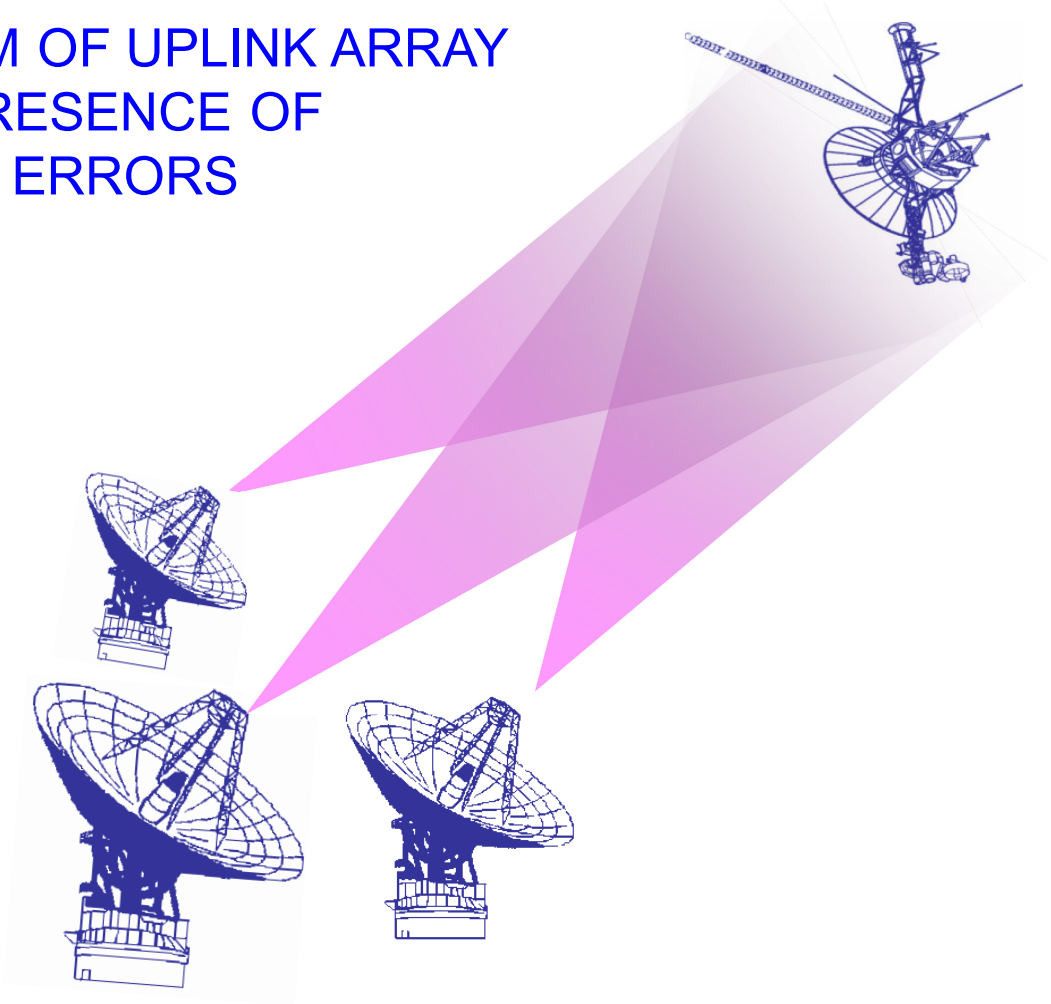
POWER SPECTRUM OF UPLINK ARRAY SIGNALS IN THE PRESENCE OF PHASE AND DELAY ERRORS



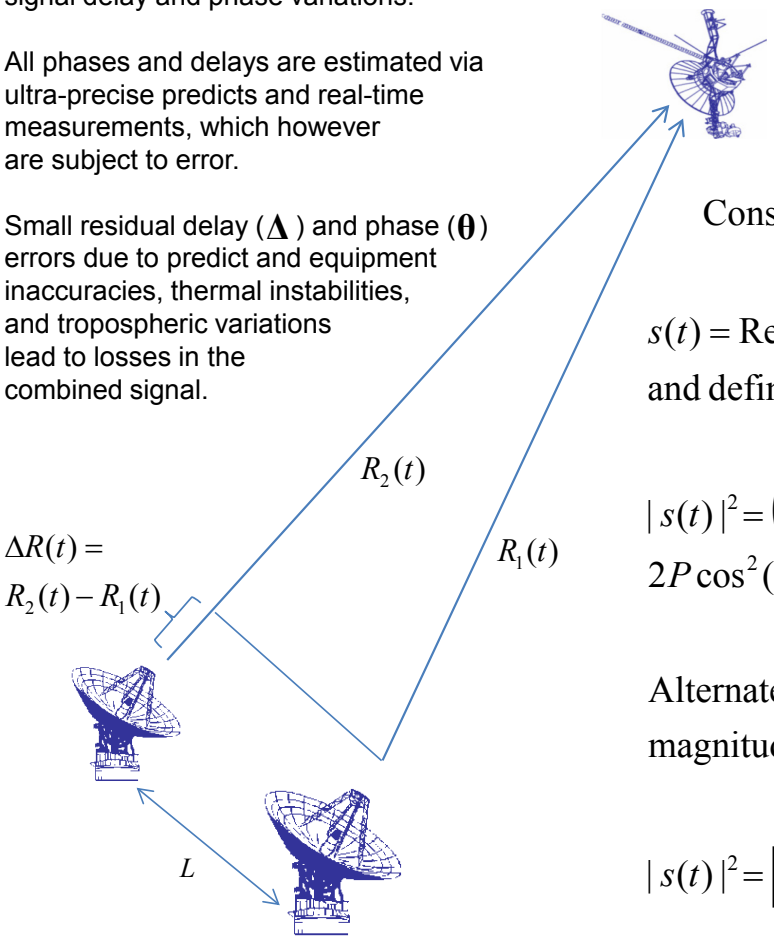
Victor Vilnrotter

Jet Propulsion Laboratory
California Institute of Technology

Time-varying range differences due to geometry are the largest components of signal delay and phase variations.

All phases and delays are estimated via ultra-precise predicts and real-time measurements, which however are subject to error.

Small residual delay ($\boldsymbol{\Delta}$) and phase ($\boldsymbol{\theta}$) errors due to predict and equipment inaccuracies, thermal instabilities, and tropospheric variations lead to losses in the combined signal.



Problem Statement and Motivation for "complex envelope" model

Consider an RF signal of the form

$$s(t) = \mathrm{Re}\left\{\sqrt{2P}\,\tilde{s}_0(t)\,e^{j\omega t}\right\} \quad \text{where } \tilde{s}_0(t) = d(t)e^{j\theta}, \quad d(t) = \pm 1$$

and define "average power" as

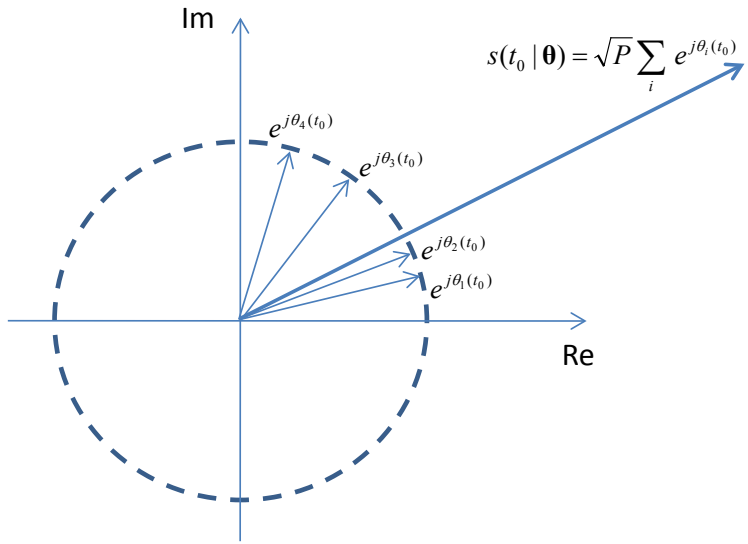
$$|s(t)|^2 = \left(\mathrm{Re}\left\{\sqrt{2P}\,\tilde{s}_0(t)\,e^{j\omega t}\right\}\right)^2 = 2Pd^2(t)\left(\mathrm{Re}\left\{e^{j(\omega t+\theta)}\right\}\right)^2 =$$
$$2P\cos^2(\omega t+\theta) = 2P\left\{\tfrac{1}{2}[1+\cos(2\omega t+2\theta)]\right\}_{LP} = P$$

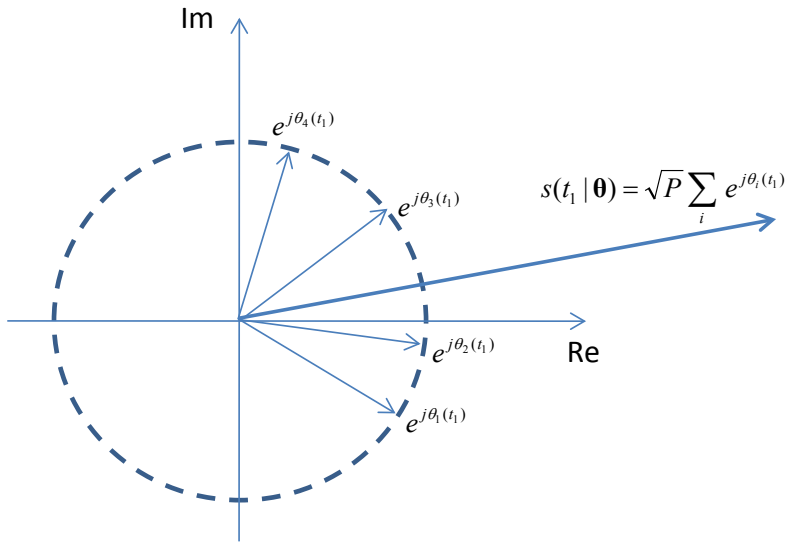
Alternately, we can define "average power" as the squared magnitude of the of the complex envelope:

$$|s(t)|^2 = \left|\sqrt{P}\,\tilde{s}_0(t)\,e^{j\omega t}\right|^2 = P\left(d^2(t)e^{j\omega t}e^{-j\omega t}\right) = P$$

The math is much simpler with the complex envelope model.

Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.



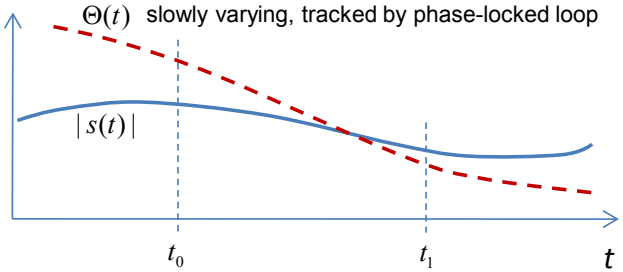


Unmodulated Array Signal Power

$$\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_N)$$

$$\Theta(t \mid \boldsymbol{\theta}) \equiv \tan^{-1}\left(\mathrm{Im}[s(t)] / \mathrm{Re}[s(t)]\right)$$

$$|s(t \mid \boldsymbol{\theta})| = \sqrt{P} \left| \sum_i \sum_k e^{j[\theta_i(t) - \theta_k(t)]} \right|$$



$$|s(t \mid \boldsymbol{\theta})|^2 = \left| \sqrt{P} \sum_{i=1}^{N} e^{j\theta_i(t)} \right|^2 = P \left(\sum_{i=1}^{N} \sum_{k=1}^{N} e^{j[\theta_i(t) - \theta_k(t)]} \right)$$

$$= P \left(\sum_{i=1}^{N} e^{j[\theta_i(t) - \theta_i(t)]} + \sum_{i=1}^{N} \sum_{\substack{k=1 \\ k \neq i}}^{N} e^{j[\theta_i(t) - \theta_k(t)]} \right)$$

$$= P \left(N + \sum_{i=1}^{N} \sum_{\substack{k=1 \\ k \neq i}}^{N} e^{j[\theta_i(t) - \theta_k(t)]} \right)$$

Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.

Conventional derivation for zero-mean, SI, equal-variance, Normal random phase:

Let θ_i and θ_k be SI $\forall i \neq k$, and let $p(\theta_i) = \frac{1}{\sqrt{2\pi\sigma_\theta^2}} e^{-\theta_i^2/2\sigma_\theta^2}$, where σ_θ^2 is the variance of each.

$$\overline{|s(t\,|\,\boldsymbol{\theta})|^2}^{\theta} = \overline{\left|\sqrt{P}\sum_{i=1}^{N} e^{j\theta_i(t)}\right|^2}^{\theta} = P\left(N + \overline{\sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq i}}^{N} e^{j[\theta_i(t)]}e^{-j[\theta_k(t)]}}^{\theta}\right) = P\left(N + \sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq i}}^{N} \overline{e^{j[\theta_i(t)]}}^{\theta}\,\overline{e^{-j[\theta_k(t)]}}^{\theta}\right) \quad \text{because } \theta_i, \theta_k \text{ are SI}$$

Averaging with respect to the Gaussian density yields:

$$\overline{e^{\pm j\theta}} = \frac{1}{\sqrt{2\pi\sigma_\theta^2}}\int_{-\infty}^{\infty} e^{\pm j\theta} e^{-\theta^2/2\sigma_\theta^2} d\theta = \frac{1}{\sqrt{2\pi\sigma_\theta^2}}\int_{-\infty}^{\infty} \cos(\theta) e^{-\theta^2/2\sigma_\theta^2} d\theta + \frac{j}{\sqrt{2\pi\sigma_\theta^2}}\int_{-\infty}^{\infty} \overset{0}{\sin(\pm\theta)} e^{-\theta^2/2\sigma_\theta^2} d\theta = e^{-\sigma_\theta^2/2}$$

Substituting, the average Uplink Array Power is given by:

$$\boxed{\overline{|s(t\,|\,\boldsymbol{\theta})|^2}^{\theta} = P\left(N + \sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq i}}^{N} \overline{e^{j[\theta_i(t)]}}^{\theta}\,\overline{e^{-j[\theta_k(t)]}}^{\theta}\right) = P\left(N + N(N-1)e^{-\sigma_\theta^2}\right)}$$

$$\lim_{\sigma_\theta^2\to 0} \overline{|s(t\,|\,\boldsymbol{\theta})|^2}^{\theta} = P\left(N + N(N-1)e^{-\sigma_\theta^2}\,\Big|_{\sigma_\theta^2\to 0}\right) = PN^2 = \max \overline{|s(t\,|\,\boldsymbol{\theta})|^2}^{\theta}$$

$$\lim_{\sigma_\theta^2\to \infty} \overline{|s(t\,|\,\boldsymbol{\theta})|^2}^{\theta} = P\left(N + N(N-1)e^{-\sigma_\theta^2}\,\Big|_{\sigma_\theta^2\to \infty}\right) = PN = \min \overline{|s(t\,|\,\boldsymbol{\theta})|^2}^{\theta}$$

Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.

Average Received Uplink Array Power: General Solution (Gaussian Approximation)

Assume $\boldsymbol{\theta}$ is a Gaussian random vector with probability density

$$p(\boldsymbol{\theta}) = \frac{\exp[-\frac{1}{2}(\boldsymbol{\theta}-\overline{\boldsymbol{\theta}})\boldsymbol{\Lambda}_\theta^{-1}(\boldsymbol{\theta}-\overline{\boldsymbol{\theta}})^T]}{(2\pi)^{N/2}\,|\Lambda_\theta|^{1/2}}$$

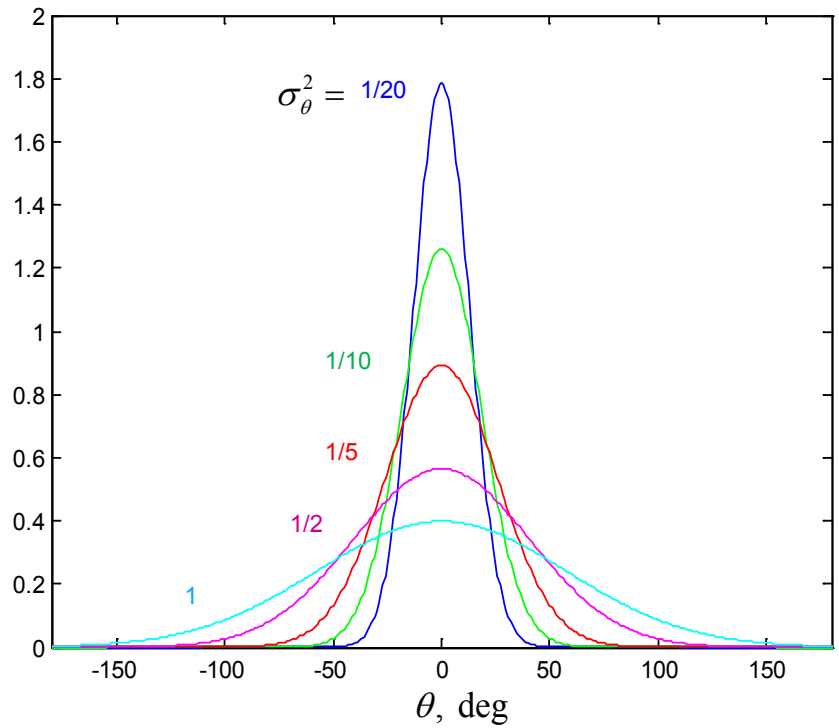
$$\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_N), \quad \overline{\boldsymbol{\theta}} = (\overline{\theta}_1, \overline{\theta}_2, \cdots, \overline{\theta}_N)$$

$$\boldsymbol{\Lambda}_{\boldsymbol{\theta}} = E\{(\boldsymbol{\theta}-\overline{\boldsymbol{\theta}})^T(\boldsymbol{\theta}-\overline{\boldsymbol{\theta}})\},$$

$$\sigma_{\theta_i}^2 = E(\theta_i - \overline{\theta}_i)^2,$$

$$\lambda_{\theta_{i,k}} = E\{(\theta_i - \overline{\theta}_i)(\theta_k - \overline{\theta}_k)\}$$

One-dimensional Gaussian density



$$\boldsymbol{\Lambda}_{\boldsymbol{\theta}} = E\{(\boldsymbol{\theta}-\overline{\boldsymbol{\theta}})^T(\boldsymbol{\theta}-\overline{\boldsymbol{\theta}})\}, \quad \boldsymbol{\Lambda}_\theta = \begin{bmatrix} \sigma_{\theta_1}^2 & \lambda_{\theta_{1,2}} & \cdots & \lambda_{\theta_{1,N}} \\ \lambda_{\theta_{2,1}} & \sigma_{\theta_2}^2 & \cdots & \lambda_{\theta_{2,N}} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{\theta_{N,1}} & \lambda_{\theta_{N,2}} & \cdots & \sigma_{\theta_N}^2 \end{bmatrix}, \quad \boldsymbol{\Lambda}_{\boldsymbol{\theta}i,k} = \begin{bmatrix} \sigma_{\theta_i}^2 & \lambda_{\theta_{i,k}} \\ \lambda_{\theta_{k,i}} & \sigma_{\theta_k}^2 \end{bmatrix}$$

Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.

For Gaussian $\boldsymbol{\theta}$, the characteristic function is defined as :

$$M_{\boldsymbol{\theta}}(\mathbf{v}) = E\{\exp[j\,\mathbf{v}\,\boldsymbol{\theta}^T]\} = \exp(-\tfrac{1}{2}\mathbf{v}\,\boldsymbol{\Lambda}_{\boldsymbol{\theta}}\,\mathbf{v}^T + j\,\mathbf{v}\,\overline{\boldsymbol{\theta}}^T)$$

For example, with $\mathbf{v} = (\nu_i, \nu_k)$, $\boldsymbol{\theta}^T = \begin{bmatrix} \theta_i \\ \theta_k \end{bmatrix}$, $\overline{e^{j(\theta_i+\theta_k)}}^{\theta} = E\{\exp[j\,\mathbf{v}\,\boldsymbol{\theta}^T]_{\mathbf{v}=\mathbf{1}}\} = M_{\boldsymbol{\theta}_{i,k}}(\mathbf{v}=\mathbf{1})$

Consider the linear transformation $\boldsymbol{\varphi}^T = \mathbf{A}\boldsymbol{\theta}^T$, hence $\overline{\boldsymbol{\varphi}}^T = \mathbf{A}\overline{\boldsymbol{\theta}}^T$

$$M_{\boldsymbol{\varphi}}(\mathbf{v}) = E\{e^{j\mathbf{v}\boldsymbol{\varphi}^T}\} = E\{e^{j(\mathbf{v}\mathbf{A})\boldsymbol{\theta}^T}\} = \exp(-\tfrac{1}{2}\mathbf{v}\,\mathbf{A}\,\boldsymbol{\Lambda}_{\boldsymbol{\theta}}\mathbf{A}^T\mathbf{v}^T + j\,\mathbf{v}\,\mathbf{A}\,\overline{\boldsymbol{\theta}}^T)$$

Alternately,

$$\boldsymbol{\Lambda}_{\boldsymbol{\varphi}} = E\{\boldsymbol{\varphi}^T\boldsymbol{\varphi}\} = E\{\mathbf{A}\boldsymbol{\theta}^T(\mathbf{A}\boldsymbol{\theta}^T)^T\} = E\{\mathbf{A}\boldsymbol{\theta}^T\boldsymbol{\theta}\mathbf{A}^T\} = \mathbf{A}\overline{\boldsymbol{\theta}^T\boldsymbol{\theta}}\mathbf{A}^T = \mathbf{A}\,\Lambda_{\theta}\mathbf{A}^T$$

yielding the same result :

$$M_{\boldsymbol{\varphi}}(\mathbf{v}) = E\{e^{j\mathbf{v}\boldsymbol{\varphi}^T}\} = \exp(-\tfrac{1}{2}\mathbf{v}\;\boldsymbol{\Lambda}_{\boldsymbol{\varphi}}\mathbf{v}^T + j\,\mathbf{v}\,\mathbf{A}\,\boldsymbol{\theta}^T) = \exp(-\tfrac{1}{2}\mathbf{v}\,\mathbf{A}\,\boldsymbol{\Lambda}_{\boldsymbol{\theta}}\mathbf{A}^T\mathbf{v}^T + j\,\mathbf{v}\,\mathbf{A}\,\boldsymbol{\theta}^T)$$

Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.

Example 1: Let $\boldsymbol{\theta} = (\theta_1, \theta_2)$, $\bar{\boldsymbol{\theta}} = (\bar{\theta}_1, \bar{\theta}_2)$, $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \mathbf{A}^T$

$$\Lambda_{\boldsymbol{\theta}} = \begin{bmatrix} \sigma_{\theta_1}^2 & \lambda_{1,2} \\ \lambda_{1,2} & \sigma_{\theta_2}^2 \end{bmatrix}, \quad \sigma_{\theta_i}^2 = \overline{(\theta_i - \bar{\theta}_i)^2}^{\theta} \quad i = 1,2, \quad \lambda_{1,2} = \overline{(\theta_1 - \bar{\theta}_1)(\theta_2 - \bar{\theta}_2)}^{\theta}$$

$$\mathbf{A}\boldsymbol{\theta}^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ -\theta_2 \end{bmatrix}, \quad \mathbf{A}\bar{\boldsymbol{\theta}}^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \bar{\theta}_1 \\ \bar{\theta}_2 \end{bmatrix} = \begin{bmatrix} \bar{\theta}_1 \\ -\bar{\theta}_2 \end{bmatrix}$$

$$\mathbf{A} \begin{bmatrix} \sigma_{\theta_1}^2 & \lambda_{1,2} \\ \lambda_{1,2} & \sigma_{\theta_2}^2 \end{bmatrix} \mathbf{A}^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \sigma_{\theta_1}^2 & \lambda_{1,2} \\ \lambda_{1,2} & \sigma_{\theta_2}^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \sigma_{\theta_1}^2 & -\lambda_{1,2} \\ -\lambda_{1,2} & \sigma_{\theta_2}^2 \end{bmatrix}$$

$$\overline{e^{j(\theta_i - \theta_k)}}^{\theta} = \exp(-\tfrac{1}{2}\mathbf{v}\,\mathbf{A}\,\Lambda_{\theta}\mathbf{A}^T\mathbf{v}^T + j\,\mathbf{v}\,\mathbf{A}\,\boldsymbol{\theta}^T)\,|_{\mathbf{v}=\mathbf{1}} = \exp[-\tfrac{1}{2}(\sigma_{\theta_1}^2 + \sigma_{\theta_2}^2 - 2\lambda_{1,2}) + j(\bar{\theta}_1 - \bar{\theta}_2) = M_{\boldsymbol{\varphi}}(\mathbf{v} = \mathbf{1})$$

With $\boldsymbol{\theta}_{i,k} = (\theta_i, \theta_k)$, $\bar{\boldsymbol{\theta}}_{i,k} = (\bar{\theta}_i, \bar{\theta}_k)$, $\mathbf{A} = diag(1,-1)$

$$M_{\boldsymbol{\varphi}_{i,k}}(\mathbf{v} = \mathbf{1}) = \exp[-\tfrac{1}{2}\mathbf{v}\,\mathbf{A}\Lambda_{\boldsymbol{\theta}_{i,k}}\mathbf{A}^T\,\mathbf{v}^T + j\,\mathbf{v}\,\mathbf{A}\,\bar{\boldsymbol{\theta}}_{i,k}^T] = \exp[-\tfrac{1}{2}(\sigma_{\theta_i}^2 + \sigma_{\theta_k}^2 - 2\lambda_{\theta_{i,k}}) + j(\bar{\theta}_i - \bar{\theta}_k)]$$

$$\overline{|s(t\,|\,\boldsymbol{\theta})|^2}^{\theta} = \overline{\left|\sqrt{P}\sum_{i=1}^{N} e^{j\theta_i(t)}\right|^2}^{\theta} = P\left(N + \sum_{i=1}^{N}\sum_{k\neq i}^{N} \overline{e^{j[\theta_i(t) - \theta_k(t)]}}^{\theta}\right) = P\left(N + \underbrace{\sum_{i=1}^{N}\sum_{k\neq i}^{N} M_{\boldsymbol{\varphi}_{i,k}}(\mathbf{v} = \mathbf{1})}_{N(N\text{-}1)\text{ terms}}\right)$$

Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.

Note that $N(N-1)$ is always even. Half the terms contain $j(\bar{\theta}_i - \bar{\theta}_k)$, the other half contains $j(\bar{\theta}_k - \bar{\theta}_i)$. Therefore we can re-write the double sum as:

$$\sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq i}}^{N} M_{\boldsymbol{\varphi}_{i,k}}(\mathbf{v}=\mathbf{1}) = \sum_{i=1}^{N-1}\sum_{k>i}^{N} M_{\boldsymbol{\varphi}_{i,k}}(\mathbf{v}=\mathbf{1}) + \sum_{k=1}^{N-1}\sum_{i>k}^{N} M_{\boldsymbol{\varphi}_{i,k}}(\mathbf{v}=\mathbf{1})$$

$$= \sum_{i=1}^{N-1}\sum_{k>i}^{N} \exp[-\tfrac{1}{2}(\sigma_{\theta_i}^2 - 2\lambda_{\theta_{i,k}} + \sigma_{\theta_k}^2) + j(\bar{\theta}_i - \bar{\theta}_k)] + \sum_{k=1}^{N-1}\sum_{i>k}^{N} \exp[-\tfrac{1}{2}(\sigma_{\theta_k}^2 - 2\lambda_{\theta_{k,i}} + \sigma_{\theta_i}^2) - j(\bar{\theta}_i - \bar{\theta}_k)]$$

$$(\bar{\theta}_k - \bar{\theta}_i) = -(\bar{\theta}_i - \bar{\theta}_k)$$

$$= 2\sum_{i=1}^{N-1}\sum_{k>i}^{N} \exp[-\tfrac{1}{2}(\sigma_{\theta_i}^2 - 2\lambda_{\theta_{i,k}} + \sigma_{\theta_k}^2)]\cos(\bar{\theta}_i - \bar{\theta}_k)]$$

Complete Solution for the Gaussian phase model):

$$\overline{|s(t\,|\,\boldsymbol{\theta})|^2}^{\theta} = P\left(N + \sum_{i=1}^{N}\sum_{k\neq i}^{N} M_{\boldsymbol{\varphi}_{i,k}}(\mathbf{v}=\mathbf{1})\right) = P\left(N + 2\sum_{i=1}^{N-1}\sum_{k>i}^{N} \exp[-\tfrac{1}{2}(\sigma_{\theta_i}^2 + \sigma_{\theta_k}^2 - 2\lambda_{\theta_{i,k}})]\cos(\bar{\theta}_i - \bar{\theta}_k)\right)$$

Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.

Special Cases:

1. Equal mean values:

If $\bar{\theta}_i = \bar{\theta}_k \quad \forall i,k$ then $(\bar{\theta}_i - \bar{\theta}_k) = 0$, $M_{\boldsymbol{\varphi}_{i,k}}[\mathbf{v}=\mathbf{1}] = \exp[-\tfrac{1}{2}(\sigma^2_{\theta_i} + \sigma^2_{\theta_k} - 2\lambda_{\theta_{i,k}})]$ and

$$\overline{|s(t\,|\,\boldsymbol{\theta})|^2}^{\theta} = P\left(N + \sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq i}}^{N} \exp[-\tfrac{1}{2}(\sigma^2_{\theta_k} + \sigma^2_{\theta_i} - 2\lambda_{\theta_{k,i}})\right)$$

2. Uncorrelated phases:

If $\lambda_{\theta_{i,k}} = E(\theta_i\theta_k) = 0 \quad \forall i,k$ then $M_{\boldsymbol{\varphi}_{i,k}}[\mathbf{v}=\mathbf{1}] = \exp[-\tfrac{1}{2}(\sigma^2_{\theta_i} + \sigma^2_{\theta_k})]$ and

$$\overline{|s(t\,|\,\boldsymbol{\theta})|^2}^{\theta} = P\left(N + \sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq i}}^{N} \exp[-\tfrac{1}{2}(\sigma^2_{\theta_k} + \sigma^2_{\theta_i})\right)$$

3. Equal variances:

If $\sigma^2_{\theta_i} = \sigma^2_{\theta_k} = \sigma^2_{\theta} \quad \forall i,k$ then $M_{\boldsymbol{\varphi}_{i,k}}[\nu=\mathbf{1}] = \exp[-\sigma^2_{\theta}]$ and

$$\overline{|s(t\,|\,\boldsymbol{\theta})|^2}^{\theta} = P\left(N + \sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq i}}^{N} \exp[-\sigma^2_{\theta}]\right) = P\left(N + N(N-1)e^{-\sigma^2_{\theta}}\right)$$

4. Deterministic Phase (zero-variance case):

If $\sigma^2_{\theta_i} = \sigma^2_{\theta_k} = \lambda_{\theta_{i,k}} = 0 \quad \forall i,k$ then $M_{\boldsymbol{\varphi}_{i,k}}(\omega) = 2\cos[\omega(\bar{\theta}_i - \bar{\theta}_k)]$ and $\overline{s(t\,|\,\boldsymbol{\theta})}^{\theta} = P\left(N + 2\sum_{i=1}^{N-1}\sum_{k>i}^{N}\cos(\bar{\theta}_i - \bar{\theta}_k)\right)$

Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.

More Realistic Tikhonov Model (SI, equal-variance case):

Assume that θ_i are Statistically Independent of θ_k $\forall i \neq k$, and let $p(\theta_i) = \dfrac{e^{\rho_\theta \cos\theta_i}}{2\pi I_0(\rho_\theta)}, -\pi < \theta_i \leq \pi$, where ρ_θ is proportional to σ_θ^{-2}.

$$\overline{|s(t\,|\,\boldsymbol{\theta})|^2}^{\theta} = \overline{\left|\sqrt{P}\sum_{i=1}^{N} e^{j\theta_i(t)}\right|^2}^{\theta} = P\left(N + \overline{\sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq i}}^{N} e^{j[\theta_i(t)]}e^{-j[\theta_k(t)]}}^{\theta}\right)$$

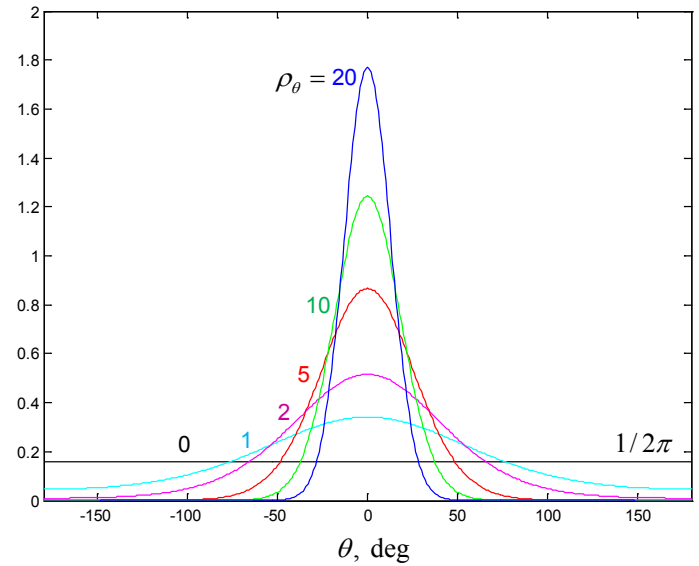
$$= P\left(N + \sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq i}}^{N} \overline{e^{j[\theta_i(t)]}}^{\theta}\,\overline{e^{-j[\theta_k(t)]}}^{\theta}\right) \quad \text{because } \theta_i, \theta_k \text{ are SI}$$

Averaging with respect to the Tikhonov density yields:

$$I_0(\rho_\theta)\overline{e^{\pm j\theta}} = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} e^{\pm j\theta} e^{\rho_\theta \cos(\theta)} d\theta$$

$$= \tfrac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\theta) e^{\rho_\theta \cos(\theta)} d\theta + \tfrac{j}{2\pi}\int_{-\pi}^{\pi} \underbrace{\sin(\pm\theta)}_{0} e^{\rho_\theta \cos(\theta)} d\theta = I_1(\rho_\theta)$$

Tikhonov, or von Mises density



The average Uplink Array Power is given by: $\overline{|s(t\,|\,\boldsymbol{\theta})|^2}^{\theta} = P\left(N + \sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq i}}^{N} \overline{e^{j[\theta_i(t)]}}^{\theta}\,\overline{e^{-j[\theta_k(t)]}}^{\theta}\right) = P\left(N + N(N-1)\left[\dfrac{I_1(\rho_\theta)}{I_0(\rho_\theta)}\right]^2\right)$

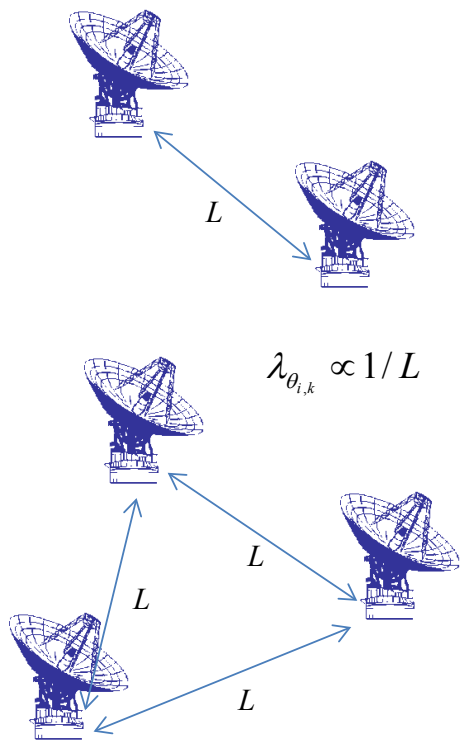
$$\lim_{\rho_\theta\to\infty} \overline{|s(t\,|\,\boldsymbol{\theta})|^2}^{\theta} = P\left(N + N(N-1)[I_1(\rho_\theta)/I_0(\rho_\theta)]\big|_{\rho_\theta\to\infty}\right) = PN^2 = \max \overline{|s(t\,|\,\boldsymbol{\theta})|^2}^{\theta}$$

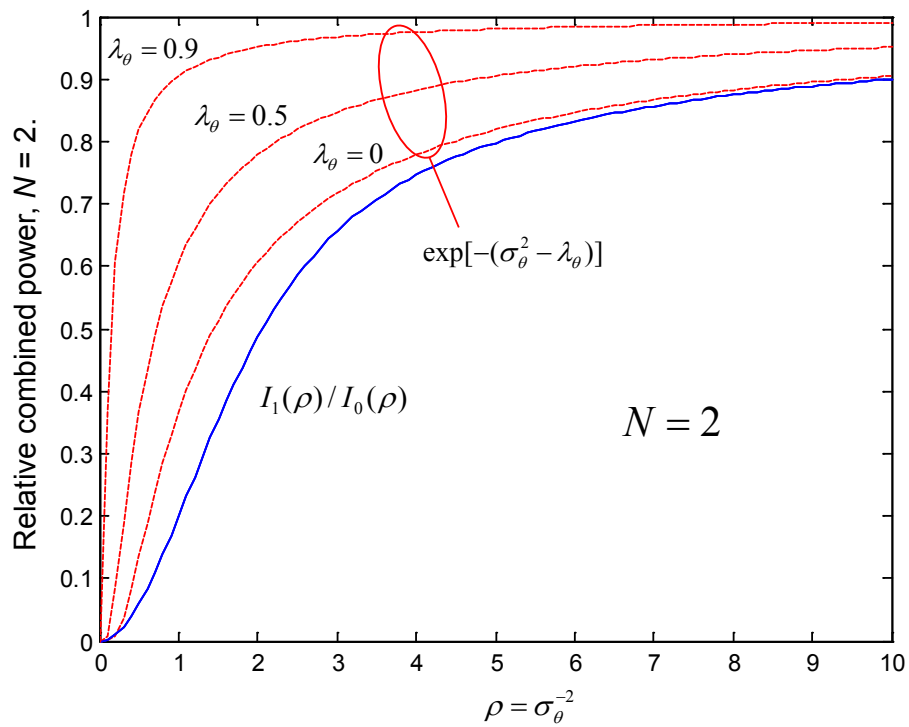
$$\lim_{\rho_\theta\to 0} \overline{|s(t\,|\,\boldsymbol{\theta})|^2}^{\theta} = P\left(N + N(N-1)[I_1(\rho_\theta)/I_0(\rho_\theta)]\big|_{\rho_\theta\to 0}\right) = PN = \min \overline{|s(t\,|\,\boldsymbol{\theta})|^2}^{\theta}$$

Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.

For N=2, 3 it makes sense to assume that phases are equally-correlated and of equal variance, in which case the average power at the spacecraft reduces to:

$$\overline{|s(t\,|\,\boldsymbol{\theta})|^2}^{\theta} = P\left(N + \sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq i}}^{N} \exp[-\tfrac{1}{2}(\sigma_{\theta_k}^2 + \sigma_{\theta_i}^2 - 2\lambda_{\theta_{k,i}})]\right) = P\left(N + N(N-1)\exp[-(\sigma_\theta^2 - \lambda_\theta)]\right)$$





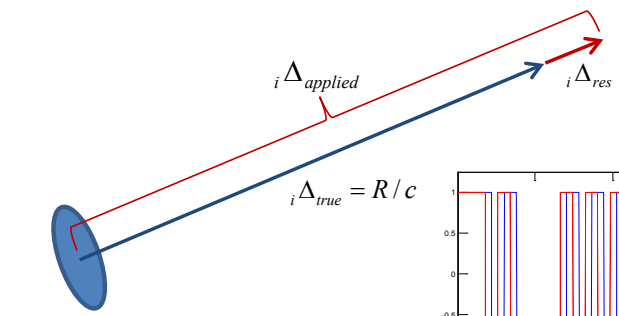
Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.

Modulated Uplink Array Signals, Negligible phase Error

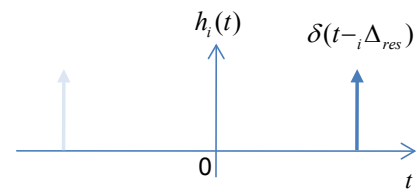
Delay impulse response:

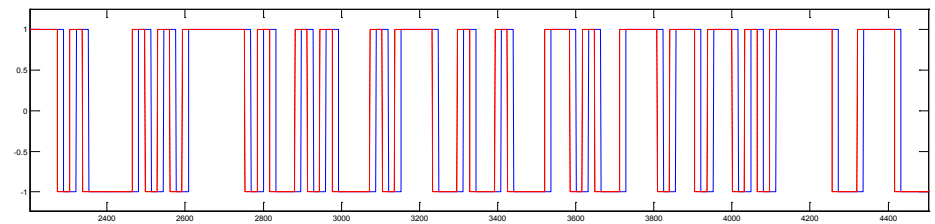
$$ {}_i h_{total}(t) = \delta(t - {}_i\Delta_{applied}) = \delta[t - ({}_i\Delta_{true} + {}_i\Delta_{res})]; \quad {}_i\Delta_{res} \equiv {}_i\Delta_{applied} - {}_i\Delta_{true} $$

$$ h_i(t) \equiv \delta(t - {}_i\Delta_{res}); \quad \text{impulse response of "residual error" measured from "true delay"} $$



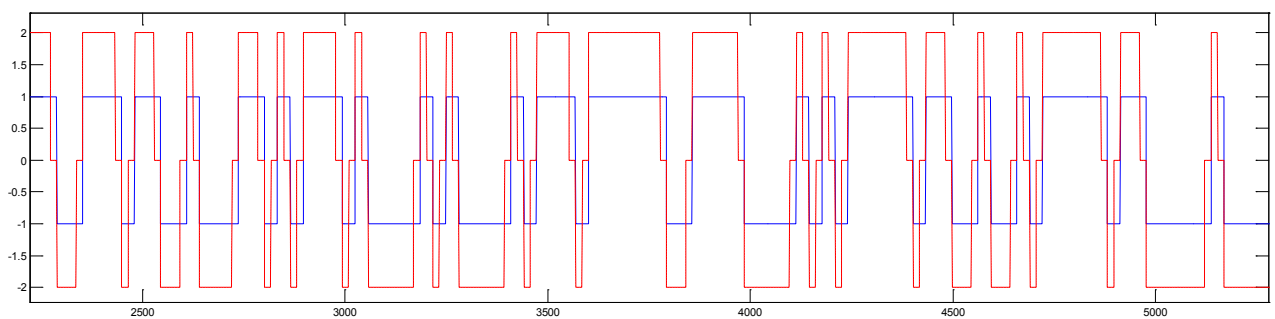
Antenna "i"





Example of received "± 1" data sequence (solid blue), and delayed version (red dash)

At the spacecraft, delayed versions of received signals add: $s(t \mid \mathbf{\Delta}) = \sum_i s_i(t - \Delta_i) = \sum_i s_0(t) * h_i(t) = s_0(t) * \sum_i h_i(t) \equiv s_0(t) * h_\Sigma(t)$



Sum of "± 1" data sequence and it's delayed version, with $\Delta = -T/2; \quad T = 32$ samples

Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.

Transfer function of the Uplink Array channel:

$$H_\Sigma(j\omega) \equiv \mathcal{F}\{h_\Sigma(t)\} = \sum_i \mathcal{F}\{h_i(t)\} = \sum_i H_i(j\omega)$$

$$h_i(t) = \delta(t-\Delta_i), \quad \therefore\ H(j\omega\,|\,\Delta_i) = \exp(-j\omega\Delta_i), \quad H_\Sigma(j\boldsymbol{\omega}\,|\,\boldsymbol{\Delta}) = \sum_i \exp(-j\omega\Delta_i)$$

$$s(t\,|\,\boldsymbol{\Delta}) = \sum_i s_i(t-\Delta_i) = s_0(t) * h_\Sigma(t) \qquad \mathcal{F}\{s(t\,|\,\boldsymbol{\Delta})\} = \mathcal{F}\{s_0(t)\}H_\Sigma(j\boldsymbol{\omega}\,|\,\boldsymbol{\Delta})$$

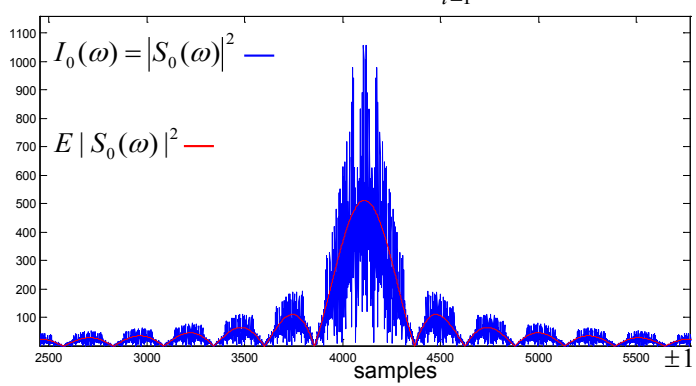
Classical Definition of Power Spectral Density: $P_s(\omega) = E\,|\,S_0(\omega)\,|^2 \equiv E\left|\mathcal{F}\{s_0(t)\}\right|^2$

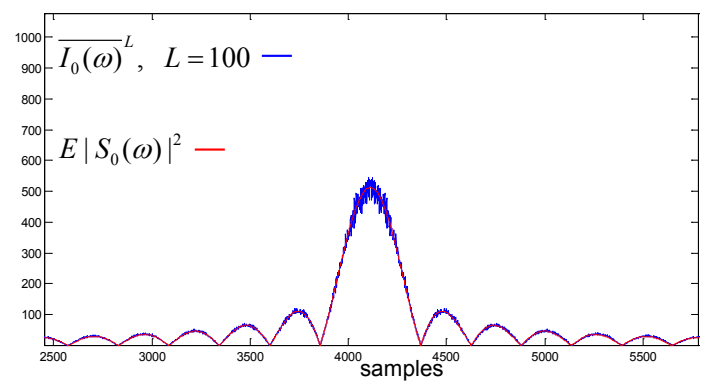
Periodogram: $I(\omega\,|\,\boldsymbol{\Delta}) \equiv \left|\mathcal{F}\{s(t\,|\,\boldsymbol{\Delta})\}\right|^2 = \left|S_0(\omega)\right|^2 \left|H_\Sigma(j\omega)\right|^2,$

Periodogram variance proportional to "spectral-level-squared" at any frequency (estimator not consistent):

$\mathrm{var}[I_0(\omega)] \propto [I_0(\omega)]^2$ where $I_0(\omega) = \left|S_0(\omega)\right|^2$ Estimator variance reduced by averaging Periodograms:

$$\overline{I_0(\omega)}^L = \sum_{l=1}^{L} |\,S_{0,l}(\omega)\,|^2 \equiv \overline{|\,S_0(\omega)\,|^2}^L; \quad \rightarrow \quad \mathrm{var}\left[\overline{I_0(\omega)}^L\right] \propto \tfrac{1}{L}\,\mathrm{var}[I_0(\omega)]$$





Examples of Sample Power Spectral Density, for equi-probable data-sequence: a) single Periodogram; b: average of 100 Period grams

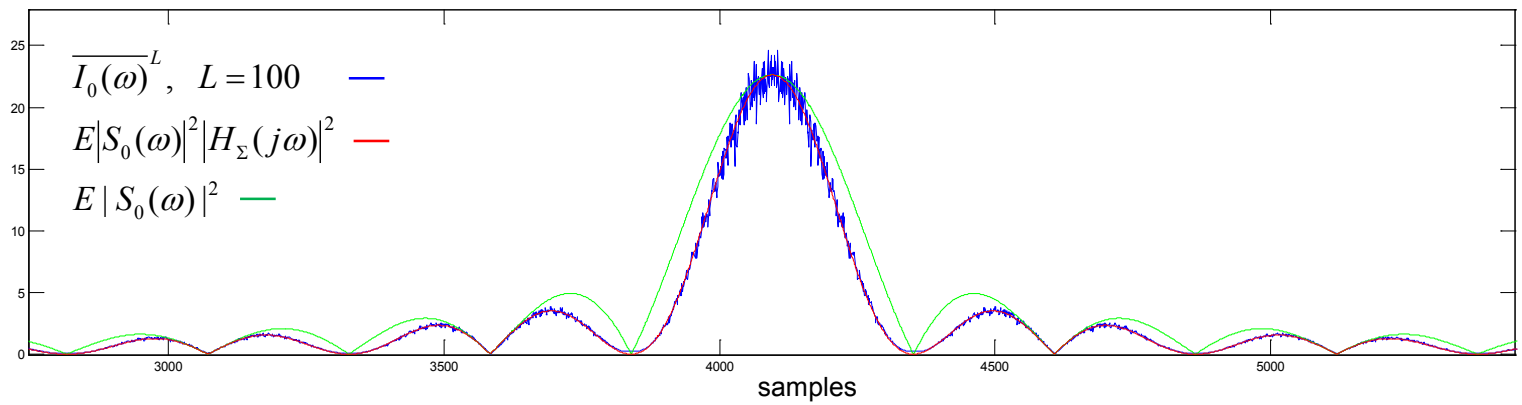
Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.

Conditional Uplink Array Channel Transfer Function, N=2:

Example 2: Consider an uplink channel with $N = 2$,
residual delays $\mathbf{\Delta} = (\Delta_1 = 0, \Delta_2 = 0.5T)$, $\sigma^2_{\Delta_i} = \sigma^2_{\Delta_k} = \lambda_{\Delta_{i,k}} = 0$

$$\left|H_\Sigma(j\omega \mid \Delta_1, \Delta_2)\right|^2 = 2 + 2\sum_{i=1}^{1}\sum_{k>i}^{2}\cos[\omega(\Delta_2 - \Delta_1)] = 2 + 2\cos[\omega(\Delta_2 - \Delta_1)]$$

$$I(\omega \mid \Delta_1, \Delta_2) = I_0(\omega)\left|H_\Sigma(j\omega)\right|^2 \propto \frac{\sin^2(\omega T/2)}{(\omega T/2)^2}(2 + 2\cos[\omega(\Delta_2 - \Delta_1)])$$



Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.

Delay-averaged Power Spectrum: Conventional derivation for zero-mean, SI, equal-variance, Normal random delays:

Let Δ_i and Δ_k be SI $\forall i \neq k$, and let $p(\Delta) = \frac{1}{\sqrt{2\pi\sigma_\Delta^2}} e^{-\Delta^2/2\sigma_\Delta^2}$, where σ_Δ^2 is the variance of each.

$$\overline{I(\omega\,|\,\boldsymbol{\Delta})}^{L,\Delta} = \overline{|S_0(\omega)|^2}^{L} \overline{\left|\sum_{i=1}^{N} e^{j\omega\Delta_i(t)}\right|^2}^{\Delta} = \overline{|S_0(\omega)|^2}^{L} \left(N + \overline{\sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq i}}^{N} e^{j\omega\Delta_i(t)} e^{-j\omega\Delta_k(t)}}^{\Delta} \right) = \overline{|S_0(\omega)|^2}^{L} \left(N + \sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq i}}^{N} \overline{e^{j\omega\Delta_i(t)}}^{\Delta}\, \overline{e^{-j\omega\Delta_k(t)}}^{\Delta} \right)$$

Averaging with respect to the Gaussian density yields:

$$\overline{e^{\pm j\omega\Delta}} = \frac{1}{\sqrt{2\pi\sigma_\Delta^2}} \int_{-\infty}^{\infty} e^{\pm j\omega\Delta} e^{-\Delta^2/2\sigma_\Delta^2} d\Delta = \frac{1}{\sqrt{2\pi\sigma_\Delta^2}} \int_{-\infty}^{\infty} \cos(\omega\Delta) e^{-\Delta^2/2\sigma_\Delta^2} d\Delta + \frac{j}{\sqrt{2\pi\sigma_\Delta^2}} \int_{-\infty}^{\infty} \underbrace{\sin(\pm\omega\Delta)}_{0} e^{-\Delta^2/2\sigma_\Delta^2} d\Delta = e^{-\omega^2\sigma_\Delta^2/2}$$

Substituting, the average Uplink Array Power Spectrum is given by:

$$\overline{I(\omega\,|\,\boldsymbol{\Delta})}^{L,\Delta} = \overline{|S_0(\omega)|^2}^{L} \left(N + \sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq 1}}^{N} \overline{e^{j\omega\Delta_k}}^{\Delta}\, \overline{e^{-j\omega\Delta_i}}^{\Delta} \right) = \overline{|S_0(\omega)|^2}^{L} \left(N + N(N-1)\, e^{-\omega^2\sigma_\Delta^2} \right)$$

Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.

Power Spectrum of Uplink Array Signals

Jet Propulsion Laboratory
California Institute of Technology

$$\overline{I(\omega\,|\,\mathbf{\Delta})}^{L,\mathbf{\Delta}} = \overline{|S_0(\omega)|^2}^{L}\left(N+\sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq 1}}^{N} M_{\mathbf{\Delta}_{i,k}}(\boldsymbol{\omega})\right) = \overline{|S_0(\omega)|^2}^{L}\left(N+2\sum_{i=1}^{N-1}\sum_{k>i}^{N}\exp[-\tfrac{1}{2}\omega^2(\sigma_{\theta_i}^2-2\lambda_{\theta_{i,k}}+\sigma_{\theta_k}^2)]\cos[\omega(\overline{\Delta}_i-\overline{\Delta}_k)]\right)$$

1. Equal mean values:

$$\text{If } \overline{\Delta}_i=\overline{\Delta}_k \quad \forall i,k \quad \text{then } (\overline{\Delta}_i-\overline{\Delta}_k)=0,\ \ M_{\mathbf{\Delta}_{i,k}}(\boldsymbol{\omega})|_{\omega_1=\omega_2=\omega}=\exp[-\tfrac{1}{2}(\sigma_{\Delta_i}^2+\sigma_{\Delta_k}^2-2\lambda_{\Delta_{i,k}})] \text{ and}$$

$$\overline{I(\omega\,|\,\mathbf{\Delta})}^{L,\mathbf{\Delta}} = \overline{|S_0(\omega)|^2}^{L}\left(N+2\sum_{i=1}^{N-1}\sum_{k>i}^{N}\exp[-\tfrac{1}{2}\omega^2(\sigma_{\Delta_k}^2+\sigma_{\Delta_i}^2-2\lambda_{\Delta_{k,i}})]\right)$$

2. Uncorrelated phases:

$$\text{If } \lambda_{\Delta_{i,k}}=E(\Delta_i\Delta_k)=0 \quad \forall i,k \quad \text{then} \quad M_{\Delta_{i,k}}(\omega)=\exp[-\tfrac{1}{2}\omega^2(\sigma_{\Delta_i}^2+\sigma_{\Delta_k}^2)] \text{ and}$$

$$\overline{I(\omega\,|\,\mathbf{\Delta})}^{L,\mathbf{\Delta}} = \overline{|S_0(\omega)|^2}^{L}\left(N+2\sum_{i=1}^{N-1}\sum_{k>i}^{N}\exp[-\tfrac{1}{2}\omega^2(\sigma_{\Delta_k}^2+\sigma_{\Delta_i}^2)]\right)$$

3. Equal variances:

$$\text{If } \sigma_{\Delta_i}^2=\sigma_{\Delta_k}^2=\sigma_{\Delta}^2 \quad \forall i,k \quad \text{then} \quad M_{\Delta_{i,k}}(\omega)=\exp[-\omega^2\sigma_{\Delta}^2] \text{ and}$$

$$\overline{I(\omega\,|\,\mathbf{\Delta})}^{L,\mathbf{\Delta}} = \overline{|S_0(\omega)|^2}^{L}\left(N+2\sum_{i=1}^{N-1}\sum_{k>i}^{N}\exp[-\omega^2\sigma_{\Delta}^2]\right) = \overline{|S_0(\omega)|^2}^{L}\left(N+N(N-1)e^{-\omega^2\sigma_{\Delta}^2}\right)$$

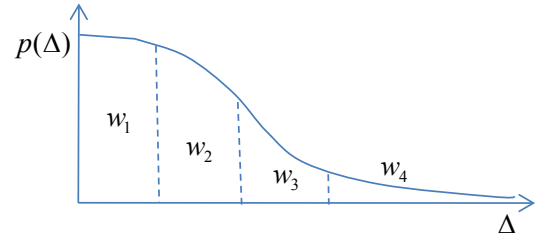
4. Deterministic delays (zero-variance case):

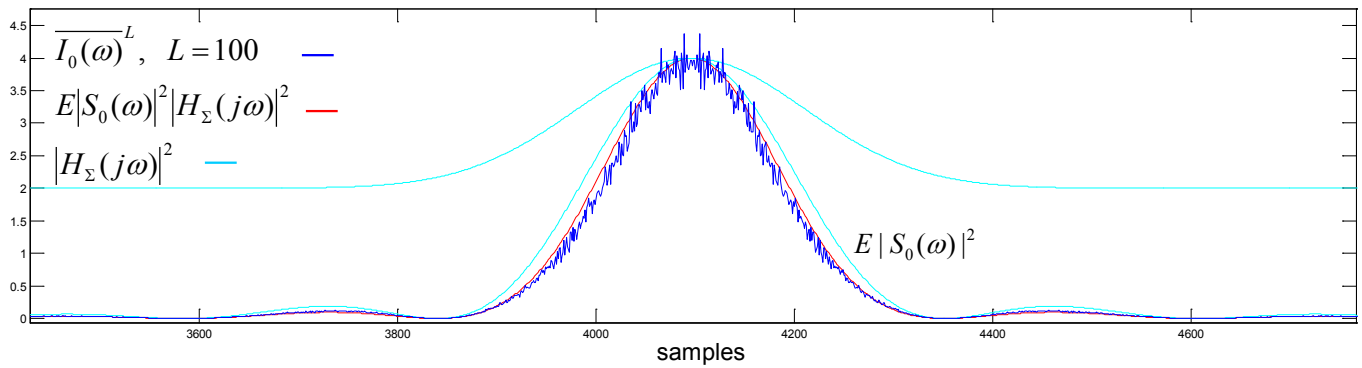
$$\text{If } \sigma_{\Delta_i}^2=\sigma_{\Delta_k}^2=\lambda_{\Delta_{i,k}}=0 \quad \forall i,k \quad \text{then} \quad M_{\Delta_{i,k}}(\omega)=2\cos[\omega(\overline{\Delta}_i-\overline{\Delta}_k)] \text{ and } \overline{I(\omega\,|\,\mathbf{\Delta})}^{\mathbf{\Delta}}=|S_0(\omega)|^2\left(N+2\sum_{i=1}^{N-1}\sum_{k>i}^{N}\cos[\omega(\overline{\Delta}_i-\overline{\Delta}_k)]\right)$$

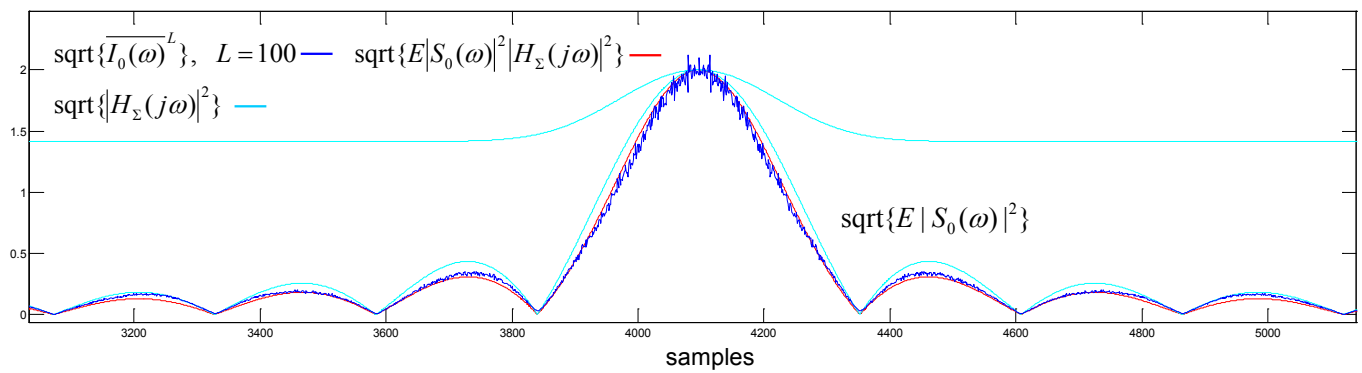
Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.

Example 3: Gaussian-integral (Erf) weighted power spectra with 4 delays used for simulation

$$\begin{bmatrix} \Delta_1 - \Delta_2 = 0 \\ \Delta_1 - \Delta_2 = T/3 \\ \Delta_1 - \Delta_2 = 2T/3 \\ \Delta_1 - \Delta_2 = T \end{bmatrix}, \quad \begin{bmatrix} w_1 = erf(1/2\sqrt{2}) = 0.3827 \\ w_2 = erf(1/\sqrt{2}) - erf(1/2\sqrt{2}) = 0.6827 - 0.3827 = 0.3 \\ w_3 = erf(3/2\sqrt{2}) - erf(1/\sqrt{2}) = 0.8664 - 0.6827 = 0.1837 \\ w_4 = 1.0 - 0.8664 = erf(\infty) - erf(3/2\sqrt{2}) = 0.1336 \end{bmatrix}$$

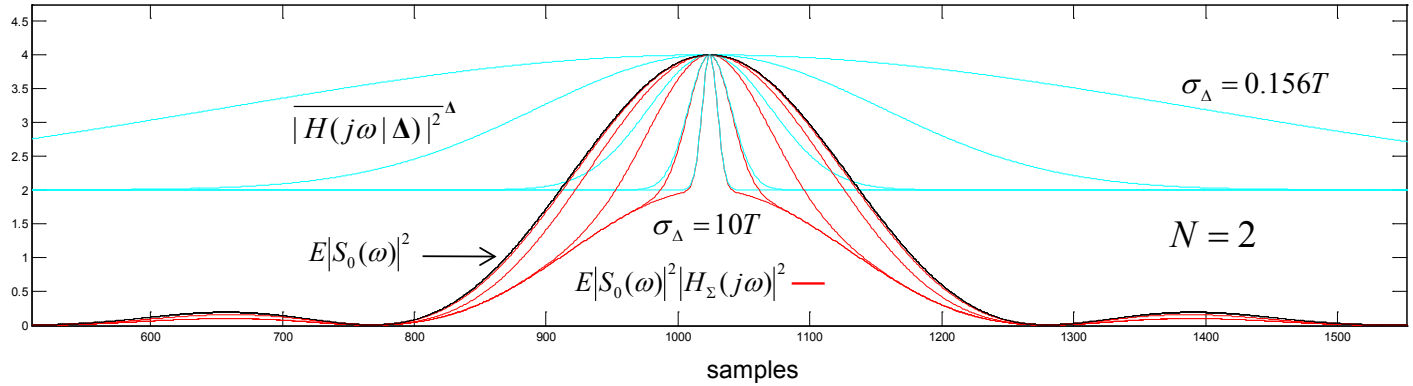


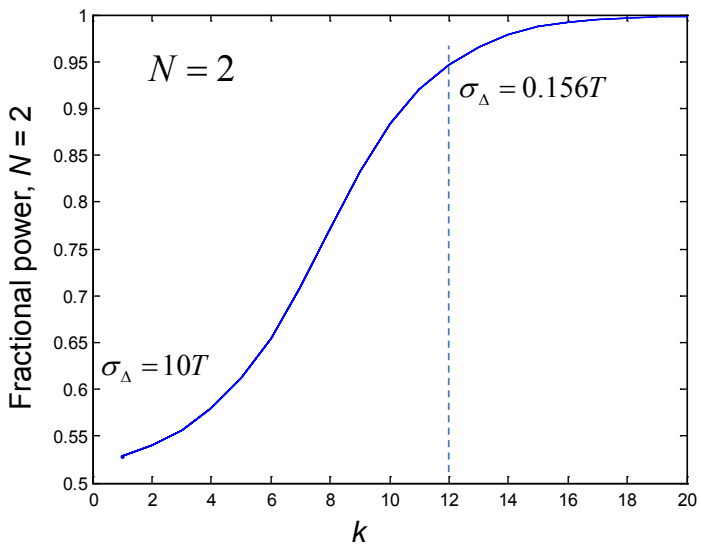




Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.

Example 4: Impact of delay error on the Power Spectral Density of Uplink Array Signals





Varied σ_Δ^2 according to the equation

$$\sigma_\Delta^2 = 100T^2 * 8^{-(k-1)}, \quad 1 < k \le 5$$

yielding standard deviations:

$$\sigma_\Delta = 10T, \quad 3.54T, \quad 1.25T, \quad 0.442T, \quad 0.156T$$

Severe decrease in BW of coherent component, as the delay-error standard deviation increases

Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.

Statistically Independent delay and phase vectors

$$s(t\,|\,\boldsymbol{\theta},\boldsymbol{\Delta}) = \sqrt{P}\sum_i e^{j\theta_i} s_0(t-\Delta_i) = \sqrt{P}\, s_0(t) * \sum_{i=1}^{N} e^{j\theta_i}\delta(t-\Delta_i) = \sqrt{P}\, s_0(t) * \tilde{h}_\Sigma(t)$$

$$H_\Sigma(j\omega\,|\,\boldsymbol{\theta},\boldsymbol{\Delta}) \equiv \sum_i \mathcal{F}\{h_i(t\,|\,\theta_i,\Delta_i)\} = \sum_i \tilde{H}_i(j\omega\,|\,\theta_i,\Delta_i) = \sum_{i=1}^{N} e^{j\theta_i} e^{-j\omega\Delta_i} = \sum_{i=1}^{N} e^{-j(\omega\Delta_i-\theta_i)}$$

assuming that $\boldsymbol{\Delta}$ and $\boldsymbol{\theta}$ are SI,

$$\overline{\left|\tilde{H}_\Sigma(j\omega\,|\,\boldsymbol{\theta},\boldsymbol{\Delta})\right|^2}^{\boldsymbol{\Delta}} = N + \sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq i}}^{N} e^{j(\theta_i-\theta_k)}\,\overline{e^{-j\omega(\Delta_i-\Delta_k)}}^{\Delta} = N + \sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq i}}^{N} e^{j(\theta_i-\theta_k)} M_{\Delta_{i,k}}(\boldsymbol{\omega})$$

$$\overline{\left|\tilde{H}_\Sigma(j\omega\,|\,\boldsymbol{\theta},\boldsymbol{\Delta})\right|^2}^{\boldsymbol{\Delta},\boldsymbol{\theta}} = N + \sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq i}}^{N} \overline{e^{j(\theta_i-\theta_k)}}^{\boldsymbol{\theta}} M_{\Delta_{i,k}}(\boldsymbol{\omega}) = N + \sum_{i=1}^{N}\sum_{\substack{k=1\\k\neq i}}^{N} M_{\boldsymbol{\varphi}_{i,k}}(\mathbf{v}=\mathbf{1}) M_{\Delta_{i,k}}(\boldsymbol{\omega})$$

$$\boxed{\overline{\left|\tilde{H}_\Sigma(j\omega\,|\,\boldsymbol{\theta},\boldsymbol{\Delta})\right|^2}^{\boldsymbol{\Delta},\boldsymbol{\theta}} = N + 2\sum_{i=1}^{N-1}\sum_{k>i}^{N} e^{-\frac{1}{2}(\sigma_{\theta_i}^2+\sigma_{\theta_k}^2-2\lambda_{\theta_{i,k}})-\frac{1}{2}\omega^2(\sigma_{\Delta_i}^2-2\lambda_{\Delta_{i,k}}+\sigma_{\Delta_k}^2)]} \cos(\overline{\theta}_i-\overline{\theta}_k)\cos[\omega(\overline{\Delta}_i-\overline{\Delta}_k)]}$$

Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.

Backup Slides

Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.

Diagonalization of the Covariance matrix:

$$\mathbf{\Lambda} = \begin{bmatrix} \sigma^2 & \lambda \\ \lambda & \sigma^2 \end{bmatrix}$$

The correlated delays can be decorrelated via a simple unitary transformation (rotation). The columns of the unitary matrix are the eigenvectors of the covariance matrix, which can be determined by first solving for the eigenvalues:

$$\det[\mathbf{\Lambda} - \alpha\,\mathbf{I}] = \begin{vmatrix} \sigma^2 - \alpha, & \lambda \\ \lambda & \sigma^2 - \alpha \end{vmatrix} = 0; \quad (\sigma^2 - \alpha)^2 - \lambda^2 = 0; \quad (\sigma^2 - \alpha)^2 = \lambda^2; \quad \sigma^2 - \alpha = \pm\lambda$$

The two eigenvalues are: $\alpha_1 = \sigma^2 - \lambda, \quad \alpha_2 = \sigma^2 + \lambda.$ Let the two orthogonal eigenvectors belonging to the two distinct eigenvalues be designated as $\boldsymbol{\beta}_1 = (\beta_{11}, \beta_{12})^T$ and $\boldsymbol{\beta}_2 = (\beta_{21}, \beta_{22})^T$, and solve by recalling that eigenvectors of a covariance matrix are merely scaled, not altered:

$$\begin{bmatrix} \sigma^2 & \lambda \\ \lambda & \sigma^2 \end{bmatrix}\begin{bmatrix} \beta_{11} \\ \beta_{12} \end{bmatrix} = \begin{bmatrix} (\sigma^2 + \lambda)\beta_{11} \\ (\sigma^2 + \lambda)\beta_{12} \end{bmatrix}, \quad \begin{bmatrix} \sigma^2\beta_{11} + \lambda\beta_{12} \\ \lambda\beta_{11} + \sigma^2\beta_{12} \end{bmatrix} = \begin{bmatrix} (\sigma^2 + \lambda)\beta_{11} \\ (\sigma^2 + \lambda)\beta_{12} \end{bmatrix} \rightarrow \beta_{11} = \beta_{12}$$

$$\begin{bmatrix} \sigma^2 & \lambda \\ \lambda & \sigma^2 \end{bmatrix}\begin{bmatrix} \beta_{11} \\ \beta_{12} \end{bmatrix} = \begin{bmatrix} (\sigma^2 - \lambda)\beta_{11} \\ (\sigma^2 - \lambda)\beta_{12} \end{bmatrix}, \quad \begin{bmatrix} \sigma^2\beta_{11} + \lambda\beta_{12} \\ \lambda\beta_{11} + \sigma^2\beta_{12} \end{bmatrix} = \begin{bmatrix} (\sigma^2 - \lambda)\beta_{11} \\ (\sigma^2 - \lambda)\beta_{12} \end{bmatrix} \rightarrow \beta_{11} = -\beta_{12}$$

Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.

Therefore, after normalization, $\boldsymbol{\beta}_1 = 2^{-1/2}(1,1)^T$, $\boldsymbol{\beta}_2 = 2^{-1/2}(1,-1)^T$, and the required unitary matrix becomes:

$$\mathbf{U} = 2^{-1/2}\begin{bmatrix}\boldsymbol{\beta}_1^T & \boldsymbol{\beta}_2^T\end{bmatrix} = 2^{-1/2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \mathbf{U}^T = \mathbf{U}^{-1}.$$

Applying this transformation to the original correlated delays yields:

$$\mathbf{U}\boldsymbol{\theta}^T = 2^{-1/2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = 2^{-1/2}\begin{bmatrix}\theta_1+\theta_2 \equiv \delta_1\\ \theta_1-\theta_2 \equiv \delta_2\end{bmatrix} \equiv \boldsymbol{\delta}^T,$$

whose components are indeed uncorrelated Gaussian random delays:

$$E\{\delta_1\delta_2\} = \tfrac{1}{2}E\{(\theta_1+\theta_2)(\theta_1-\theta_2)\} = \tfrac{1}{2}E\{\theta_1^2-\theta_2^2\} = \tfrac{1}{2}(\sigma_{\theta_1}^2-\sigma_{\theta_2}^2) = 0$$
$$E\{\delta_1^2\} = \tfrac{1}{2}E\{(\theta_1+\theta_2)^2\} = \tfrac{1}{2}E\{\theta_1^2+\theta_2^2+2\theta_1\theta_2\} = \sigma_{\theta_1}^2+\lambda$$
$$E\{\delta_2^2\} = \tfrac{1}{2}E\{(\theta_1-\theta_2)^2\} = \tfrac{1}{2}E\{\theta_1^2+\theta_2^2-2\theta_1\theta_2\} = \sigma_{\theta_2}^2-\lambda$$

$$\boldsymbol{\Lambda}^* \equiv \begin{bmatrix}\sigma^2+\lambda & 0\\ 0 & \sigma^2-\lambda\end{bmatrix}$$

Alternately, applying the rotation operation to the covariance matrix directly, we get:

$$2^{-1/2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\sigma^2 & \lambda\\ \lambda & \sigma^2\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}2^{-1/2} = 2^{-1}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\sigma^2+\lambda & \sigma^2-\lambda\\ \lambda+\sigma^2 & \lambda-\sigma^2\end{bmatrix} = 2^{-1}\begin{bmatrix}2(\sigma^2+\lambda) & 0\\ 0 & 2(\sigma^2-\lambda)\end{bmatrix} = \boldsymbol{\Lambda}^*$$

Note: applying the rotation operation to the diagonal covariance matrix restores the original correlations:

$$2^{-1/2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\sigma^2+\lambda & 0\\ 0 & \sigma^2-\lambda\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}2^{-1/2} = 2^{-1}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\sigma^2+\lambda & \sigma^2+\lambda\\ \sigma^2-\lambda & \lambda-\sigma^2\end{bmatrix} = 2^{-1}\begin{bmatrix}2\sigma^2 & 2\lambda\\ 2\lambda & 2\sigma^2\end{bmatrix} = \boldsymbol{\Lambda}$$

Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.

With $\mathbf{\Lambda}^* = \begin{bmatrix} \sigma^2 + \lambda & 0 \\ 0 & \sigma^2 - \lambda \end{bmatrix}$, $M_{\boldsymbol{\theta}}(\mathbf{v} = \mathbf{1}) = \exp(-\tfrac{1}{2}\mathbf{A}\mathbf{\Lambda}^*\mathbf{A}^T + j\,\mathbf{A}\,\overline{\boldsymbol{\theta}}^T) =$

$$\exp\{-\tfrac{1}{2}[(\sigma_1^2 + \lambda) + (\sigma_2^2 - \lambda)] + j\omega(\theta_1 - \theta_2)\} = \exp[-\tfrac{1}{2}(\sigma_1^2 + \sigma_2^2) + j(\overline{\theta}_1 - \overline{\theta}_2)]$$

which is identical to the uncorrelated case.

If the phase error is caused primarily by tropospheric variations, we can think of the unitary transformation as **moving the array elements further apart**, thus causing decorrelation of delays and phases and hence degrading array performance. Equivalently, applying the unitary transformation to a diagonal covariance matrix (representing large array diameter) could be thought of as decreasing the size of the array. This notion applies most directly to 2 and 3 element arrays, but the concept also applies to arrays consisting of a large number of elements.

Copyright 2010 California Institute of Technology. Government sponsorship acknowledged.