, a DSGE would not be a good modeling choice for Athena.

In an environment of intentional international intervention, however, policy variables will surely change. To the extent possible, the models must encompass the major policy variables and their effects on the economy. Historical data must be used carefully, as simple extrapolation is useless; today's data may be the best predictor of tomorrow's data but not necessarily next month's. The economy's decision makers also know better than to rely on trends and will not behave as if they did. We will return to this topic in Section 1.3 below.

Sectors	<i>goods</i>	<i>black</i>	<i>pop</i>	<i>actors</i>	<i>region</i>	<i>world</i>	Row Sums (Revenues)	Base Case Quantities	
<i>goods</i>	813	20	2,800	14,000	2,500	113	20,246	2.025E+10	
<i>black</i>	zero	0	12	1,400	906	23,800	26,118	5.224E+04	
<i>pop</i>	12,000	22	36	4,640	454	70	17,222	4.306E+07	
<i>actors</i>	2,025	2,612	3,336	0	2,593	3	10,569	500 REM in \$M / yr	
<i>region</i>	2,714	2,612	5,853	2,859	1,727	1,502	17,266	0.3 graft (fraction)	
<i>world</i>	2,695	2,612	5,685	2,960	8,635	(blank)	22,587	1.05 AFwb (fraction)	
Column Sums (Expenditures)	20,246	7,878	17,722	25,859	16,815	25,488		62,000 PFwb in \$/feedstock ton	
Base Case Prices	1	500,000	400	All SAM entries are in \$M/year.					
	\$/ GoodsBKT	\$/ tonne	\$/ work-yr						

**Figure 2. Numerical Tableau.** The SAM shown in Figure 1 could contain numbers like these, which were used in testing. This example shows that there are differences between Revenues and Expenditures resulting from exogenous income to the populace (in addition to wages) and from transfers to actors of the *black* sector's net revenues and some of the foreign aid intended for the region. The *world* sector doesn't balance because there is a net inflow of money. Some of the additional input data needed by the model is shown to the right. As in Figure 1, a blue font identifies cells containing required input data.

## 1.2 Equilibrium?

How can an "equilibrium" model possibly be useful in this context?

Answer: First, interpret all of the value flows as average rates rather than annual amounts. Second, model any phenomena that are slow to react or that have delayed reactions elsewhere in the simulation, not in the CGE.

Thus, only prices, production quantities, wages, and the number of jobs—and those other variables that depend only on these—are assumed to reach equilibrium each time the CGE is evaluated, which is nominally once a week. Even the number of jobs is assumed to be stable under the variations caused by people being afraid to go shopping or to work.

Sectors	<i>goods</i>	<i>black</i>	<i>pop</i>	<i>actors</i>	<i>region</i>	<i>world</i>	Row Sums (Revenues)	Base Case Quantities	Exported Quantities	In-Region Quantities	Revenue to allocate
<i>goods</i>	813	20	2,800	14,000	2,500	113	\$20,246	2.02E+10	1.13E+08	2.01E+10	\$20,246
<i>black</i>	zero	0	12	1,400	906	23,800	\$26,118	5.22E+04	4.76E+04	4.64E+03	\$7,878
<i>pop</i>	12,000	22	36	4,640	454	70	\$17,222	4.31E+07	1.75E+05	4.29E+07	\$17,722
<i>actors</i>	2,025	2,612	3,336	0	2,593	3	\$10,569		REM in \$ / yr :	500	\$25,859
<i>region</i>	2,714	2,612	5,853	2,859	1,727	1,502	\$17,266		graft: (fraction)	0.3	\$16,815
<i>world</i>	2,695	2,612	5,685	2,960	8,635	(blank)	\$22,587		AFwb: (fraction)	1.05	\$25,488
Column Sums (Expenditures)	\$20,246	\$7,878	\$17,722	\$25,859	\$16,815	\$25,488			PFwb in \$ / feedstock tonne :	62,000	
Base Case Prices	1	500,000	400	All SAM entries are in \$M/year.							
	\$ / GoodsBKT	\$ / tonne	\$ / work-yr								

**Figure 3. CGE Tableau.** One of the things that makes an ordinary SAM easier to construct is that the row sums (revenues) have to equal the column sums (expenses) for all of the sectors. One of the things that makes our SAM harder to construct is that this condition holds only for the *goods* sector because we have omitted some of the cash flows from the SAM and account for them explicitly and separately. The last column on this revised tableau, which takes these additional cash flows into account, shows the revenue that each sector actually has available to spend. The numbers in this column do match the expenditures for all sectors.

These simple steps are the key to embedding the well-established CGE modeling methodology into a broad simulation such as Athena.

Everything else is modeled elsewhere in Athena. A few examples: Whether and when investment funds produce new production capacity will be dealt with in the ground model, which might have some “investment” funds being used to repair collateral damage rather than building new production capacity. The available labor force will be supplied by the demographics model, which will consider migration and refugees, subsistence agriculture and urban drift, insurgent recruitment activities, neighborhood security, and wages, among other things. Effective tax rates and how the political actors spend their income will be modeled in the politics model. Eventually, even seasonal variation might be included in production functions.<sup>12</sup>

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<sup>12</sup> Disclaimer: This paper refers to Athena’s conceptual design, not to the current state of implementation, which is ongoing. Functionality is being implemented in stages. Some details will inevitably differ in subsequent versions.

### 1.3 *Shape vs. Size*

#### 1.3.1 *Shape*

Most of the parameters in a CGE model describe the economic behavior of the participants in the economy—that is, how they allocate their income to their expenses. Among producers in a competitive economy, these allocations are largely driven by the technology they use. Among consumers, they are largely driven by cultural expectations. In the absence of changes in technology and culture, data from nearby regions and recent times can be expected to be useful, if selected carefully. We have chosen to call these the *shape* parameters.

More than technology must be the same for historical shape data to be useful, even for producers. For example, the Iraqi economy under Saddam Hussein was highly subsidized by oil revenues. Production decisions did not seek to minimize costs, but to provide full employment for supporters.<sup>13</sup> In such a case, if reconstruction goals included establishment of a competitive economy, shape data from a nearby country with a competitive economy would provide a better assessment of potential unemployment issues than recent data from the region itself. If reconstruction goals had included maintenance of the *status quo*, different models for the producers' budget decisions than are contained in this paper would be needed.

On the other hand, the effects of introducing *changes* in technology—such as replacement of oxen by tractors—could conceivably be approximated by careful changes in shape parameters, using data from regions that use the new technology. However, a great deal of research and validation should be performed before any such results were considered to be more than educated guesses as to the economic consequences of such changes.

#### 1.3.2 *Size*

In our model, the product of the number of consumers in a region and their per capita consumption requirement—which depends upon per capita income, added to export demand, determines the required size of the economy. If it is any smaller, there will be shortages, any larger and there will be overages and consequent waste. We have chosen to call the coefficients associated with the relevant submodels the *size* parameters.

Nominal population breakdowns and per capita consumption can be estimated from base case data. Demands for exported goods and labor are assumed to be part of the data that describes the scenarios being studied.

Athena's demographics model tracks the movements of population groups, such as displacements of refugees, that result from things that happen in the simulation and may later simulate migrations that occur for economic or other reasons. It supplies the economics model with the numbers of available workers and consumers. It also keeps track

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<sup>13</sup> "The sale of factories to private investors [during reconstruction after the Gulf War] would certainly result in layoffs. Who would keep four thousand workers at the vegetable oil company when the director himself had admitted that it only needed a thousand?" [Chandrasekaran 2007], p 126.

of the number of people engaged in subsistence agriculture. Although they are not part of the cash economy, they are included in the computation of the gross domestic product, the GDP, nominally valued as if they were paid the poverty-level wage.

The size of the economy is also affected by production constraints of various kinds. The ground model tracks production capacity in each sector of the economy in each neighborhood and the availability of feedstocks. The attitudes model keeps track of whether neighborhood security is so low that people are afraid to go out.

Remittances and foreign aid affect the amount of cash consumers have to spend and consequently the size of the economy, as well.

### **1.4 Numeraire**

The *dollar*, or the symbol \$, is used in this document to represent the unit of money. This choice was made for the convenience of American readers. Any other symbol—such as that which is used in the region of interest—could have been used instead. Only the numerical values of prices would change, none of their ratios would. To put that another way, every monetary value could be expressed in terms of, say, the number of work-years of an average worker, instead of in dollars, but money is a much more convenient unit of account.

But what is a \$ worth?<sup>14</sup> <sup>15</sup> The role of the *numeraire* is to answer that question.

Using the per capita consumption of goods in a base case to size the economy makes it convenient to use the average price of labor in the base case as the numeraire.<sup>16</sup>

With the base case average wage as the numeraire, the average wage is fixed. Other prices, however, vary as the modeled economy differs from the base case. Thus, there are changes in purchasing power, which are captured by the consumer price index, the *CPI*. The GDP captures changes in the economy as a whole. The GDP deflator is approximated by the quotient  $GDP/CPI$ .

Since the price of black market goods is assumed to be exogenously determined by the international market, it might seem a natural choice for the numeraire. However, that price is not suitable because the value seen inside the region, although expressed in \$/tonne for the convenience of Athena's users, is driven by the exchange rate, which is, in turn, driven by the regional economy—not the other way around. There is also a practical reason: the base consumer demand for black market goods could, at least in principle, be zero, which would lead to attempts to divide by zero. Technically, if the quality of the available data

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<sup>14</sup> This is an issue even in the real world. If the value of the \$ is not tied to a specific amount of something, prices can change arbitrarily. From 1946 to 1971, for example, the U.S. Treasury defined the value of a dollar by fixing the exchange value of an ounce of gold at \$35. A reviewer (Robert Swaim) pointed out that this practice was abandoned as fluctuations in the availability of the commodity, gold, induced fluctuations in the economy. He noted that it seems unlikely that an economy of interest would tie its currency to a commodity, even though commodities may be used as a medium of exchange.

<sup>15</sup> There is further discussion of this issue in a slightly different context in footnote 76 on page 45.

<sup>16</sup> We believe this is one of the novel aspects of our approach.

warranted doing so, the price of black market goods in the world market should be adjusted by the varying exchange rate between the value of a \$ in the world market and the value of a \$ in the regional market, which changes with inflation in the regional economy.

### **1.5 Inflation and Escalation**

When the model's numeraire is held constant, as in Athena 4, there is no inflation.<sup>17</sup>

Although inflation can be introduced into the model by changing the numerical value of the numeraire over time, additional modeling will be required to properly account for inflation and its effects, many of which belong in the attitudes model, some of which affect support for actors.

There can, however, be changes in the relative prices. The way in which individual prices vary with time is called *escalation*.

Economists call prices that show the effects of inflation "*nominal*" prices. When these effects are removed, they speak of "*real*" prices and "*constant year Y dollars*", where "year Y" is given a particular value. Sometimes the context makes it unnecessary to include the phrase "year Y".

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<sup>17</sup> A reviewer (Jeff Hanes) pointed out that inflation is often used by governments to increase their available revenue and that the populace behaves differently than if there were no inflation.

## 2 Selection of Sectors

A CGE is an aggregate model of the flows of value between the participants in an economy. One of the key modeling issues is the level of aggregation that is most useful. Those transactions that are grouped together into a *sector* will be treated as if they always occur in fixed proportions. Thus, unless the relative amounts of, say, illegal drugs and illegal arms are of particular interest to the user of the model, illegal drugs and arms can be aggregated into a single *black market* sector.

Alternatively, a CGE can be thought of as a model of the world economy that is resolved to some level of detail. Thus, there must always be a “rest of the world” sector.<sup>18</sup> To continue the previous example, suppose the user becomes interested in the distinction between the illegal drugs and illegal arms industries. Then the modeler can expand the model by splitting the *black market* sector into *illegal drugs* and *illegal arms* sectors, creating the appropriate detailed models of behavior, and gathering the appropriate disaggregate data.

The complexity of the CGE is heavily dependent on the number of sectors. Every sector defines both a row and a column in a matrix. The columns describe how the participants in the sector spend their incomes, which requires assumptions about their behavior and, usually, an analytical solution of an optimization problem. The more sectors there are, the more ways their incomes are split, so the number of equations increases approximately as the square of the number of sectors. Furthermore, too much resolution (that is, too many sectors) is unlikely to give meaningful results in the semi-chaos that is normal for stability and reconstruction operations. Thus, there is a strong incentive to keep the number of sectors to a practical minimum.

On the other hand, there should be sectors to deal with all relevant issues. Athena relies on the CGE to find wages, jobs, and shortages. The people provide labor, which is an essential factor of production, so they must be included in the CGE. However, since we have a model of how wages are distributed across the population,<sup>19</sup> we can address issues such as how many people are earning less than the defined poverty wage without having sectors to represent several categories of workers (such as white collar, blue collar, etc.). Since Athena has rules for the effects of shortages of *water*, *food*, and (*electrical*) *power* on civilian attitudes, those sectors may be split out in a later version of Athena. Until then, Athena's “environmental situations” model provides an adequate substitute.

After we have gained some experience with the six-sector model, it should not be difficult to increase resolution by splitting the *goods* and *black* sectors as desired—always keeping in mind that not only the amount of data but the amount of computation both grow as the square of the number of sectors. The advantage to splitting the *goods* or *black* sectors is that relative proportions among sub-sectors can then change as the economy seeks its equilibria.

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<sup>18</sup> The transactions within that sector, while far from zero, are of little interest to the economy within the region and are ignored.

<sup>19</sup> See [Lawler 2010].

Splitting new *kinds* of sectors out of the *region* or *world* sectors will be more of a challenge, but the modeling procedure is given in Section 4.

### **2.1 Routinely Neglected Sectors**

Black markets are, by definition, illegal, and are often omitted from “official” statistics. Nevertheless, in the regions of interest, they may contribute significantly to the economy. In the six-sector CGE presented in this paper as an example, we limit our attention to those black market goods traded on the world market because those are most relevant to the use cases we expect to address, as they provide important income to some of the political actors. Other black market items are assumed to be part of the “all other goods” sector, and parameter calibration should take this assumption into account. Future use cases may make it desirable to model consumer black markets explicitly in a later version.<sup>20</sup>

Some portions of the economy, such as the people who are engaged in subsistence agriculture, may be isolated from the rest and not participate in the cash economy. It is not necessary to create sectors in these cases, but they should be considered in the GDP.<sup>21</sup>

Barter transactions, on the other hand, are already included in the CGE, theoretically, at least. If barter is a significant part of the economy, care should be taken to include the value of barter transactions in the data used to calibrate model parameters.

Estimation of the extents to which migrants, pilgrims, and displaced persons (including refugees) are consumers and are in the available workforce is within the scope of the demographics model, but not the CGE.

### **2.2 Regional Political Actors**

The regional political actors operate within the regional economy. Some of them may get additional funding from other actors, such as the national government or from foreign sources. Conceivably, each one could be modeled as an economic sector, but that would be a poor modeling choice, as their behavior is determined by their goals and tactics.<sup>22</sup> Making each actor a sector would not only make the CGE considerably larger, but the CGE would have to be reconstructed for every scenario.

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<sup>20</sup> A reviewer (Jeff Hanes) pointed out that sometimes the black market is the only truly free market, so prices would surely be endogenous. Nevertheless, *international* drug, arms, and human trafficking are outside the scope of a model of regional economics, so their prices are exogenous to the regional model.

<sup>21</sup> A simple, straightforward way include their contribution in the CGE is to add the product of the number of people so engaged and the wage associated with the poverty level to the GDP.

<sup>22</sup> It is reasonable to expect that most of the actors' goals will be politically motivated, aimed at gaining control of neighborhoods, but those goals can include accumulating significant amounts of personal wealth. To the extent that personal wealth is spent outside the region or deposited in offshore bank accounts, those funds do not reenter the economy. While they could be treated as components of the payments from the *actors* sector to the *world* sector, it is not necessary to do so. Accounting for these transactions would show as a reduction in the imbalance between revenues and expenses for the *actors* sector, but it would have no effect on the regional economy. Imbalances are to be expected during simulation as the actors use revenues in some time periods to fund activities in other time periods.

The approach selected for Athena is to aggregate all of the regional political actors into a single sector, then pass the buck to the politics model. The politics model, which has access to any of the economic details it needs, such as how much each of the other sectors pays to the *regional political actors* sector, is responsible for determining each individual actor's income.

The politics model also keeps track of each actor's savings so they do not have to balance their budgets at each and every tick of the simulation clock and can save up for major activities. In the politics model, actors select tactics that they think will contribute to the achievement of their goals, but their choices are constrained by their assets, of which available accumulated cash is not the only kind.<sup>23</sup>

The politics model then aggregates the actors' expenditures, sector by sector, and reports that back to the CGE.

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<sup>23</sup> In the current model design, political actors are also constrained by the limited number of troops and by the communications asset packages they have available. Later versions may impose additional constraints, such as the production capacity they control.

## 3 Some Implementation Issues

### 3.1 Calibration

An astounding feature of the CGE methodology is that all of its many shape parameters can be easily estimated from the data in a SAM for a base case. The easiness results from the fact that each shape parameter is associated with a different cell in the SAM and the “association” implies an equation that is linear in the parameter. Thus, the parameters can be calibrated by solving those equations one after another. The equations are not quite independent, but the linkage is so weak that it is only necessary to solve some of them before the others.

Calibration equations for each of the sectors in the six-sector model are given in Section 4.5.

### 3.2 Latent Demand and Constrained Supply

If production is allowed to adjust to demand, which is the normal equilibrium assumption in CGEs, the solution will be that supply equals demand. However, if production is unable to adjust for any reason—such as an insufficient number of machines or an insufficient work force—prices will rise until the demand has decreased to the production capacity. Those price changes, however, will not be just in the sector with insufficient production capacity, but will ripple through the economy.

That is not a problem; that's what the CGE is designed to model. However, the resultant demands for products and factors will be those that correspond to constrained production.

#### 3.2.1 Long-run Computation

To compute the *latent* demand, the demand there would be if there were enough time to build the production capacity to meet all demands, including those that would arise from the not-yet-built factories, a long-run perspective must be taken, as is usually done with CGEs.

#### 3.2.2 Medium-run Computation

To determine the number of jobs in the economy, the CGE must be run again, taking existing production capacity limits, existing manpower limits, and existing feedstock<sup>24</sup> limits into account. The time scale for the “medium run” is thus defined by the way it is used: the time it takes to repair or replace damaged production capacity. Obviously, this is more notional than precisely quantitative.

#### 3.2.3 Short-run Computation

To determine the highly transient effects of workers staying home from work and of consumers not going shopping due to low security for their group in the neighborhood, we assume that the number of jobs will not be affected, but that production and consumption will be. This computation requires a third run through the CGE.

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<sup>24</sup> In this context, *feedstocks* are those ingredients that are necessary for production, are obtained from outside the region, and are not explicitly contained in the model.

### ***3.3 Switches and Multiple Passes through the CGE***

The CGE consists of a system of nonlinear simultaneous equations. As illustrated in the previous section, what the equations should be depends upon some assumptions. In the previous section, all production equations were switched between constrained and unconstrained versions to estimate the effects of various constraints on demands and prices.

The same approach can be used to investigate the effects of establishing or removing production quotas or price controls, even minimum wage policies. Ultimately, these issues could be studied either sector by sector or across the entire economy by introducing user-controlled switches (or, perhaps, model-controlled, adaptive switches) and making multiple passes through the CGE.

### ***3.4 Distinguish Quantity Produced from Quantity Demanded***

When the CGE is solved by an iterative procedure, as in Athena, which uses Gauss-Seidel iteration,<sup>25</sup> quantities produced equal quantities demanded only after the algorithm has converged—and not always even then. Consequently, separate variables must be assigned in the models (and in the program code).

It may be that the distinction is not needed if the program is written in GAMS<sup>26</sup>, which is the case for many of the CGE models in the professional literature, as this issue does not seem to get any attention. (Athena's CGE is not expressed in GAMS.)

### ***3.5 Inventories and Reserves***

The data in a CGE cell usually represents a years' worth of payments from the sector that defines the column to the sector that defines the row. In Athena, however, these cells are interpreted as the rates at which cash flows. (Goods and services flow in the other direction, from the row sector to the column sector, but services can be a bit abstract when the recipient is, for example, a regional political actor or the government.)

When sector revenues are not equal to sector expenses, there must be a change in the inventories of goods and in the amounts of cash reserves. If the economy were truly in

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<sup>25</sup> The Gauss-Seidel iteration algorithm for solving a set of linear or non-linear equations requires an initial approximate solution. The equations are written so that every independent variable appears on the left side of one and only one equation—though they may also appear on the right-hand side. The equations are evaluated in succession, always using the most recently computed values of the variables in the right-hand sides. This process is repeated until the values of the variables on the left-hand sides converge. If the equations are linear, convergence is assured if the coefficient matrix is symmetric positive-definite or strictly or irreducibly diagonally dominant [Wikipedia Gauss-Seidel 2010]. If the equations are not linear, unique convergence is not assured, but will usually result when the dominant terms are used to isolate the variables. Leontief relationships are usually assumed to be linear, but that assumption is not crucial when the Gauss-Seidel algorithm is used, as is illustrated by the design optimization process described in [Chamberlain 1990]. Cobb-Douglas relationships are not linear, but they are “well-behaved”, in that they are monotonic with their independent variables, continuous, and have continuous derivatives of all orders.

<sup>26</sup> GAMS stands for the General Algebraic Modeling System, and can be seen at <http://www.gams.com>. This reference should not be construed as a recommendation for or against its use in this context or any other.

equilibrium, there would be no such changes and revenues would exactly equal expenses for all sectors. Usually this is accounted for in the CGE by a *savings/investment* sector that accepts any discrepancy between revenues and expenses as savings, then redistributes the savings as investments, leaving no change in inventories and reserves.

In an Athena-like model, changes in inventories and reserves are likely to be significant. They should be allowed for when assessing sector expenditure choices, but the accumulations of differences are tracked outside of the CGE, as will be described in Section 4.5.4.6 for the regional political actors.

### **3.6 The “Laugh Test”**

A model that evokes laughter from a user who is savvy in the ways of the world but unfamiliar with the jargon and conventions of the academic fields from which the model was derived has an uphill battle to gain acceptance and use.

*Unemployment*—In the real world, unemployment is never zero. Consequently, unemployment computations must include normal turbulence in the labor market to avoid unwanted snickers.

*Profit*—To an economist, the amount of company income commonly thought of as “profit” that is needed to keep producers in the business is considered to be a cost rather than a profit, and the term “profit” is restricted to the difference between revenues and costs. To a non-economist, the sequence of statements used to derive price and production quantity equations—“Producers maximize profits. Profits are zero.”—does not pass the laugh test. Consequently, the term “profit” should either be very carefully and prominently defined or, perhaps, avoided. In Athena, the term “net revenues” is used for what economists call “profit”, and use of the term “profit” is avoided.

*Endowment*—In common terms, an “endowment” is the part of an institution’s income derived from donations.<sup>27</sup> In the context of CGEs, economists use the term to refer to the exogenously specified amount of a factor of production that an economic entity simply has, such as hectares of arable land or work-hours/person per year.<sup>28</sup> The region’s production capacity limits and its supply of available labor embody the endowment concept, but the term is avoided in Athena. Athena also uses the term “assets”, but somewhat more broadly, as it encompasses all those things an actor can use to execute his tactics and is not limited to factors of production.

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<sup>27</sup> This definition of *endowment* is part of the definition given in [Webster 1993].

<sup>28</sup> This definition of *endowment* is paraphrased from the definition given in [Deardorff 2011].

## 4 Recipe for an Athena-Like CGE

### 4.1 Select Sectors of Interest

As discussed in Section 2, the number of sectors selected should be kept to a minimum, yet include those needed to address relevant issues.

Thus, it is likely the model should include one or more black market sectors. Subsistence agriculture and urban drift, on the other hand, change slowly and should be modeled in the demographics model, not in the CGE.

Derivation of the equations for a CGE is a complicated, error-prone process and relevant data for regions that were recently (or still are) in economic chaos is difficult to obtain. It is highly recommended that models start with as few sectors as possible and that resolution be increased only after some experience has been gained with both sector modeling and data gathering. When sectors are split (see Section 5), new models are needed for all new sectors and demand equations for all other sectors must be reassessed.

### 4.2 Derive Sector Product Quantity Equations

The total demand quantity rate<sup>29</sup> for each sector  $i$ 's product,  $QD_i$ , is the sum of every sector  $j$ 's demand for that product,  $QD_{ij}$ :

$$QD_i = \sum_j QD_{ij}$$

Each producer would like to produce its product at exactly the same average rate,  $QS_i$ , as it is consumed to avoid overproducing or wasting his product. (This is the first of several behavioral assumptions.) If it could do that, then  $QS_i = QD_i$ . However, it may be constrained by production quotas, shortages of feedstocks, or production capacity limits. Consequently, total production rates are given by:

If production is not constrained:

$$QS_i = QD_i$$

If production is constrained:

$$QS_i = \max \left[ 0, \min \left( QD_i, CAP_i, \frac{MF_{wi}}{AF_{wi}} \right) \right]$$

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<sup>29</sup> CGEs are usually constructed with an implicit time scale of one year. For use in a dynamic simulation model, however, it is more convenient to think of the CGE's value flows and other annual amounts as *rates* of flow, rather than as annual amounts. Nevertheless, the "rate" qualifier will be omitted from most of the discussion to keep the terminology familiar.

Technically, since the Athena simulation is time-stepped, rather than event-driven, the rates are assumed constant over a time step and, after being multiplied by the length of the time step, become changes in amounts. The length of the time step can affect the accuracy of the simulation, but the model, which is slightly more abstract than the implemented simulation code, treats time as passing continuously and transactions as occurring smoothly at instantaneously average rates. Real-world transactions, of course, occur in discrete chunks at irregular times.

where

$CAP_i$  is sector  $i$ 's production capacity, the maximum production rate achievable with the currently available infrastructure (facilities, equipment, etc.). During the simulation, this value can change due to collateral damage, investments coming to fruition, and so on.

$MF_{wi}$  is the maximum amount of sector  $i$ 's imported feedstock currently available, expressed as an average rate. If the product of sector  $i$  does not have a required feedstock, this term is omitted.

$AF_{wi}$  is the amount of imported feedstock needed to produce each unit of sector  $i$ 's product.<sup>30</sup>

The  $\max[0, \sim]$  operation is “bullet-proofing” to ensure that the solution is never negative during or at the conclusion of the iteration process.

### **4.3 Derive Sector Product Price Equations**

Not all prices are determined competitively. Some prices—e.g., internationally marketed illegal drugs—may be exogenous (that is, dictated by the world market). Then, quantity supplied is the smaller of capacity and demand.

In competitive markets, product prices drive quantities demanded and quantities produced. Thus, the price equations for products adjust prices until supply is the smaller of capacity and demand.

In the two-product, one-factor, six-sector model, there are only three prices. The price of internationally traded black market goods is taken to be exogenous, the price of labor is used as the numeraire, and the price of goods is adjusted to make the possibly constrained supply equal demand. The goods price equation is derived in Section 4.5.2.5. When the model is expanded to a larger number of sectors, there will be more price equations.

### **4.4 Derive Sector Factor Quantity Equations**

The total demand for each sector's factor of production is the sum of the amounts demanded by all sectors (some self-demand is okay).<sup>31</sup>

If a single economic entity, such as the populace, offers multiple factors of production, such as *arable land* in addition to labor, sectors must be included in the model to account for supply and demand of each of those factors. As with *available labor*, the supply of these factors is exogenous, and payments for their use add to the revenues of the populace. The populace then computes its expenditures in the same way as is described in Section 4.5.3.6.

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<sup>30</sup> This formulation assumes there are no substitutes for the feedstock (like a Leontief requirement).

<sup>31</sup> When a sector is highly aggregated, more of its output will be used within the sector as input for other products in that aggregate. This internal consumption causes a difference between gross output and net output of the sector; that difference is the self-demand. Thus, it may be desirable to include some self-demand to obtain realistic sector sizes. However, it was pointed out by one of the reviewers (Robert Swain) that doing so may risk model instability. Since the risk is larger for larger amounts of within-sector demands, values exceeding perhaps 10% of gross output suggest the need to increase model resolution by splitting such sectors. Consider, for example, the absurd extreme of a one-sector CGE: the self-demand would be 100% of the gross.

The limitations on available amounts of factor quantities are exogenous. The amounts that are used/supplied (such as the number of jobs) are the smaller of the demand and the available supply.

Consumers choose demand amounts to maximize a utility function, which is chosen as part of the specific model; but they spend exactly their after-tax income. In a later version of Athena, savings and investment will be modeled outside the CGE as is done for the regional political actors.

#### **4.5 Derive Sector-by-Sector Equations**

Every sector's total revenue is the sum of its revenue from each sector. This is not quite as trivial a statement as it appears to be because every source of revenue for sector  $i$  is also an expense to some sector  $j$  and must be considered from both points of view. Remittances<sup>32</sup>, for example, are unlikely to be overlooked when considering how much income families have, but they are easy to overlook when considering the payments made by the *rest of the world* sector.

The general procedure for derivation of a sector's demand equations is to divide the sector's total revenue among its sector-by-sector expenditures in accord with an assumption about how the sector behaves. Applying that behavior assumption may lead to an optimization problem that can be solved to produce a model of the sector's behavioral decisions.

The general idea is that producers are usually assumed to make decisions that maximize profit, consumers are assumed to make decisions that maximize their "utility", and political actors are assumed to spend according to budgets that are consequences of their political agendas.

The subsections below illustrate this general procedure by going into considerable detail for several different kinds of sectors, using Athena's six-sector model to illustrate. The six sectors are as follows:

- All Goods in the Region (other than black market goods), aka *goods* or  $g$ —The *goods* sector product is an aggregate of all goods, other than those on the international black market, that are produced in the region. The relative amounts of each kind of good are in fixed proportions; to study relative proportions, a higher-resolution model (that is, one with more sectors) would be required. The unit of production is a notional basket of goods and services of that mix, called a *GoodsBasket*, with the size of the basket being just enough that it has a value of one dollar in the base case. (Thus, the number of *GoodsBaskets* produced in a unit of time in the base case can be determined from the dollar value of goods produced during that time.)
- The Black Market, aka *black* or  $b$ —The black market product is an aggregate of illegal drugs traded on the international market, in fixed proportions. Quantities are expressed in terms of the number of metric tons (that is, *tonnes*) of refined opium ready to ship at the port of export that would have the same monetary value as the aggregate. Trafficking in illegal drugs is the use case, which provides focus, but other illegal

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<sup>32</sup> Remittances are the monies sent home by people who are working outside the region.

goods,<sup>33</sup> such as illegal arms or humans—or a combination of all black market goods—could be considered instead, providing appropriate data is used. For now, grey markets<sup>34</sup> are assumed to have negligible cash flows or to be treated as part of the white-market *goods* sector.

- The Populace, aka *pop* or *p*—The populace sector consists of the people who live, work, and consume in the region. They provide a wide variety of different kinds of labor, but the relative amounts are in fixed proportions in this model; to study relative wages and unemployment between different kinds of jobs within Athena, a higher-resolution model would be required.<sup>35</sup> Quantities of labor are expressed in terms of the number of work-years per year of a mix of workers some of whom do each of the jobs in the mix, earning an average wage of  $P_p$ .
- Regional Political Actors, aka *actors* or *a*—The regional political actors sector consists of those actors<sup>36</sup> whose revenues are derived, at least in part, from economic activity in the region and whose behavior is modeled in Athena's model of regional politics. This sector must include the government as one of the actors so that the destination of tax revenues does not depend upon whether the region is less than, equal to, or more than a single country. The CGE computes the actors' total revenues and passes this information to the politics model. The politics model decides, partly autonomously and partly in response to user input, how the actors spend those revenues and passes that information back to the CGE. If the expenditures do not equal the revenues, it is the responsibility of the politics model to keep track of accumulations.
- The Rest of the Region, aka *region* or *r*—The rest of the region sector encompasses all economic activity involving economic entities within the region that is not accounted for elsewhere. This sector allows us to cleanly distinguish the region from the rest of the world.
- The Rest of the World, aka *world* or *w*—The rest of the world sector encompasses all economic activity between economic entities within the region and the rest of the world. All economic activity that does not affect any of the other sectors is ignored.

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<sup>33</sup> When discussing black markets, it is tempting to assert that using the term “goods” to describe their products fails the laugh test and to seek some other word. We have resisted that temptation.

<sup>34</sup> Grey markets deal in legal goods, but operate outside normal distribution channels and often involve distributors who have no relationship with the producers.

<sup>35</sup> See [Lawler 2010] for a study of the relationships between survival, poverty, and the average wage. Her results reduce the need for such a larger model.

<sup>36</sup> According to [FM 5-0 2010], “An actor is an individual or group within a social network who acts to advance personal interests. Relevant actors may include states and governments; multinational actors such as coalitions; and regional groupings, alliances, terrorist networks, criminal organizations, and cartels. They may also include multinational and international corporations, nongovernmental organizations, and other actors able to influence the situation either through, or in spite of, the appropriate civil, religious, or military authority.”

Sectors	<i>goods</i>	<i>black</i>	<i>pop</i>	<i>actors</i>	<i>region</i>	<i>world</i>	Row Sums (Revenues)	Base Case Quantities
<i>goods</i>	$BX_{gg}$	$BX_{gb}$	$BX_{gp}$	$BX_{ga}$	$BX_{gr}$	$BX_{gw}$	$BREV_g$	$BQD_g$
<i>black</i>	$BX_{bg} = 0$	$BX_{bb}$	$BX_{bp}$	$BX_{ba}$	$BX_{br}$	$BX_{bw}$	$BREV_b$	$BQD_b$
<i>pop</i>	$BX_{pg}$	$BX_{pb}$	$BX_{pp}$	$BX_{pa}$	$BX_{pr}$	$BX_{pw}$ does not include REM	$BREV_p$	$BQD_p$
<i>actors</i>	$BX_{ag}$	$BX_{ab}$ does not include BNR.b	$BX_{ap}$	$BX_{aa}$	$BX_{ar}$	$BX_{aw}$ does not include ( <i>graft</i> * FAR)	$BREV_a$	$BREM$
<i>region</i>	$BX_{rg}$	$BX_{rb}$	$BX_{rp}$	$BX_{ra}$	$BX_{rr}$	$BX_{rw}$ contains all of FAR	$BREV_r$	<i>graft</i>
<i>world</i>	$BX_{wg}$	$BX_{wb}$ does not include cost of feedstocks	$BX_{wp}$	$BX_{wa}$	$BX_{wr}$	$BX_{ww} = 0$ (blank)	$BREV_w$	$AF_{wb}$
Column Sums (Expenditures)	$BEXP_g$	$BEXP_b$	$BEXP_p$	$BEXP_a$	$BEXP_r$	$BEXP_w$		$PF_{wb}$
Base Case Prices	$BP_g$	$BP_b$	$BP_p$	The "B" prefix implies Base Case data.				

**Figure 4. Base Case Variables in the SAM.** Cells in the SAM are labeled  $BX_{ij}$ , where  $i$  is the row,  $j$  is the column, and the  $B$  prefix indicates that these are base case variables. Cells in the CGE matrix have the same labels, but without the  $B$  prefix. Note that  $BX_{pw}$  includes only the value of exported labor, but not  $BREM$ , the base case remittances.  $BREM$  must be entered explicitly as additional data so consumer spending can be computed.  $BX_{aw}$  and  $BX_{rw}$  contain  $FAA$  (foreign aid to actors) and  $FAR$  (foreign aid to the region). Furthermore,  $BX_{wa}$  does not include the cost of imported feedstocks. In the CGE's tableau, a portion (*graft*) of the  $FAR$  (foreign aid to the region) is transferred to  $X_{aw}$ , and  $X_{rw}$  is what's left of the  $FAR$ . Also in the CGE, the *black* sector's net profit is added to  $X_{ab}$  and the cost of imported feedstocks is added to  $X_{wa}$ . There are a few additional variables needed to complete the database:  $AF_{wb}$  and  $PF_{wb}$  are the Leontief coefficient and the price for the black market's imported feedstock.

Note the modeling assumption that goods production requires no black market goods ( $X_{bg} = BX_{bg} = 0$ ). That assumption is embedded in the current code; if the *goods* sector does have a substantial requirement for *black* goods, the CGE equations and the associated code should be modified.

#### 4.5.1 Notation

Up to this point in the discussion, sectors have been identified generically by the subscripts  $i$  and  $j$ , depending upon whether they were being considered as suppliers ( $i$ ) or purchasers ( $j$ ). In the detailed descriptions of the six Athena sectors that follow,  $i$  and  $j$  will

appear mostly as indices of summation. In subscripts, they will have been replaced by specific values:  $g, b, p, a, r,$  or  $w$ , depending on which specific sector is intended.

So as to avoid interrupting the flow of development with definitions of variables, the definitions are collected here. Note that, as described in the previous paragraph,  $QS_b$ , for example, means  $QS_i$  with  $i$  taking on the value  $b$  (indicating the black market).

$A_{ij}$  is the Leontief production function's parameter that specifies how many units of sector  $i$ 's product is needed to make each unit of sector  $j$ 's product.

$A_{gp}$  is the unconstrained per capita demand for goods, in GoodsBaskets/year per capita. This Leontief relationship implies that the outputs of other sectors (such as opium and labor) cannot be substituted for goods. Since the *goods* sector is a composite "basket" of goods, the basket is sized so that it contains one dollar's worth of goods. As a result,  $A_{gp}$  can be interpreted as the portion of the *average annual base case cost of living*, in \$ per year per capita,<sup>37</sup> that results from purchasing goods. It does not include the contributions to the cost of living from the other sectors (that is, it also excludes taxes and tax-like payments).

When calibrating the model, it may be possible to find that number itself. An alternative approach would be to identify what is required for the average consumer to live for a year—say 1.46 cubic meters of drinking water at 76 \$/m<sup>3</sup>, 5300 KWh of energy at \$0.18/KWh, 820,000 kcal of food at 0.0035 \$/kcal, 0.61 phone-years of telecommunications at \$300/phone-year, and \$100/year of everything else, then compute  $A_{gp}$  by summing the products of these price-quantity pairs, yielding 4,231 \$/year for this hypothetical average consumer.<sup>38</sup> Then, at 1 \$/GoodsBasket, the requirement  $A_{gp}$  would be for 4,231 GoodsBaskets/year per capita. If there are 180,000,000 consumers, the base case expenditure on goods would come to 762 billion \$.

$AF_{wi}$  is the amount of imported feedstock needed to produce each unit of sector  $i$ 's product, in feedstock units/product  $i$  units. In particular,  $AF_{wb}$  is the per-tonne amount of feedstock required for black market goods. This Leontief relationship implies that nothing can be substituted for the feedstock.

$A_j$  is the Cobb-Douglas production coefficient, a scaling parameter used in the Cobb-Douglas production function; the value is calibrated from base case data by inverting the Cobb-Douglas formula.<sup>39</sup>

<sup>37</sup> Note that this is not a *Cost of Living Index*, but the cost of living itself.

<sup>38</sup> For a reality check, this number should be compared to the officially defined "poverty" level and to the average wage in the region. If the *goods* portion of the average cost of living exceeds the poverty level, then the average consumer is better off than poverty. If even the *goods* portion of the average cost of living exceeds the average wage, however, then the average consumer is not making enough to meet his needs.

<sup>39</sup> If the equation were rewritten with all of the quantities expressed as ratios to their values in the base case,  $A_j$  in that rewritten equation would be 1. To put it another way, if we had base case values of the demands,  $BQD_{ij}$ , the calibrated value would be given by  $A_j = 1/(\prod_i BQD_{ij}^{f_{ij}})$ . However, there is no need to gather that much data, as only the product is used. Furthermore, the tax-like payments would change the expression somewhat. A derivation of the correct formula to use is given on page 30.

$ATI_j$  is what's left of sector  $j$ 's revenues after making all tax and tax-like payments, in \$/year.<sup>40</sup>

$B\sim$  (that is, a  $B$  prefix) is the value of that variable (that is: of whatever replaces " $\sim$ ") in the base case. For example,  $BREV_i$  is the base case value of  $REV_i$ , the revenue rate for sector  $i$ . It would be superfluous to list all these variables here, but most of them are shown in Figure 4.

$C$  is the denominator of the Consumer Price Index computation, the \$ cost of the defined market basket of consumer purchases at base case prices. (The numerator is  $SUM$ .)

$C_j$  is the per-unit cost of production in sector  $j$ , in \$/unit.

$CAP_i$  is sector  $i$ 's production capacity, the maximum production rate achievable with the currently available infrastructure (facilities, equipment, etc.), in product  $i$  units/year. During the simulation, this value can change due to collateral damage, investments coming to fruition, and so on.

$CF_{ij}$  is the cost of feedstock for sector  $j$  obtained from sector  $i$ . Only  $CF_{wb}$  is currently used in the model.

$COST_j$  is the total cost of production in sector  $j$ , in \$/year.

$CPI$  is the Consumer Price Index, the ratio of the cost of a predefined market basket of consumer purchases at present prices to the cost at base case prices.

$EXP_j$  is the total expenditures of sector  $j$ , in \$/year.

$EXPORTS_i$  is sector  $i$  export rate, expressed in product  $i$  units/year.

$f_{ij}$  is a model parameter that is the fraction of sector  $j$ 's budget that is paid to sector  $i$ . This shape parameter is normally calibrated from the base case. It will appear in several contexts, one of the most notable of which is as the exponents in the Cobb-Douglas production function. These functions express how percentage changes in ingredient quantities affect the percentage change of the output product quantity. If there are no economies of scale,  $\sum_i f_{ij} = 1$ .<sup>41</sup>

$FAA$  is the portion of foreign aid that goes directly to the *actors* sector, in \$/year.

$FAR$  is the portion of foreign aid that goes directly to the *region* sector, in \$/year.

$GDP$  is the Gross Domestic Product, the market value of all final goods and services produced in the region, in \$/year.

*graft* is the fraction of  $FAR$  that winds up in the pockets of the actors. Some of this may be graft, skimmed off the foreign aid that was intended to help support the regional

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<sup>40</sup> Since this variable,  $ATI_j$ , includes tax-like payments in addition to tax payments, the term "disposable income" might have been a better choice than "after-tax income".

<sup>41</sup> On the other hand, there can be economies or diseconomies of scale if the parameter  $A_g$  depends on  $QS_g$  even if the sum of the  $f_{ij}$  is 1.

economy into the actors' coffers. No doubt some of this funds legitimate administrative expenses, so calling it *graft* may be unfair.

$k_j = 1 - (t_{aj} + t_{rj} + t_{wj})$  is the fraction that is left of sector  $j$ 's sales (or wages) after making all tax and tax-like payments.

$L$  is the "Lagrangian", a modified version of the objective function sometimes used in solution of constrained optimization problems. See Section 4.5.2.5.

$\lambda$  is the "Lagrange multiplier" used with the Lagrangian. See Section 4.5.2.5.

$LF$  is the number of people in the labor force.

$MF_{ij}$  is the maximum amount of sector  $j$ 's feedstock that can be obtained from sector  $i$ , in feedstock units/year. In the current version of the model, we use only  $MF_{wb}$ .

$NR_i$  is sector  $i$ 's net revenue, in \$/year.

$PF_{ij}$  is the price of sector  $j$ 's feedstock when obtained from sector  $i$ , in \$/feedstock unit. In the current version of the model, we use only  $PF_{wb}$ .

$P_i$  is the price per unit of the product of sector  $i$ , in \$/product  $i$  unit.

$QD_i$  is the total demand for the product of sector  $i$ , in product  $i$  units /year.

$QD_{ij}$  is sector  $j$ 's demand for the product of sector  $i$ , in product  $i$  units /year.

$QF_{ij}$  is the amount of sector  $j$ 's feedstock obtained from sector  $i$ , in feedstock units/year. In the current version of the model, we use only  $QF_{wb}$ .

$QGDP_i$  is the quantity of the product or factor supplied and sold by sector  $i$  (for those sectors that have a product or factor), in product  $i$  units /year, that contributes to the gross domestic product (GDP).

$QS_i$  is the quantity of the product or factor supplied and sold by sector  $i$  (for those sectors that have a product or factor), in product  $i$  units /year. We assume that all units produced are sold, even though some may be consumed within the sector.<sup>42</sup>

$QS_p$  or  $QS_{pop}$  is the number of jobs, as distinguished from the amount of available labor,  $LF$ , some of whom may be unemployed.

*Real Unemployment* is the difference between the number of available workers and the number of jobs.

$REM$  is what the populace receives from remittances, in \$/year.

$REV_i$  is sector  $i$ 's total revenue from all sources, in \$/year.

$SA$  is the value of subsistence agriculture to the economy, that is, its contribution to the gross domestic product, in \$/year.<sup>43</sup> It is exogenous to the CGE.

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<sup>42</sup> Those products that are sold within the same sector in which they were produced are counted as both revenues and expenses. Hence, they cancel out on the balance sheet, but they affect the quantity produced. When the production function allows economies or diseconomies of scale, the quantity produced will affect price.

*SUM* is numerator of the Consumer Price Index computation, the \$ cost of the defined market basket of consumer purchases at present prices. (The denominator is *C*.)

$t_{ij}$  is the tax or tax-like fraction of sector  $j$ 's sales revenue (or wages) that is paid to sector  $i$ .  
If  $i = world$ , this is the fraction of sector  $j$ 's revenue spent on imports other than feedstocks.

*Turbulence* is number of people who are between jobs and not "really" unemployed. Economists call this "frictional unemployment". This is always greater than zero.

*TurFrac* is the fraction of the labor force who are "between jobs".

*U* is the so-called "utility function" that expresses consumer preferences for purchases.

*Unemployment* is the sum of the *Real Unemployment* and the *Turbulence*, and is always greater than zero.

$X_{ij}$  is the payment received by sector  $i$  from sector  $j$ , in \$/year.

To reduce confusion, we have used the "mathematical asterisk", \*, almost everywhere to indicate multiplication instead of the lighter style of implying multiplication by proximity of symbols because many of the symbols we use are composed of several characters. We trust that readers with experience in programming will be familiar with the practice.

#### **4.5.2 Example: Producers with Competitive Markets**

Producers are assumed to make production decisions to maximize their profits. To anticipate what their decisions will be by solving a mathematical optimization problem, we assume that a particular *production function* describes the average state-of-the-art of their technology.<sup>44</sup>

The production function says how much of the product can be made from known amounts of the ingredients. Some production function formulas that might be assumed to be applicable, such as the *Cobb-Douglas* formula, allow substitution among ingredients. Others, such as the *Leontief* formula, do not. A wide variety of formulas, with solutions to

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<sup>43</sup> The value is computed in the demographics model as the product of the number of people engaged in subsistence agriculture and the poverty wage divided by the average subsistence family size. Suppose, for example, the poverty line wage was about \$280/year per person, as in [Lawler 2010]. The number of people supported by subsistence agriculture will be supplied by the demographics model.

<sup>44</sup> A nuance that is usually ignored is that some producers are more efficient than others and consequently have positive net revenues, while others have negative net revenues. But even most of the less efficient producers make enough to stay in business, so *zero* net revenues must be associated with *average* returns on investment, executive salaries, etc. Consequently, the production function parameters should describe the *average* state-of-the-art, not that which would be implied by the best technology available. When the parameters are obtained by calibration to real-world data, the issue is moot, but any attempt to estimate the parameters from data about the technology must be carefully tempered.

the associated optimization problems, can be found in the literature,<sup>45</sup> but these two are the most commonly used and are used in Athena's six-sector model.<sup>46 47</sup>

In the real world, as opposed to the world of the model, producers may not behave in this way. As noted earlier, for example, Saddam Hussein operated Iraq's state-run industries at a loss (with oil-profit subsidies) to ensure full employment of his supporters. A model to describe that kind of behavior would be quite different than is currently in the Athena design. The master plan for the Athena economics model includes switches that will allow the analyst to select among a wide variety of such choices, including price controls, production quotas, employment quotas, and whatever else might be needed. Ultimately, Athena might even be able to control some of the switches in response to simulated changes in conditions. Such enhancements to the model are, however, highly speculative and are unlikely to be implemented unless a sponsor expresses a need to study such cases.

If there is a major feedstock, its effects can be incorporated in the model as is done for the black market sector in Section 4.5.3.2 below.

#### 4.5.2.1 Goods: Revenues

The only source of revenue for the *goods* sector is sales of its product:

$$REV_g = P_g * QS_g.$$

#### 4.5.2.2 Goods: Production Requirements

Assume that the production technology is described by a Cobb-Douglas production function,<sup>48</sup> and that only goods and labor from within the region are required. Thus,  $f_{bg} = f_{ag} = f_{rg} = f_{wg} = 0$ , and the production function is given by

$$QS_g = A_g * QD_{gg}^{f_{gg}} * QD_{pg}^{f_{pg}}.$$

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<sup>45</sup> See, for example, [Griffin 1987].

<sup>46</sup> The current Athena model of the *goods* sector assumes a Cobb-Douglas production function and the model of the *pop* sector assumes a Cobb-Douglas consumer preference function. The model of the *black* sector assumes a Leontief production function and a Leontief requirement for feedstock. Later versions of Athena may use other assumptions.

<sup>47</sup> One of our advisors (Dr. Les Servi) suggested we consider using a *CRESH* function (see [Hanoch 1971]), which has a number of desirable properties, including an elasticity of substitution that can be set somewhere between the *zero* of the Leontief function and the *one* of the Cobb-Douglas. In response to our concern that it would be difficult to find appropriate values of that elasticity, he pointed out that assuming an arbitrary number such as 0.2 would generally fit better than either extreme. We decided not to use it now because the formulation is considerably more complicated and difficult to explain; it may be appropriate to introduce it in some later version of Athena to improve fidelity.

<sup>48</sup> In a Cobb-Douglas production function, inputs may be substituted for each other according to the formula  $QS_j = A_j \prod_i QD_{ij}^{f_{ij}}$ . In words, this says that the quantity supplied is the product of a technology-driven parameter and the product of the amounts of each ingredient raised to an exponent, with the sum of the exponents exactly equal to 1.0. Thus, if the quantity supplied is fixed, a smaller amount of one of the ingredients can be compensated for by a larger amount of another.

Let's consider the *goods* sector's payments to each sector, starting with those that produce products or offer factors of production.

*Goods*—The self-consumption of goods within the *goods* sector is  $QD_{gg}$ , an amount that will be determined as part of the producer's optimization process. The within-sector payment is<sup>49</sup>

$$X_{gg} = P_g * QD_{gg}.$$

*Black*—No payments. Black market goods are not used in the production of other goods.<sup>50</sup> Some of the workers may purchase black market goods, but that is accounted as an expense of the *pop* sector.

*Pop*—The labor required to produce  $QS_g$  of goods is  $QD_{pg}$ , an amount that will also be determined during optimization. The wages paid by the *goods* sector are

$$X_{pg} = P_p * QD_{pg}.$$

#### 4.5.2.3 Goods: Other Payments

Section 4.5.2.1 described *goods'* payments to other sectors as a result of its production requirements, but there are additional expenses.

*Actors*—The regional actors do not contribute directly to the production of goods, but they may, depending on the scenario, collect taxes, kickbacks, bribes, "protection", or other fees—or they may subsidize other sectors. These are all modeled as a tax-like payment<sup>51</sup>:

$$X_{ag} = t_{ag} * P_g * QS_g.$$

*Region*—The *Rest of the Region* does not contribute directly to the production of goods, but it does supply the production capacity. Assuming the cost of using that production capacity is a tax-like multiple of the portion that is used gives

$$X_{rg} = t_{rg} * P_g * QS_g.$$

*World*—The *Rest of the World*, supplies imported goods, the cost of which is modeled as a tax-like fraction of sales revenue:

$$X_{wg} = t_{wg} * P_g * QS_g.$$

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<sup>49</sup> It is tempting to use the unit cost,  $C_g$ , instead of the unit price,  $P_g$ , in this equation on the assumption that within-sector transfers would be at cost. There are two reasons for not doing so: First, the assumption may not be valid even within a company, much less a sector consisting of most of the goods in the country. Second, in an ideally competitive environment, the two values will, in equilibrium—that is, after the computation has converged—be identical.

<sup>50</sup> If this assumption is invalid, a  $QD_{bg}^{f_{bg}}$  term must be added to the production function before proceeding further.

<sup>51</sup> Subsidies correspond to negative tax-like payments.

#### 4.5.2.4 Goods: Cost and Net Revenue

Since we have assumed that black market goods are not needed to produce other goods,<sup>52</sup> the total cost of production in the *goods* sector is

$$COST_g = X_{gg} + X_{pg} + X_{ag} + X_{rg} + X_{wg}$$

Using the expressions developed in the previous subsections, this may be written as

$$COST_g = P_g * (t_{ag} + t_{rg} + t_{wg}) * QS_g + P_g * QD_{gg} + P_p * QD_{pg}$$

Consequently, the net revenue is given by

$$NR_g = REV_g - COST_g = P_g * k_g * QS_g - P_g * QD_{gg} - P_p * QD_{pg}$$

#### 4.5.2.5 Goods: Expenditures—The Goods Producer's Optimization Problem

Assume the producers will choose  $QS_g$ ,  $QD_{gg}$ , and  $QD_{pg}$  to maximize their net revenue.<sup>53</sup> Now, the production function,  $QS_g = A_g * QD_{gg}^{f_{gg}} * QD_{pg}^{f_{pg}}$ , is assumed to describe the relationship between the amount of output and the amounts of inputs that are consistent with the production technology being used.

One way to solve this optimization problem is to use the technique of Lagrange multipliers.<sup>54</sup>

First, form the *Lagrangian*,  $L$ , a modified version of the optimization objective that incorporates the production function:

$$L = NR_g - \lambda * (QS_g - A_g * QD_{gg}^{f_{gg}} * QD_{pg}^{f_{pg}})$$

where  $\lambda$  is the *Lagrange multiplier*. More explicitly, the Lagrangian can be written as

$$L = P_g * k_g * QS_g - P_g * QD_{gg} - P_p * QD_{pg} - \lambda * (QS_g - A_g * QD_{gg}^{f_{gg}} * QD_{pg}^{f_{pg}})$$

Next, set the partial derivatives of  $L$  with respect to the decision variables and to  $\lambda$  to zero, then solve:<sup>55</sup>

Taking the derivative with respect to  $QS_g$ :

$$\frac{\partial L}{\partial QS_g} = P_g * k_g - \lambda = 0 \Rightarrow \lambda = P_g * k_g$$

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<sup>52</sup> This is probably a good assumption with the restricted set of black market goods we are considering here—those that are traded on the international market, such as drugs, arms, and humans. Domestic black market goods are combined with the rest of the goods in the *goods* sector.

<sup>53</sup> This is a standard assumption, and is often categorized as assuming the producer is a “rational decision maker”. A more accurate statement is that it merely simplifies the problem by assuming the producer has no other objectives, such as maintaining a stable workforce or maximizing market share or favoring a preferred supplier.

<sup>54</sup> For descriptions of the technique, see [Jensen 2010] or [Wikipedia Lagrange 2010].

<sup>55</sup> The sequence of symbols “= 0 ⇒” means “which, when set to zero, can be solved to yield”. Technically, one could insist on making sure the second derivatives are negative to ensure that the solutions are indeed minima, rather than maxima or inflection points. However, most production functions—such as *Cobb-Douglas*—are “well-behaved”, and the check is unnecessary.

Taking the derivative with respect to  $QD_{gg}$ :

$$\frac{\partial L}{\partial QD_{gg}} = -P_g + \lambda * \frac{f_{gg}}{QD_{gg}} * A_g * QD_{gg}^{f_{gg}} * QD_{pg}^{f_{pg}} = 0 \Rightarrow QD_{gg} = \lambda * \frac{f_{gg}}{P_g} * QS_g$$

Using the value for  $\lambda$  from the first of these two equations, the second becomes

$$QD_{gg} = \frac{P_g}{P_g} * k_g * f_{gg} * QS_g$$

Taking the derivative with respect to  $QD_{pg}$ :

$$\frac{\partial L}{\partial QD_{pg}} = -P_g + \lambda * \frac{f_{pg}}{QD_{pg}} * A_g * QD_{gg}^{f_{gg}} * QD_{pg}^{f_{pg}} = 0 \Rightarrow QD_{pg} = \lambda * \frac{f_{pg}}{P_g} * QS_g$$

Using the value for  $\lambda$ , this becomes

$$QD_{pg} = \frac{P_g}{P_p} * k_g * f_{pg} * QS_g$$

Taking the derivative with respect to  $\lambda$  recovers the production function:

$$\frac{\partial L}{\partial \lambda} = QS_g - A_g * QD_{gg}^{f_{gg}} * QD_{pg}^{f_{pg}} = 0 \Rightarrow QS_g = A_g * QD_{gg}^{f_{gg}} * QD_{pg}^{f_{pg}}$$

Using the optimal values of input and supplied quantities and  $\lambda$  in the production function (which was, of course, recovered when the partial derivative with respect to  $\lambda$  was set to zero) will yield an equation for the smallest value of the price of goods consistent with the parameters that describe the technology and the prices of those inputs that meets all expenses, which include enough of what non-economists call "profit" that producers will stay in business. That is, the process produces an estimate of the cost of production,  $C_g$ . Since we assume a competitive market, the net revenues must be zero, and the market price,  $P_g$ , can be no higher than that minimum, which is  $C_g$ .

Since we have assumed there are no economies of scale, the resultant cost (and price) does not depend on the supplied quantity. Thus, applying the process described above, we have

$$QS_g = A_g * \left( \frac{P_g}{P_g} * k_g * f_{gg} * QS_g \right)^{f_{gg}} * \left( \frac{P_g}{P_p} * k_g * f_{pg} * QS_g \right)^{f_{pg}}$$

Rearranging terms gives

$$QS_g = A_g * (P_g * k_g * QS_g)^{f_{gg} + f_{pg}} * \left( \frac{f_{gg}}{P_g} \right)^{f_{gg}} * \left( \frac{f_{pg}}{P_p} \right)^{f_{pg}}$$

The assumption that there are no economies of scale implies that  $f_{gg} + f_{pg} = 1$ , so dividing through by  $QS_g$  gives

$$1 = A_g * P_g * k_g * \left( \frac{f_{gg}}{P_g} \right)^{f_{gg}} * \left( \frac{f_{pg}}{P_p} \right)^{f_{pg}}$$

Solving this equation for the cost (and price) by dividing both sides by everything on the right-hand side except  $P_g$  gives

$$C_g = P_g = \frac{1}{A_g * k_g} * \left(\frac{P_g}{f_{gg}}\right)^{f_{gg}} * \left(\frac{P_p}{f_{pg}}\right)^{f_{pg}}$$

Solving for  $P_g$  and again using the assumption that  $f_{pg} = 1 - f_{gg}$ , yields<sup>56</sup>

$$P_g = \frac{P_p / f_{pg}}{(A_g k_g f_{gg}^{f_{gg}})^{1/f_{pg}}}$$

As promised in footnote 39 on page 19, this equation can also be used to calibrate the Cobb-Douglas production coefficient,  $A_g$ , by using base case prices:

$$A_g = \frac{1}{BP_g * k_g} * \left(\frac{BP_g}{f_{gg}}\right)^{f_{gg}} * \left(\frac{BP_p}{f_{pg}}\right)^{f_{pg}}$$

Using this, the equation for  $P_g$  may be rewritten in terms of base case prices as

$$P_g = BP_g \left(\frac{P_p}{BP_p}\right)^{\frac{f_{pg}}{1-f_{gg}}}$$

Once again using the assumption that  $f_{pg} = 1 - f_{gg}$ , we note that the ratio of the two prices is fixed at the ratio of the base case prices:

$$\frac{P_g}{BP_g} = \frac{P_p}{BP_p}$$

Since the average wage,  $P_p$ , is the *numeraire*, it is constant. Consequently, so is the price of a fixed basket of goods.

This rather surprising result applies only to the six sector model because it has only one commodity, *goods*, and one factor of production, *pop*. With more commodities and/or more factors, trades may be made among commodities and/or among the factors and only the aggregate commodity price will be proportional to the aggregate factor price. This topic is discussed further in Section 5.

The unconstrained quantity supplied is the sum of the demands:

$$QS_g = QD_g$$

where  $QD_g = \sum_j QD_{gj}$ .

The producer may not be able to produce that much if there is insufficient production capacity. If the optimal amount of production exceeds the capacity limit, then that limit itself produces the maximum profit.<sup>57</sup> If the above derivation is repeated with  $CAP_g$

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<sup>56</sup> If  $f_{gg}$  is zero, the middle term does not appear in the equation, so there is no division by zero. Instead, since  $f_{pg} = 1 - f_{gg}$ , we have  $f_{pg} = 1$ , so  $P_g = P_p / (A_g * k_g)$ . Implementation warning: The exponent approaching zero makes the term approach one faster than the divisor approaching zero makes it approach infinity. But the computer might not see it that way.

<sup>57</sup> Formal proof of this statement is beyond the scope of this paper.

replacing  $QS_g$ , the resultant formula for ingredient quantities is  $QD_{ig} = f_{ig} * (P_g/P_i) * k_g * CAP_g$ . Thus, in the “medium-run” case,

$$QS_g = \max[0, \min(QD_g, CAP_g)]$$

#### 4.5.2.6 Consumer and Labor Security Factors

If consumers are afraid to shop, that is modeled as a reduction in the number of consumers, which reduces the demand for goods. If some workers are afraid to go to work, that is modeled as a reduction in the available work force. Consequently, the “short-run” case has different inputs but the same formula for  $QS_g$  as the “medium-run” case.

#### 4.5.2.7 Goods: Calibration

The formula for  $X_{gg}$  can be written as  $X_{gg} = f_{gg} * k_g * REV_g$ . In the base case,  $X_g$  has the value  $BX_g$  and  $REV_g$  has the value  $BREV_g$ , so the equation can easily be solved for  $f_{gg}$ . But  $k_g$  depends on the tax-like rates  $t_{ag}$ ,  $t_{rg}$ , and  $t_{wg}$ , so they must be determined first.

The tax-like rates appear by themselves in the formulas for  $X_{ag}$ ,  $X_{rg}$ , and  $X_{wg}$ . Thus,

$$t_{ag} = BX_{ag}/BREV_g$$

$$t_{rg} = BX_{rg}/BREV_g$$

$$t_{wg} = BX_{wg}/BREV_g$$

Now, since,  $k_g = 1 - (t_{ag} + t_{rg} + t_{wg})$ ,

$$f_{gg} = BX_{gg}/(k_g * BREV_g)$$

Since we assume that the *goods* sector buys nothing from the *black* sector,  $X_{bg} \equiv 0$ , and  $f_{bg} = \text{blank}$ . (That is, we don't use this parameter.)

From the equation for  $X_{pg}$ , we have

$$f_{pg} = BX_{pg}/(k_g * BREV_g)$$

### 4.5.3 Example: Producers with Exogenous Prices (e.g., International Markets)

Black market producers are also assumed to make production decisions to maximize their profits. In this case, we assume a Leontief formula for the production function.<sup>58</sup> All of their revenue comes from sales.

As long as we are using a low-resolution CGE—that is, one with a relatively small number of sectors, the goods (and we use the term loosely) aggregated in the black market sector should be limited to those for which the price is exogenously determined in the world market, not within the region. This includes such things as illegal drugs, illegal arms, and human trafficking. One result of this limitation is that price will usually greatly exceed the per-unit cost of production, leading to a significant amount of net revenue. We assume

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<sup>58</sup> In a Leontief production function, input requirements are assumed to be strictly proportional to the output quantity. That is,  $QD_{ij} = A_{ij} * QS_j$  for every ingredient  $i$ . The associated production function can be written as  $QS_j = \min_i(QD_{ij}/A_{ij})$ .

that the recipients of that net revenue are among the actors in the *actors* sector. If not, those funds go to the *world* sector.

If the CGE becomes sufficiently sophisticated to distinguish between black-market and white-market consumer goods, then it will be necessary to consider how purchasing preferences shift as the region stabilizes. As long as that distinction is not made, preference shifts between black-market consumer goods and white-market consumer goods will take place entirely *within* individual cells in the CGE and will have no effect on the Cobb-Douglas coefficients.

Opium production depends heavily on poppy feedstocks that may be grown in other countries,<sup>59</sup> so our model allows the analyst to consider the effects of interdicting the flow of feedstocks. This approach can, of course, be used as an archetype for other sectors that depend heavily on an imported feedstock. If there is no need to consider interdicting the feedstock for a particular scenario, the feedstock can be treated as an imported good and the parameter  $AF_{wb}$  can be set to zero.

#### 4.5.3.1 *Black: Revenues*

The only source of revenue for the *black* sector is sales of its product:

$$REV_b = P_b * QS_b.$$

#### 4.5.3.2 *Black: Production Requirements*

As we did with the *goods* sector, let's consider the *black* sector's payments to each sector.

*Goods*—Assume no appreciable amount of the other modeled products (i.e., labor) can be substituted for the materials used while processing feedstock into finished black market goods. Then, the quantity of *goods* needed by the *black* sector is given by

$$QD_{gb} = A_{gb} * QS_b$$

where  $A_{gb}$  is a model parameter that describes the processing technology and can, at least in principle, be calibrated from SAM (Social Accounting Matrix) data or derived from an understanding of the processes used. The payment from the *black* sector to the *goods* sector is given by

$$X_{gb} = P_g * QD_{gb} = P_g * A_{gb} * QS_b.$$

*Black*—Assume the self-consumption of product within the black market,  $QD_{bb}$ , whether it is used in the production process itself or is used by the producers themselves, is a small fraction,  $A_{bb}$ , of the total amount produced and that that small amount cannot be replaced by any other input:

$$QD_{bb} = A_{bb} * QS_b$$

The value of the product exchanged within the sector is given by

$$X_{bb} = P_b * QD_{bb} = P_b * A_{bb} * QS_b$$

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<sup>59</sup> See [Wikipedia Opium 2011].

*Pop*—Assume that *goods* cannot be substituted for labor in converting the black market feedstock to black market goods. Thus, *black's* demand for labor is

$$QD_{pb} = A_{pb} * QS_b$$

and the total payment is

$$X_{pb} = P_p * QD_{pb} = P_p * A_{pb} * QS_b.$$

where  $P_p$  is the average wage.

*World*—The *Rest of the World* sector does not supply production capacity, but it does collect taxes and tax-like payments at a rate assumed to be a fraction of the sales,  $t_{wb}$ . This fraction includes any imports other than a main feedstock. The *world* sector may also supply an imported feedstock, at an exogenous price of  $PF_{wb}$ . Feedstock is a natural for a Leontief relationship, because inputs cannot be substituted for one another and input quantity requirements are proportional to the output quantity, but we must also ensure that the demand does not exceed the maximum supply available ( $MF_{wb}$ ), so the amount of feedstock used is given by<sup>60</sup>

$$QF_{wb} = \max(0, \min(A_{Fb} * QS_b, MF_{wb})).$$

The cost of this feedstock is

$$CF_{wb} = PF_{wb} * AF_{wb} * QS_b$$

and the total payment from *black* to *world* is given by

$$X_{wb} = t_{wb} * P_b * QS_b + PF_{wb} * QF_{wb} = t_{wb} * P_b * QS_b + CF_{wb}.$$

Feedstock could be obtained from more than one other sector (for example, both from the *region* and from the *world*). If that were the case, the producer would also have to choose how much to purchase from each supplier and there would be additional terms in the above equation.<sup>61</sup>

#### 4.5.3.3 *Black: Other Payments*

Section 4.5.3.1 described *black's* production requirements in terms of the demands it places on other sector's products and the payments it makes to those sectors. We have already seen that some of those payments are not driven by the production technology. This section discusses the remaining non-production-related payments to sectors.

*Actors*—The *Regional Actors* sector does not produce a product used by the black market. Instead, it receives the profits. Thus, the payment from *b* to *a* is

$$X_{ab} = t_{ab} * P_b * QS_b + NR_b$$

---

<sup>60</sup> The use of the function  $\max(0, \sim)$  here and elsewhere are to ensure that spurious negative quantities do not occur during iteration

<sup>61</sup> The economically optimal solution to the subproblem is trivial: He should buy as much as he needs from the cheapest supplier and none from the more expensive ones—unless the cheapest supplier cannot supply enough. In that case, he should buy the rest from the next cheapest, and so on until a limit has been reached. This complication has been avoided in this version by assuming that essentially all of the feedstock comes from only one sector, the *world*.

where the net revenue,  $NR_b$ , is derived in the next subsection.

*Region*—The *Rest of the Region* does not produce a product used by the black market, but it does supply production capacity. Assume the cost of using that production capacity is a tax-like multiple of the portion that is used (that is,  $QS_b$ ), as contrasted with the amount that is made available ( $CAP_b$ ). Then, the payment from  $b$  to  $r$  is

$$X_{rb} = t_{rb} * P_b * QS_b.$$

#### 4.5.3.4 *Black: Cost and Net Revenues*

The total cost of production in the black market is the sum of the production requirements and the other payments, excluding the net revenues, which are paid to the actors:

$$COST_b = (X_{bb} + X_{gb} + X_{pb} + X_{wb}) + ([X_{ab} - NR_b] + X_{rb}).$$

Using the expressions for the  $X_{ib}$  from the previous subsections, the total cost of production may be rewritten as<sup>62</sup>

$$COST_b = P_b A_{bb} QS_b + P_g A_{gb} QS_b + P_p A_{pb} QS_b + (t_{wb} + t_{rb}) P_b QS_b + P F_{wb} A_{Fb} QS_b$$

This expression may also be written as the product of the per-unit cost and the quantity supplied:

$$COST_b = C_b * QS_b,$$

so the per-unit cost is

$$C_b = P_b * (A_{bb} + t_{rb} + t_{wb}) + P_g * A_{gb} + P_p * A_{pb} + P F_{wb} * A_{Fb}$$

The world market determines the price, which is likely to be significantly higher than the cost. Thus, the net revenue may be significantly greater than zero. The net revenue is computed by

$$NR_b = REV_b - COST_b = (P_b - C_b) * QS_b$$

As noted above, this profit is collected and used by the political actors as determined by the Politics model.

#### 4.5.3.5 *Black: Expenditures—The Black Marketer's Optimization Problem*

Assume the black marketer is comfortable enough with his market position and his relationship to the authorities that he is able to concentrate on making choices that maximize his net revenue, rather than be distracted by the vicissitudes of operating outside the law. Bribes, on the other hand, may be considered as normal costs of doing business. Bribes and the costs of imports other than feedstocks are included in the tax-like parameters  $t_{rb}$  and  $t_{wb}$ .

Then, taking the prices as given, whether they are exogenous or determined by the market,<sup>63</sup> the black marketer's optimization problem is to find the values of  $QD_{bb}$ ,  $QD_{gb}$ ,

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<sup>62</sup> This equation is so long that we eschewed the mathematical asterisk to keep it on one line.

<sup>63</sup> If the market being modeled does not determine prices, they must be exogenous. That is, the black marketer does not get to choose the price as part of his optimization problem.

$QD_{pb}$ ,  $QF_{wb}$ , and  $QS_b$  that maximize his net revenue, subject to the constraint that the amount produced is not negative:<sup>64</sup>  $QS_b \geq 0$ .

Since neither the price nor the per-unit cost are affected by the producer's decision variables, and assuming the price exceeds the per-unit cost, net revenue is maximized by maximizing the supplied quantity and by wasting no inputs. Sales are limited by the demand, so the optimum supplied quantity in the long-run case is given by

$$QS_b = QD_b.$$

In the medium and short run, the producer may not be able to produce that much if there is insufficient production capacity or if the feedstock is in short supply. Thus, in a constrained environment,

$$QF_{wb} = \max(0, \min(A_{Fb} * QS_b, MF_{wb}))$$

and

$$QS_b = \max(0, \min(QD_b, CAP_b)).$$

The requirement that no inputs are wasted produces the following:

$$QD_{bb} = A_{bb} * QS_b,$$

$$QD_{gb} = A_{gb} * QS_b$$

$$QD_{pb} = A_{pb} * QS_b$$

$$QF_{wb} = A_{Fb} * QS_b$$

#### 4.5.3.6 *Black: Calibration*

Since we assume the *black* sector has a Leontief production function and gives its net revenue to the *actors* sector, its CGE cell formulas are different than those for the *goods* sector. The calibration procedure, however, is pretty much the same: express the expenditure equations in terms of base case data, then solve for the unknown shape parameters by inspection, solving the easiest ones first.<sup>65</sup>

From the equation for  $X_{gb}$ , and since supplied quantities equal demanded quantities in equilibrium, we have  $BQS_b = BQD_b$ , so

$$A_{gb} = BX_{gb} / (BP_g * BQD_b)$$

From the equation for  $X_{bb}$ , we have

$$A_{bb} = BX_{bb} / (BP_b * BQD_b)$$

---

<sup>64</sup> Since both price and cost are exogenously determined, we need to allow for the case in which the cost exceeds the price. In that case, an algorithm that tries to maximize net revenues would suggest producing a negative quantity if it did not have to satisfy this constraint. If cost exceeded price in reality, the producer would simply go out of business.

<sup>65</sup> Procedural suggestion: Write the equations for  $X_{ib}$  from Section 4.5.3 on a white board in a single color, then replace the variables that are known from the base case by their base case equivalents, using a different color. The unknown variables will stand out in the original color, making it easier to decide on an effective solution order.

From the equation for  $X_{pb}$ , we have

$$A_{pb} = BX_{pb} / (BP_p * BQD_b)$$

Since the base case SAM differs from the CGE in that the *black* net revenues are *not* included in  $BX_{ab}$ , even though they are included in  $X_{rb}$ , we have

$$t_{ab} = BX_{ab} / (BP_b * BQD_b)$$

From the equation for  $X_{rb}$ , we have

$$t_{rb} = BX_{rb} / (BP_b * BQD_b)$$

From the equation for  $X_{wb}$ , which includes paying for feedstocks, we have

$$t_{wb} = (BX_{wb} - PF_{wb} * AF_{wb} * BQD_b) / (BP_b * BQD_b)$$

#### 4.5.4 Example: The *Populace* Sector, the Factor of Production and the Consumers

In many CGEs, this sector is called the “Households” sector. We have avoided that term on the grounds that it seems vaguely “too western”. Regardless, this sector supplies the labor factor of production.<sup>66</sup> Via Athena’s demographics model, this sector also supplies the consumers, who, together with per capita consumption requirements (and exports), determine the size of the economy:

$$QD_{gp} = A_{gp} * consumers.$$

The demographics model also supplies the number of available workers,  $CAP_p$ .

In the regions in which we are interested, a substantial fraction of the populace engages in subsistence agriculture. While their participation in the cash economy is negligible, their contribution to the GDP (Gross Domestic Product) is included in Athena by valuing their work as if they were paid at the wage corresponding to the “poverty line”, which is a region-dependent exogenous value, usually defined by the governments in the region [Lawler 2010].<sup>67</sup>

##### 4.5.4.1 *Pop: Revenues*

The populace has another source of income besides payment for labor: remittances.<sup>68</sup> When added to the value of exported labor,<sup>69</sup> these constitute the content of  $X_{pw}$ , which is the payment from the *world* sector to the *pop* sector. The total revenue is given by

$$REV_p = P_p * QS_p + REM$$

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<sup>66</sup> Labor is the only factor of production explicitly considered in this version of Athena, though capital infrastructure, including land, is represented by production capacity constraints and is assumed to be managed within Athena, but in the ground model, outside the CGE. Athena’s political actors can use some of their financial assets to build infrastructure if they choose to do so; such investments take time to come to fruition but can be expected to have immediate effects on civilian satisfaction levels and on political influence. See Sections 5.3 and 5.5 for further discussion.

<sup>67</sup> If it is desired to omit imputed values when computing the GDP, this wage can be set to zero.

<sup>68</sup> *Remittances* are the funds sent home by people who have left the region “to find their fortunes”.

<sup>69</sup> The quantity of exported labor is included in the total amount of labor supplied,  $QS_p$ .

#### 4.5.4.2 Pop: The Size of the Labor Force and the Number of Jobs

The demographics model determines the size of the labor force,  $LF$ , based on the population, wage rates, subsistence agriculture, insurgency recruitment incentives, force group activities, etc. The market, not the populace, determines the number of jobs. Hence, even in the long-run case, the number of people employed (never less than zero) is

$$QS_p = \max\left(0, \min(LF, QD_p)\right).$$

#### 4.5.4.3 Pop: Taxes, Tax-Like Payments, and After-Tax Income

Assume that workers pay taxes and tax-like payments to the political actors,  $a$ , the rest of the region,  $r$ , and the rest of the world,  $w$ , as fractions  $t_{ap}$ ,  $t_{rp}$ , and  $t_{wp}$  of wages, but that they get to keep all of the  $REM$  they receive.<sup>70</sup> Then, the populace's after-tax income is given by

$$ATI_p = REV_p - (t_{ap} + t_{rp} + t_{wp}) * P_p * QS_p.$$

Using  $k_p = 1 - (t_{ap} + t_{rp} + t_{wp})$  to simplify the notation, this may be written as

$$ATI_p = k_p * P_p * QS_p + REM.$$

#### 4.5.4.4 Pop: Consumption Choices and the Utility Function

Assume that the amount of satisfaction consumers get from their purchases can be described by a so-called *utility function* that depends on the amounts purchased. Assume further that they make their choices to maximize the value of that function<sup>71</sup> while spending all of their after-tax income.<sup>72</sup>

If the utility function for consumers is assumed to be a Cobb-Douglas function:

$$U = QD_{bp}^{f_{bp}} * QD_{gp}^{f_{gp}} * QD_{pp}^{f_{pp}},$$

the populace's optimization problem is to find the values of  $QD_{bp}$ ,  $QD_{gp}$ , and  $QD_{pp}$  that maximize  $U$ , subject to the condition that total expenditures equal the after-tax-income:

$$\sum_{i=b,g,p} P_i * QD_{ip} = ATI_p.$$

If the CGE is to distinguish between black-market and white-market consumer goods, it will be necessary to consider how the Cobb-Douglas parameters for those sectors change as the region becomes more stable. Since the six-sector example does not make that distinction, those preference shifts are irrelevant, as they take place entirely within a single cell (purchases of *goods* by the *populace*).

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<sup>70</sup> Alternatively, if  $REM$  is subject to tax-like payments, it can be added to  $REV_p$  instead of to  $ATI_p$ .

<sup>71</sup> This assumption is fine in theory, but in practice, one should realize that it can only be a rough approximation to even the economic portion of the quality of life portion of satisfaction. Nevertheless, we will follow the common practice of assuming that the utility function guides consumer choices of expenditures. The attitudes, demographics, and politics models deal with many of the other factors that motivate people.

<sup>72</sup> Consumer savings is handled in a later version of the model in much the same way as actors' savings.

#### 4.5.4.5 Pop: Expenditures—The Consumer's Optimization Problem

We can also solve this constrained optimization problem by the method of Lagrange multipliers. First, introduce a Lagrange multiplier  $\lambda$  for the constraint equation and define the Lagrangian function:

$$L = QD_{bp}^{f_{bp}} * QD_{gp}^{f_{gp}} * QD_{pp}^{f_{pp}} - \lambda * (\sum_{i=b,g,p} P_i * QD_{ip} - ATI_p)$$

Taking partial derivatives with respect to the decision variables,  $QD_{ip}$ , and setting them to zero gives

$$\frac{\partial L}{\partial QD_{ip}} = \frac{f_{ip}}{QD_{ip}} * U - \lambda * P_i = 0 \text{ for } i = b, g, p$$

which implies

$$P_i * QD_{ip} = \frac{U}{\lambda} * f_{ip}$$

Summing both sides of this equation over all values of the index  $i$  and noting that the sum of the  $f_{ip}$  is 1, we have

$$\sum_i P_i * QD_{ip} = \frac{U}{\lambda} * \sum_i f_{ip} = \frac{U}{\lambda}$$

By the after-tax-income constraint, which is recovered when the derivative with respect to  $\lambda$  is set to zero, the left-hand side equals  $ATI_p$ , so  $\frac{U}{\lambda} = ATI_p$ . Therefore,

$$P_i * QD_{ip} = f_{ip} * ATI_p \text{ for } i = b, g, p$$

Finally, for  $i = b, g, p$ , we have expressions for the optimal consumption quantities and the associated expenses:

$$QD_{ip} = f_{ip} * ATI_p / P_i$$

and

$$X_{ip} = P_i * QD_{ip} = f_{ip} * ATI_p$$

Furthermore,

$$X_{ap} = t_{ap} * P_p * QS_p$$

$$X_{rp} = t_{rp} * P_p * QS_p$$

$$X_{wp} = t_{wp} * P_p * QS_p$$

#### 4.5.4.6 Pop: Calibration

From the definition of the after-tax income and the above equations for  $X_{ip}$ , we have

$$t_{ip} = BX_{ip} / (BREV_p - REM) \text{ for } i = a, r, w$$

$$k_p = 1 - (t_{ap} + t_{rp} + t_{wp})$$

$$BATI_p = \sum_i k_p * BP_i * BQD_{ip} + REM$$

$$f_{ip} = BX_{ip} / BATI_p \text{ for } i = b, g, p$$

#### 4.5.5 Example: A Sector with No Product—Regional Political Actors

The behavior of the regional political actors is exogenously specified via budget percentages. Expenditures are computed in the politics model, where the actors are trying to accomplish their goals within the resources they have available. Those resources include accumulated savings in addition to income, but also include manpower and communications assets. Other resources may be included in later versions of Athena—but the politics model, not the economics model, will manage their use.

When the revenues and expenditures of a sector are both endogenous to the CGE, they must be equal or the economy will not be in balance. When the use of some revenues is modeled exogenously to the CGE, the economy does not have to be in balance. Such is the case for both the *actors* and the *world* sectors.

Whether and when the actors choose to save their money and whether, when, and how they choose to spend it is modeled in the politics model. If some of the actors are accumulating personal wealth and spending it outside the country, those expenditures should show up as payments to the *world* sector; if they are spending it on personal consumption, those expenditures should show up as payments to the *goods* sector; but if they are “stuffing it in their mattresses”, the economy will simply be out of balance. The expectation is that they will merely spend some of their revenues at different times than they receive them. This can lead to variations in employment.

When imports and exports are not in balance, or if more foreign aid is coming into the country than is going out (or vice versa), the economy will be growing or shrinking. This may show up as changes in the regional debt (which is not tracked in this version of Athena).

##### 4.5.5.1 Actors: Revenues

Revenues are obtained by summing payments and by collecting a portion of the foreign aid to the region:

$$REV_a = \sum_j X_{aj} + graft * FAR$$

As explained in Section 4.5.3.3,  $X_{ab}$  includes the profits made in the black market sector. Also,  $X_{rw}$  is  $FAR$ , the foreign aid that goes directly to the actors.

##### 4.5.5.2 Actors: Expenditures

Total expenditures are obtained by summing the amounts the actors spend on each sector:

$$EXP_a = \sum_i X_{ia}$$

where each of the expenditures is given by

$$X_{ia} = f_{ia} * REV_a$$

The budget fractions  $f_{ia}$ , are obtained from the politics model and do not necessarily sum to one, as discussed earlier in this subsection.

#### 4.5.5.3 Actors: Calibration

From the above equation for  $X_{ia}$ , we have

$$f_{ia} = BX_{ia}/BREV_a \text{ for all } i$$

#### 4.5.6 The Rest of the Region

##### 4.5.6.1 Region: Revenues

This sector supplies production capacity to the *black* and *goods* sectors and collects tax-like revenues in return for the use of this capital.

If the region of interest is less than the entire country, this sector can also account for all of the interactions between the region, the other sectors, and the rest of the world.

It also gets to keep the part of the foreign aid that is not extracted by the actors:

$$REV_r = \sum_j X_{rj} + (1 - graft) * FAR$$

##### 4.5.6.2 Region: Expenditures

The *region's* revenues are distributed by an exogenous budget vector described by the fractions  $f_{br}$ ,  $f_{gr}$ , and  $f_{pr}$ . Assume these expenditures are used to purchase the outputs of those sectors: drugs, goods, and labor. Then:

$$X_{br} = f_{br} * REV_r, QD_{br} = X_{br}/P_b$$

$$X_{gr} = f_{gr} * REV_r, QD_{gr} = X_{gr}/P_g$$

$$X_{pr} = f_{pr} * REV_r, QD_{pr} = X_{pr}/P_p$$

To complete the budget, some portion of the revenues may be given to some of the actors, spent on other things within the region, and some may go like taxes to the *world* sector for imported goods:

$$X_{ar} = f_{ar} * REV_r$$

$$X_{rr} = f_{rr} * REV_r$$

$$X_{wr} = f_{wr} * REV_r$$

If the budget must be balanced,  $\sum_i f_{ir} = 1$ . If it is not balanced, the regional debt (or surplus) will change. This change must be accounted for elsewhere in the economics model because the CGE addresses cash flows only.

Total expenditures are then given by:

$$EXP_r = \sum_i X_{ir}.$$

##### 4.5.6.3 Region: Calibration

From the above equations for  $X_{ia}$ , we have

$$f_{ir} = BX_{ir}/BREV_r \text{ for all } i$$

### 4.5.7 The Rest of the World

Revenues for the *Rest of the World* sector from within the region do not necessarily balance its expenditures in the region. Its net revenue,  $NR_w = REV_w - EXP_w$ , is known as the *balance of trade* and is not necessarily zero.

#### 4.5.7.1 World: Revenues

This sector has no product, so there is no price or quantity to compute. It does, however, supply imported feedstocks for the black market, the accounting for which was described in Section 4.5.3.1, and it includes tax-like payments to account for imports of foreign goods.

$$REV_w = \sum_j X_{wj}$$

#### 4.5.7.2 World: Expenditures

Total expenditures are given by:

$$EXP_w = \sum_i X_{iw}$$

The components of the *world's* expenditures in the region are the exogenously specified annual \$ amounts of exports,  $X_{iw}$  for  $i = b, g$ , and  $p$ , and include various kinds of foreign aid. For simplicity, local prices are assumed to prevail in the export markets, except for black market goods, in which the reverse is true.

Export quantities are obtained by dividing the dollar values by the prices:

$$QD_{iw} = X_{iw}/P_i \text{ for } i = b, g$$

To compute  $QD_{pw}$ , which is the amount of local labor "exported to" the *world* sector for embassy employees, local contractors, and so on,  $REM$  must be subtracted from  $X_{pw}$  before the division:

$$QD_{pw} = (X_{pw} - REM)/P_p$$

In addition,

$$\begin{aligned} X_{aw} &= FAA + FAR * graft \\ X_{rw} &= FAR * (1 - graft) \end{aligned}$$

where  $FAA$  is the foreign aid that goes directly to the actors,  $FAR$  is that part of foreign aid that goes directly to the region, and  $graft$  is the fraction of  $FAR$  that winds up in the pockets of the actors.

Furthermore,

$$X_{ww} = \text{blank}.$$

That is, while cash flows in the world that do not involve any of the sectors in the region are not zero, they are outside the scope of the model and are treated as if they were zero.

Aid is often diverted from the donor's intentions to other uses; this issue is examined in detail in [Doroudi 2010].

The attitudes model will consider how well the aid correlates with need, as reflected by environmental situations.

### 4.5.7.3 World: Calibration

Since SAM data does not perfectly match CGE data, we need to extract  $FAA$  and  $FAR$ :

$$FAA = BX_{aw}$$

$$FAR = BX_{rw}$$

There are no additional model parameters to calibrate from the base case.

## 4.6 Derive Equations for Outputs of Interest

### 4.6.1 Shortages, Overages, and Idle Capacity

The following statements apply to all sectors that either produce a product or provide a factor of production:

- Shortages and overages are two sides of the same issue. If less is produced in the medium-run case than is demanded in the long-run case, there is latent demand. If less is produced in the short-run case than is demanded in the long-run case, there is a shortage.
- On the other hand, if more is produced in the short-run case than is demanded in the long-run case, there is an overage.
- If capacity exceeds production, there is idle capacity.

### 4.6.2 Unemployment



**Figure 5. Unemployment.** The Labor Force provided by the demographics model is the number of people who want to work in the current economic climate. It overstates the number of people actually available to work because some fraction of the labor force is always between jobs,<sup>73</sup> even in the best or worst of times. Idle capacity in the populace sector corresponds to the number of people who want jobs but cannot get them, which can be thought of as the *Real Unemployment* level.<sup>74</sup> To obtain *Unemployment*, the *Turbulence* must be added back in.

<sup>73</sup> Economists call this “frictional unemployment”.

<sup>74</sup> Economists call this “cyclical unemployment”, but that term suggests more stability than we expect to see.

### 4.6.3 Consumer Price Index

The consumer price index, *CPI*, is defined as the ratio of the cost of a specified “market basket” of consumer goods at present prices to the cost of that same market-basket-full of goods in a base case. Thus,

$$CPI = \frac{SUM}{C} = \frac{P_b * BQD_{bp} + P_g * BQD_{gp} + P_p * BQD_{pp}}{BP_b * BQD_{bp} + BP_g * BQD_{gp} + BP_p * BQD_{pp}}$$

where

$$SUM = \sum_{i=b,g,p} P_i * BQD_{ip}$$

depends on current prices and base case quantities, while

$$C = \sum_{i=b,g,p} BP_i * BQD_{ip}$$

depends only on the base prices and quantities.

It is shown in Section 5 that prices do not change from their values in the base case in the six-sector model, so the CPI computation will not be useful until the CGE has been expanded to contain multiple *goods* and/or *pop* sectors.

### 4.6.4 Gross Domestic Product

The gross domestic product, *GDP*, is defined as the market value of all final goods and services.

Subsistence agriculture, *SA*, can make a significant contribution to the *GDP*, but our concentration and emphasis is on the market economy, for that is where an economy can be seen to grow or shrink.

Assuming that people engaged in subsistence agriculture are managing to survive, it is reasonable to value their work at the poverty level. Computation of this value requires knowing the number of people engaged in this activity, which is tracked in the demographics model. It also requires the wage that is associated with the poverty level. Consequently, the demographics model supplies the value of *SA*.

The *GDP* is the value of final goods and services. As illustrated in Figure 6, it can be computed as the sum of the expenditures for regional goods and services by consumers, actors, the rest of the region, and the rest of the world (exports), plus the value of subsistence agriculture, less the value of imported feedstocks that are used in manufacturing those final goods, less the value of other imports.

Sectors	<i>goods</i>	<i>black</i>	<i>pop</i>	<i>actors</i>	<i>region</i>	<i>world</i>
<i>goods</i>			+	+	+	+
<i>black</i>			+	+	+	+
<i>pop</i>			+	+	+	+
<i>actors</i>						
<i>region</i>						
<i>world</i>	-	-	-	-	-	

+ Subsistence Agriculture

**Figure 6. Gross Domestic Product.** The GDP is the market value of all final goods and services produced in the region. Exports (but not *REM*, *FAA*, or *FAR*) add to the GDP and imports, including black market feedstocks, are subtracted. The value of subsistence agriculture is included.

When production is constrained, the supply of goods and services may be less than the demand. Assuming production is allocated first to the producing sectors, *goods* and *black*, with anything left over contributing to the GDP, we have

$$\begin{aligned}
 QGDP_g &= \max[0, (1 - k_g * f_{gg}) * QS_g - A_{gb} * QS_b] \\
 QGDP_b &= \max[0, (1 - A_{bb}) * QS_b] \\
 QGDP_p &= \max\left[0, QS_p - \frac{P_g}{P_p} * k_g * f_{pg} * QS_g - A_{pb} * QS_b\right]
 \end{aligned}$$

This assumption is not totally satisfactory, as the real-world response to shortages is quite complicated.<sup>75</sup>

Since the only feedstocks in the current model are those used in the black market, the *GDP* can be computed by

$$GDP = \sum_{i=g,b,p} P_i * QGDP_i + SA - PF_{wb} * QF_{wb} - \sum_j X_{wj}$$

<sup>75</sup> The first responses to shortages are to increase the use of alternative goods, services, and factors. But such substitutions generally cost more and are also subject to shortages. Eventually, there are no more substitutions that can be made and the shortage has become more than a mere inconvenience. Aggregating all goods into a single *goods* sector glosses over those details and reveals the beyond-substitutability-shortage situation that must be dealt with in the Athena model proper as well as here. Shortages of water, food, and electrical power are already modeled in Athena, but they are not yet triggered by the economics model.

#### 4.6.5 Deflated GDP and Deflated GDP Per Capita

The *GDP* is computed in terms of so-called “nominal” prices, which are subject to changes in the economy that lead to changes in the value of the dollar. That is to say, they are subject to inflation. To discern actual changes in the amounts of final goods and services that are produced, inflation must be removed. The appropriate factor for doing so accounts for differential inflation among all of the goods and services that make up the *GDP*. Computation of that factor is beyond the scope of this model, but the *CPI* provides a convenient, pretty good approximation. The deflated *GDP* is given by

$$DGDP = GDP/CPI.$$

Expressed per person, the *DGDP per capita* is given by

$$DGDP \text{ per capita} = DGDP/Population.$$

Although inflation is not addressed in Athena 4, so  $DGDP = GDP$ , the per capita value of the GDP might as well be stated in deflated terms.

## 5 Limitations of the Six-Sector Model

As was pointed out on page 27, all prices in the six-sector CGE, with a single commodity, *goods*, and a single factor of production, *pop* (that is, labor), are constant. This rather surprising<sup>76</sup> result comes about as the result of several assumptions:

- Production of the product of the *goods* sector has constant returns to scale
- Production of the product of the *goods* sector requires only other *goods* and *pop*
- The price of the product of the *black* sector is exogenously controlled by the international, not the local, market
- To establish the value of the \$, some price—or combination of prices—must be chosen to be the *numeraire*.

Since none of the prices change during a simulation, the *GDP per capita* is a more meaningful figure of merit than the average wage.

### 5.1 About Reducing Demand by Raising Prices

Strictly speaking, the six-sector CGE is not a one-commodity, one-factor model, as remittances increase the *pop* sector's income above what they make as wages and the *region*, *actors*, and *world* sectors, as well as the *goods* sector, provide jobs.

Consequently, it may be theoretically possible, when production capacity is insufficient to meet demand or there is a shortage of labor, to raise prices<sup>77</sup> enough to reduce demands to the capacity limits.

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<sup>76</sup> In retrospect, it should not have been surprising. Consider a so-called “Robinson Crusoe economy” [Wikipedia Robinson Crusoe 2012] in which Crusoe produces a single commodity, coconuts, and provides a single factor of production, labor. His production function dictates the number of coconuts he gets per day. Let's call that production level  $C$  coconuts/day. That ratio is, in essence, his wage:  $C$  coconuts/day. If he decides to introduce money into his economy, he can decide on a wage rate, say  $S$  \$/day, but then the price of coconuts must be  $S$  \$/day divided by  $C$  coconuts/day, giving a price of  $(S/C)$  \$/coconut. If he decides his time is worth 50 \$/day and he wants to eat 4 coconuts/day, then the price of coconuts must be  $50/4 = \$12.50/\text{coconut}$ . If he decides his time is worth 100 \$/day, then coconuts cost  $\$25/\text{coconut}$ .

Thus, the value of the \$ — that is, the *numeraire* — is the price he chose for his wage:  $S$ . With more commodities or more factors of production, a *numeraire* is still needed and relative prices of commodities and of factors are not fixed, but the total value of all of the commodities produced will still equal the total value of all of the factors used.

<sup>77</sup> Or one might say “*the price*”, since the ratio of the price of goods to the price of labor is fixed. Production capacity limits might cause the price of goods to increase to drive down the demand for goods; wages would follow in lockstep. Labor shortages might drive the average wage up to reduce the demand for labor; the price of goods would follow in lockstep. If there were shortages in both production capacity and labor supply, the more severe constraint would be binding; the other would follow.

Leaving the algebra “as an exercise for the student”, setting  $QD_g = CAP_g$  and  $QD_p = CAP_p$  and solving for  $P_g$  and  $P_p$ , respectively, gives

$$P_g = \frac{f_{ga} * REV_a + f_{gr} * REV_r}{CAP_g - (QD_{gg} + QD_{gb} + QD_{gp} + QD_{gw})}$$

$$P_p = \frac{f_{pp} * REM + f_{pa} * REV_a + f_{pr} * REV_r}{CAP_p - (QD_{pg} + QD_{pb} + k_p * f_{pp} * QS_p + EXPORTS_p)}$$

In both of these equations, if the denominator is zero, so is the numerator, and there is no solution. In this case, however, there would be no discrepancy between capacity and demand, so these equations would be moot.

These results are very peculiar, in that all the burden for reducing demand to supply by increasing prices would fall on the actors, the rest of the region, and the populace's use of remittances rather than on the interplay between the *goods* and *pop* sectors. It is seldom a right answer to have a tail wag the dog.

In the real world, the price changes that would take place to make demands equal to the capacities would take place within the *goods* and *pop* sectors; higher resolution—more sectors—will be required to capture this phenomenon. Shortages in one commodity or factor will be mitigated by substitution of other commodities or factors, but their capacities are also limited, so inescapable shortages can still occur. Athena assumes that capacity estimates take the mitigating effect of implicit substitutions into account.<sup>78</sup>

Thus, the six-sector model will detect shortages, but the price escalations that would occur in the real world are not modeled in Athena 4—nor are the other social consequences of shortages, which will have to await later versions.

## 5.2 About Increasing the Number of Goods Sectors

More *goods* sectors would allow for substitution between goods, but it would also make it easier to compare the effects of alternative investment strategies. From the point of view of modeling, splitting out more *goods* sectors would be relatively easy, but the amount of data needed goes up roughly as the square of the number of sectors.<sup>79</sup>

Models for additional *goods* sectors can be patterned after Section 4.5.2.

If there is more than one commodity in the model, shortages will tend to drive prices up even while substitution is taking place. Governments (in the real world, but not in Athena 4) can decide to set price controls. When they do, domestic black markets tend to spring

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<sup>78</sup> If, for example, the demand for copper frying pans exceeds the supply, people can buy stainless steel or aluminum frying pans. But the production capacity for these may also be insufficient to meet the demand, so the consequences may be mitigated, rather than eliminated. Furthermore, a model should take into account that substitutes usually are not quite as good in some respect—more expensive, shorter lifetime, etc.

<sup>79</sup> In reality, it's not quite that bad, as the commodity vs. commodity block of the CGE tableau typically contains many zeroes. Consequently, the growth in data requirements is probably closer to  $5 * N$  than  $N^2$ . As the number of sectors increases, the multiplier 5 probably increases, too, but less rapidly than linearly with  $N$ .

into existence, offering the scarce commodity at a higher price. This phenomenon could be modeled by splitting sectors into white and black market segments.

Templates for such interdependent sectors are *not* included in Section 4.5. Domestic black markets derived from shortages must be modeled very differently than the international black markets with exogenous prices that are described in that section.

### **5.3 About Increasing the Number of Black Market Sectors**

There are at least three different major international black markets: illegal drugs, illegal arms, and humans. When more than one of these is operating in a region of interest, it is awkward to describe their combination as a “black basket”.

They differ slightly in character, but can be patterned after Section 4.5.3.

### **5.4 About Increasing the Number of Populace Sectors**

For use in Athena-based studies, it is probably unnecessary to distinguish between different kinds of workers than the two that are already included in the model: those who are engaged in subsistence agriculture and those who are not. If it were desired to recognize that there is some economic interaction between them, an explicit sector should be defined for the subsistence agriculturalists who do interact, treating the remainder (if any) as they are now in the model.

Whether it would be useful to track differences in wages and consumption patterns in the cash economy between agricultural workers, factory workers, office workers, service workers, professionals, or whatever depends, of course, on the study to be performed.

These sectors could be patterned after Section 4.5.4, but data requirements could explode if too much resolution is desired.

It may be noted that the different kinds of workers do not have to match the number or kinds of consumers so long as the total expenses of all the populace sectors equal the total revenues of all of the populace sectors—adjusted for any non-zero savings transactions.

### **5.5 About Including Other Factors of Production**

#### **5.5.1 Capital**

Capital is already included in the Athena's ground model, though the model could be strengthened. Capital is not a sector in the CGE and should not be. It is represented at a neighborhood level by production capacity limits, which are aggregated into  $CAP_g$  and  $CAP_b$ . The black sector's capacity is also constrained by the rate at which it can get its imported feedstock (if any).

Differentiation between various goods will be accompanied by differentiating between their capital stocks. We plan for Athena to have infrastructure models to estimate how their capacity limits are affected by investment and sabotage, which are among the tactics available to actors. The infrastructure models must also deal with depreciation, the time value of money, and other factors.

Capital investment in the model assumes replication of existing technology rather than innovation, so that the CGE shape parameters do not change. Capital investments in the

real world are sometimes accompanied by changes in labor productivity. Such innovation would change the shape parameters or the production functions, which are calibrated from the base case, so new base cases would have to be identified.

### **5.5.2 Land and Natural Resources**

Land, natural resources, and other factors whose value depends on their accumulated amount, or "stock", should be treated like capital, and not be sectors in Athena's CGE.

### **5.5.3 Intellectual Capital, Entrepreneurship, etc.**

Intellectual capital, entrepreneurship, and other psychological factors are beyond the scope of Athena. They affect production functions' form and parameter values, often profoundly. Fortunately, few would have effects within Athena's three-year planning horizon.

## ***5.6 About Increasing the Number of Institutional Sectors***

Any institutions that are needed to describe the economy in the region, such as banks, should be modeled as actors.

It may be necessary to identify some tactics that only certain actors can use. For example, can only a Central Bank affect inflation by manipulating the money supply?

## ***5.7 How to Increase Resolution by Splitting Sectors***

Every CGE is, in a sense, a model of the economy of the entire world.<sup>80</sup> Consequently, sectors can only be split out of sectors that are already in the model.

### *Step 1. Identify the new child sectors and their parents*

Even if a parent sector keeps the same name it had before the split, it will encompass a reduced set of transactions in the real world—though there may be more entries in the now-larger CGE matrix.

### *Step 2. Examine every sector for its payments to the new sectors*

In most cases, this examination will be a straightforward revision of the current optimization to include the new sectors as recipients of revenue. The expectation is that whatever was allocated to each parent sector will now be allocated to its children. However, increases in resolution may be accompanied with better understanding of the modeled economy, so the new model may not match this expectation.

### *Step 3. Define the new sectors' optimization problems*

Sectors that produce a product are likely to require an assumed production function to describe the technology of production. The optimization problem may be to select input and output amounts to maximize net revenues or some other objective, subject to the technology described by the production function. Feedstocks, inventories, and investments may require special handling.

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<sup>80</sup> That's a bit of an overstatement, of course, because the *world, world* transaction is not modeled.

Sectors that supply a factor of production or a natural resource are likely to be owned. It may be convenient to give the sector a generic name that reflects the owner. The optimization problem may be to select expenditures to maximize a utility function that describes the owner's preferences, subject to the limitations imposed by his assets. Again, feedstocks, inventories, and investments may require special handling.

Sectors that serve a special purpose in the model are likely to require special handling.

*Step 4. Analytically solve the new sectors' optimization problems*

When child sectors are simply higher-resolution pieces of the parent sector, their optimization problems are likely to be identical to their parent's. Then the child sector can be modeled in the same way as the parent, and it will not be necessary to solve a new analytical problem.

These analytical results supply the rest of the revenue information needed.

Using these analytical results in the production functions will generally yield equations for prices of sector products.

*Step 5. Update the code*

Revise the code to incorporate the new models. It is likely that there will be more to do than the above description suggests.

*Step 6. Test*

It is important to conduct not only verification tests, which ensure that the code does what it is expected to do in realizing the design, but validation tests, which ensure that the results are reasonable when compared to what would be expected to happen in the real world, as well.

The testing protocol should include extreme values of all variables to ensure the program is robust. When feasible, it should also include regression tests to ensure that the new functionality does not break existing code.

### **5.8 What if the Region of Interest is Not a Country?**

So long as the "official" government of the region—the one that collects taxes—is modeled as one of the actors, the model described here is scalable. That is, no changes to the model are required to deal with a region that is smaller than a country.

If the region consists of several countries, every "official" government in the region should be modeled as an actor. Unless we are content to treat the group of countries as a de facto "union", the model would have to be expanded to contain separate sectors for each country. Thus, if there were three countries in the region, say A, B, and C, a minimal model would contain the following 14 sectors:

- Goods in A
- Black in A
- Pop in A
- Rest of region A

- Goods in B
- Black in B
- Pop in B
- Rest of region B
- Goods in C
- Black in C
- Pop in C
- Rest of region C
- Actors
- Rest of the world.

It might be possible to dispense with some of the *black* sectors, but it might also be necessary to include other regionally specific sectors. It would probably be desirable to have separate *actors* sectors for each country and also have an *international actors* sector.

Additional sectors might be desired as discussed in Sections 5.2–5.4.

## 6 Appendix: Balancing a SAM

by Brian J. Kahovec

In this appendix, we describe a process that will balance a Social Accounting Matrix. The entries of a SAM, denoted by  $SAM_{ij}$ , are the annual value of transactions that economic sector  $i$  receives from sector  $j$ . We say that a SAM is balanced if the total expenditures equal the total revenues for each sector.

An unbalanced SAM implies an economy that is not in equilibrium. If revenues exceed expenses, value is entering the economy from outside the region. If expenses exceed revenues, value is leaving the economy. These conditions may indeed be the case for the region of interest, and it may vary by sector, so a balanced SAM is not an absolute requirement for validity.

However, data collection is a real-world process, so it would be remarkable if any sector's row sum equaled its column sum even if the sector is, in fact, balanced. For that to happen, the data gatherers would have to have found consistent reports of every cash flow (at the resolution of the SAM) in the region!

Consequently, Athena's users have the option of balancing the SAM or not.

### 6.1 The X Matrix

The X matrix is an unbalanced, partial SAM that does not include some cash flows such as graft and remittances. The following example will be used in later computations.

**Table 1. X Matrix**

	<i>goods</i>	<i>black</i>	<i>pop</i>	<i>actors</i>	<i>region</i>	<i>world</i>	Total Revenues
<i>goods</i>	1087	5	97880	0	785	21410	121167
<i>black</i>	0	0	50	0	0	800	850
<i>pop</i>	95475	80	0	20000	300	0	115855
<i>actors</i>	4000	80	6086	0	0	3737	13903
<i>region</i>	0	0	0	0	0	1085	1085
<i>world</i>	20092	0	13000	0	0	0	33092
Total Expenditures	120654	165	117016	20000	1085	27032	

### 6.2 The T Matrix

The T matrix is a matrix of the cash flows that were omitted from the X matrix.<sup>81</sup> Adding this matrix, shown below, to the unbalanced, partial social accounting matrix X gives us an unbalanced, complete SAM.

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<sup>81</sup> The graft transactions were omitted on the grounds that they will not appear in official data, so an explicit estimate of the fraction can be expected to be more reliable. Remittances were omitted so the X matrix entry can be used to compute the number of jobs. The black market feedstock was omitted to facilitate explicit consideration of feedstock interdiction.

Table 2.  $T$  Matrix

	<i>goods</i>	<i>black</i>	<i>pop</i>	<i>actors</i>	<i>region</i>	<i>world</i>
<i>goods</i>	0	0	0	0	0	0
<i>black</i>	0	0	0	0	0	0
<i>pop</i>	0	0	0	0	0	+REM
<i>actors</i>	0	$R_b - E_b$	0	0	0	+graft · $X_{rw}$
<i>region</i>	0	0	0	0	0	-graft · $X_{rw}$
<i>world</i>	0	+ $CF_{wb}$	0	0	0	0

where  $E_b$  is the total expenditures (column sum) of the black sector in the initial  $X$  matrix and  $R_b$  is the total revenues (row sum) of the black sector.  $REM$  is the amount that the populace receives from remittances. The variable *graft* is the percentage of foreign aid to the region that ends up in the pockets of the actors.  $CF_{wb}$  is the cost of the *black* sector's feedstock obtained from the world sector. The following matrix is an unbalanced, complete SAM. We use the notation  $T(X)$  to emphasize that  $T$  was computed from  $X$ .

 Table 3.  $X+T(X)$  Matrix

	<i>goods</i>	<i>black</i>	<i>pop</i>	<i>actors</i>	<i>region</i>	<i>world</i>	Total Revenues
<i>goods</i>	1087	5	97880	0	785	21410	121167
<i>black</i>	0	0	50	0	0	800	850
<i>pop</i>	95745	80	0	20000	300	194	116319
<i>actors</i>	4000	765	6086	0	0	3846	14697
<i>region</i>	0	0	0	0	0	977	977
<i>world</i>	20092	595	13000	0	0	0	33538
Total Expenditures	120924	1445	117016	20000	1085	27227	

### 6.3 The Diagonal Similarity Scaling Algorithm

Schneider's DSS algorithm [Schneider 1990] will balance any square matrix  $A$ . We begin by choosing the sector with the largest difference between the revenues and expenditures, so that the algorithm will converge in the fewest iterations. We refer to this sector as sector  $q$ . Then we adjust the elements in the rows and columns of sector  $q$  so that it is balanced. Then at the next iteration, we choose a new sector  $q$  and change its elements so that the new  $q$  is balanced. This, in turn, unbalances the previous sector  $q$ . But, as we iterate, the matrix will converge to a balanced matrix.

We define  $\alpha$  so that

$$\alpha \cdot R_q = \frac{E_q}{\alpha}$$

where  $E_q$  is the total expenditures (column sum) of sector  $q$  and  $R_q$  is the total revenues (row sum) of sector  $q$ .

Solving for  $\alpha$  yields

$$\alpha = \sqrt{\frac{E_q}{R_q}}$$

For each element  $A_{ij}$  in  $A$ , we compute

$$A_{ij} = \begin{cases} \alpha \cdot A_{ij} & \text{if } i = q, j \neq q \\ \frac{A_{ij}}{\alpha} & \text{if } i \neq q, j = q \\ A_{ij} & \text{otherwise} \end{cases}$$

This will balance the  $q$  sector and reduce the discrepancy for each of the other sectors, but not eliminate it, so we'll have to iterate. At each iteration, we recompute the total revenues and expenditures of each sector. Then we recompute  $\alpha$  and  $A_{ij}$ . We repeat this process until, for each sector  $i$ ,  $|E_i - R_i|$  is less than some epsilon.

### 6.4 Reordering the Sectors

We would like the sectors with the greatest change to be the sectors with the least reliable data. Can that be accomplished? Would ordering the sectors from softest to hardest data and then running the algorithm in that order make this happen?

To test this hypothesis, I adjusted the algorithm so that it would run six times—once for each sector—in an order that I chose. Then the algorithm would run as above until it converged to a solution. I reordered the sectors and performed this test several times. It did not matter how the sectors were ordered, the algorithm always produced the same final matrix, though with differing numbers of iterations. So it appears that the result of this algorithm is insensitive to the ordering of sectors and that we cannot use it to change the softer data more than the harder data.

### 6.5 The BAL Matrix

The *BAL* matrix is the result of balancing the SAM. Using the DSS algorithm to balance the  $X + T(X)$  matrix from Section 1.2 yields the following matrix whose entries have been rounded to integers.

**Table 4. BAL Matrix**

	<i>goods</i>	<i>black</i>	<i>pop</i>	<i>actors</i>	<i>region</i>	<i>world</i>	Total Revenues
<i>goods</i>	1087	4	96211	0	775	23300	121377
<i>black</i>	0	0	63	0	0	1123	1187
<i>pop</i>	97131	63	0	17330	301	215	115040
<i>actors</i>	4696	696	7024	0	0	4914	17330
<i>region</i>	0	0	0	0	0	1076	1076
<i>world</i>	18462	424	11742	0	0	0	30628
Total Expenditures	121337	1187	115040	17330	1076	30628	

Notice that the total revenues equal the total expenditures for each sector. The matrix below shows the relative change in each entry of the SAM. Each element is computed by subtracting  $BAL - (X + T(X))$  and dividing by the corresponding element in the unbalanced SAM,  $X + T(X)$ .

**Table 5.  $\Delta$  Matrix**

	<i>goods</i>	<i>black</i>	<i>pop</i>	<i>actors</i>	<i>region</i>	<i>world</i>
<i>goods</i>	0%	-20%	-2%	0%	-1%	9%
<i>black</i>	0%	0%	26%	0%	0%	40%
<i>pop</i>	1%	-21%	0%	-13%	0%	11%
<i>actors</i>	17%	-9%	15%	0%	0%	28%
<i>region</i>	0%	0%	0%	0%	0%	10%
<i>world</i>	-8%	-29%	-10%	0%	0%	0%

If these deltas are too large, then the user may want to reconsider the initial  $X$  matrix before proceeding to input either the unbalanced  $X + T(X)$  or the balanced  $BAL$ .

### 6.6 *Currently in Athena*

At this point, Athena does not compute  $T$ . The user must compute  $T$  and add it to  $X$ . If the user wishes to use an unbalanced SAM in the Athena scenario, he should input the unbalanced  $X + T(X)$  matrix. If the user wants a balanced SAM, he should input the  $BAL$  matrix.

### 6.7 *Later Versions of Athena*

To be more user-friendly, later versions of Athena might compute  $T$  and add it to  $X$ . Then, in order to balance the matrix, we must revise the DSS algorithm. Balancing the matrix and removing the additions (because Athena will put them back in) will give us a usable SAM for our scenario.

### 6.8 *The Revised Diagonal Similarity Scaling Algorithm*

The  $T$  matrix makes it necessary to revise Schneider's DSS algorithm. We begin with a matrix  $A$  and a sector  $q$ . Recall that sector  $q$  is the sector with the largest difference between the revenues and expenditures.

We compute  $\alpha$  as

$$\alpha = \sqrt{\frac{E_q}{R_q}}$$

For each element  $A_{ij}$  in  $A$ , we compute

$$A_{ij} = \begin{cases} \alpha \cdot A_{ij} & \text{if } i = q, j \neq q \\ \frac{A_{ij}}{\alpha} & \text{if } i \neq q, j = q \\ A_{ij} & \text{otherwise} \end{cases}$$

Up to this point, the algorithm has not been changed. The addition of the  $T(X)$  matrix to  $X$  requires us to add a constraint. In particular, the value of  $A_{wb}$  depends on the cost of the feedstock, which depends on the total revenue of the *black* sector. So at the end of each iteration, we set  $A_{wb}$  to its initial value plus the current cost of feedstock. At the  $k^{\text{th}}$  iteration, we compute

$$A_{wb}^k = A_{wb}^0 + CF_{wb}^k$$

where  $CF_{wb}^k$  is computed as

$$CF_{wb}^k = PF_{wb} \cdot AF_{wb} \cdot \frac{R_b^k}{P_b}$$

From here, we recompute the total revenues and expenditures, determine a new  $q$  sector, and repeat the process. Performing the revised algorithm on the  $X + T(X)$  matrix from the previous example yields the following balanced matrix.

**Table 6.** *BAL* Matrix

	<i>goods</i>	<i>black</i>	<i>pop</i>	<i>actors</i>	<i>region</i>	<i>world</i>	Total Revenues
<i>goods</i>	1087	3	96145	0	776	23390	121402
<i>black</i>	0	0	93	0	0	1653	1746
<i>pop</i>	97198	43	0	17232	302	216	114990
<i>actors</i>	4726	478	7064	0	0	4964	17232
<i>region</i>	0	0	0	0	0	1079	1079
<i>world</i>	18391	1222	11688	0	0	0	31301
Total Expenditures	121402	1746	114990	17232	1079	31301	

The matrix below shows the relative change in each entry of *BAL*. It is computed the same way as the previous  $\Delta$  matrix.

**Table 7.**  $\Delta$  Matrix

	<i>goods</i>	<i>black</i>	<i>pop</i>	<i>actors</i>	<i>region</i>	<i>world</i>
<i>goods</i>	0%	-40%	-2%	0%	-1%	9%
<i>black</i>	0%	0%	86%	0%	0%	107%
<i>pop</i>	2%	-46%	0%	-14%	1%	11%
<i>actors</i>	18%	-38%	16%	0%	0%	29%
<i>region</i>	0%	0%	0%	0%	0%	10%
<i>world</i>	-8%	105%	-10%	0%	0%	0%

## 6.9 Solving for the SAM

Now we will compute a new matrix from *BAL* so that when the matrix of additional cash flows is added to it, we will end up with a balanced SAM. The  $T$  matrix will be computed just as it was before, but this time using the final matrix, which we will refer to as  $Y$ . Then we will solve for  $Y$  using the equation  $Y + T(Y) = BAL$ . The following equations show how we solve for each of the elements in  $Y$  that will be affected by  $T$ .

The defining equation for  $BAL_{rw}$  is

$$Y_{rw} - graft \cdot Y_{rw} = BAL_{rw}.$$

Hence,

$$Y_{rw} = \frac{BAL_{rw}}{1-raft}.$$

But if  $raft = 1$ ,  $Y_{rw} = 0$ .

The defining equation for  $BAL_{aw}$  is

$$Y_{aw} + graft \cdot Y_{rw} = BAL_{aw}.$$

Hence,

$$Y_{aw} = BAL_{aw} - graft \cdot Y_{rw} = BAL_{aw} - \frac{graft}{1-graft} \cdot BAL_{rw}.$$

But if  $graft = 1$ ,  $Y_{aw} = BAL_{aw}$ .

The defining equation for  $BAL_{pw}$  is

$$Y_{pw} + REM = BAL_{pw}.$$

Hence,

$$Y_{pw} = BAL_{pw} - REM.$$

The defining equation for  $BAL_{wb}$  is

$$Y_{wb} + CF_{wb} = BAL_{wb}.$$

Hence,

$$Y_{wb} = BAL_{wb} - CF_{wb}$$

The defining equation for  $BAL_{ab}$  is

$$Y_{ab} + (R_b - E_b) = BAL_{ab} \text{ where } R_b = \sum_i Y_{bi} \text{ and } E_b = \sum_i Y_{ib}.$$

Since  $Y_{ab}$  is in the summation  $\sum_i Y_{ib}$ ,  $Y_{ab}$  cancels and we are unable to solve for it. This means that there are infinitely many solutions, so we choose the value from the original matrix  $X$ . Hence,

$$Y_{ab} = X_{ab}$$

For the entries not listed above,  $T(Y)_{ij} = 0$  and  $Y_{ij} = BAL_{ij}$ . This gives us a usable matrix that will balance once the cash flows are added back in.

**Table 8.** Y Matrix

	<b>goods</b>	<b>black</b>	<b>pop</b>	<b>actors</b>	<b>region</b>	<b>world</b>	Total Revenues
<b>goods</b>	1087	3	96145	0	776	23390	121402
<b>black</b>	0	0	93	0	0	1653	1746
<b>pop</b>	97197	43	0	17232	302	22	114796
<b>actors</b>	4726	80	7064	0	0	4856	16726
<b>region</b>	0	0	0	0	0	1079	1079
<b>world</b>	18391	0	11688	0	0	0	30079
Total Expenditures	121402	126	114990	17232	1079	31119	

The matrix below shows the relative change in each entry of the SAM. Each element is computed by subtracting  $Y - X$  and dividing by the corresponding entry in  $X$ .

**Table 9.  $\Delta$  Matrix**

	<i>goods</i>	<i>black</i>	<i>pop</i>	<i>actors</i>	<i>region</i>	<i>world</i>
<i>goods</i>	0%	-40%	-2%	0%	-1%	9%
<i>black</i>	0%	0%	86%	0%	0%	107%
<i>pop</i>	2%	-46%	0%	-14%	1%	0%
<i>actors</i>	18%	0%	16%	0%	0%	30%
<i>region</i>	0%	0%	0%	0%	0%	-1%
<i>world</i>	-8%	0%	-10%	0%	0%	0%

Again, if these deltas are too large, the user may want to reconsider the initial  $X$  matrix before proceeding to input either the won't-be-balanced  $X$  or the will-be-balanced  $Y$ .

If the user wishes to use a balanced SAM in the Athena scenario, he should input  $Y$ . But, if he wants an unbalanced SAM, he should input  $X$ .

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— Robert G. Chamberlain  
— William H. Duquette

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