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**SUBJECT:** Small Body GN&C Research Report: A Robust Model Predictive Control Algorithm with Guaranteed Resolvability

## **ABSTRACT**

A robustly stabilizing MPC (model predictive control) algorithm for uncertain nonlinear systems is developed that guarantees the resolvability of the associated finite-horizon optimal control problem in a receding-horizon implementation. The control consists of two components; (i) feed-forward, and (ii) feedback part. Feed-forward control is obtained by online solution of a finite-horizon optimal control problem for the nominal system dynamics. The feedback control policy is designed off-line based on a bound on the uncertainty in the system model. The entire controller is shown to be robustly stabilizing with a region of attraction composed of initial states for which the finite-horizon optimal control problem is feasible. The controller design for this algorithm is demonstrated on a class of systems with uncertain nonlinear terms that have norm-bounded derivatives, and derivatives in polytopes. An illustrative numerical example is also provided.

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# 1 Introduction

The incorporation of feedback with open-loop guidance enables the ability to precisely track desired trajectories, even in the presence of unknown disturbances and nonlinearities in the dynamics used to compute those open-loop trajectories. The ability to resolve for the open-loop trajectory during a trajectory maneuver allows for model updates to be incorporated into the problem such that the open-loop trajectory becomes more optimal. This allows for separate inner- and outer-loop receding horizon implementations to achieve a degree of robustness.

The existence of an initially feasible open-loop guidance solution provides a framework from which the feedback control can both be developed and provide information on the invariant ellipsoids (tubes) that bound the propagation of the actual state about the nominal desired state. With this knowledge, guarantees are given on both the convergence of the controller trajectory, as well as the resolvability of the open-loop portion through the technique presented herein. This resolvability provides a means of implementing receding-horizon on the open-loop guidance to account for real-time knowledge of model parameters. In addition, the implementation of optimal control for the guidance and feedback makes the receding-horizon implementation achieve a sense of optimality otherwise impacted by uncertainty in models and parameters.

## 2 Description of the System and MPC Control Algorithm

Consider an uncertain nonlinear dynamical system with state  $x \in \mathbb{R}^n$  and control  $u \in \mathbb{R}^m$  as

$$\dot{x} = f(x, u, t) \tag{1}$$

where  $\mathbf{X} \subseteq \mathbb{R}^n$  and  $\mathbf{U} \subseteq \mathbb{R}^m$  are sets defining the state and control constraints, respectively. We will refer to dynamics in (1) as the *real dynamics*. Our objective is to obtain a control input  $u(\cdot)$  such that the closed loop system for (1) is *asymptotically stable about the origin*,  $x = 0$ , with a *region of attraction*  $\mathcal{R}_a \subseteq \mathbf{X}$  such that

$$x(t) \in \mathbf{X}, u(t) \in \mathbf{U}, \forall t \geq t_0, \quad \text{when } x(t_0) \in \mathcal{R}_a. \tag{2}$$

We propose a model predictive control (MPC) approach where the control input is composed of two components:

- Feed-forward control input,  $u_o$ ,
- Feedback control input,  $u_f$ ,

such that

$$u(t) = u_o(t) + u_f(t). \tag{3}$$

Suppose that there exist sets  $\mathbf{X}_o$ ,  $\mathbf{X}_f$  and  $\mathbf{U}_o$ ,  $\mathbf{U}_f$ , which all contain the origin of the respective space they belong to, such that<sup>2</sup>

$$\begin{aligned} \mathbf{X}_o + \mathbf{X}_f &\subseteq \mathbf{X}, \\ \mathbf{U}_o + \mathbf{U}_f &\subseteq \mathbf{U}. \end{aligned} \tag{4}$$

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<sup>2</sup>For sets  $A$  and  $B$ ,  $C = A + B$  implies that: If  $a \in A$  and  $b \in B$  then  $a + b \in C$ .

The feed-forward component of the control input will be determined by online solution of a finite horizon optimal control problem, and the feedback part will be determined off-line as a control policy.

For feed-forward control design, knowledge of the following nominal plant model is assumed:

$$\dot{z} = F(z, u_o, t), \quad (5)$$

where  $F$  is an approximate model of  $f$  in (1) and  $z \in \mathbb{R}^n$  is the state driven by this control through the nominal model dynamics. Model (5) is used to solve the following finite-horizon optimal control problem (FHC):

**FHC**

$$\min_{u_o(\cdot)} J(u_o; t_s, T, z(t_s)) = \int_{t_s}^{t_s+T} h(z(\tau), u_o(\tau)) d\tau + V(z(t_s + T))$$

$$\text{subject to } \begin{cases} \dot{z} = F(z, u_o, t) \\ z(t) \in \mathbf{X}_o, \quad \forall t \in [t_s, t_s + T] \\ u_o(t) \in \mathbf{U}_o, \quad \forall t \in [t_s, t_s + T] \\ z(t_s + T) \in \Omega_o \\ x(t_s) - z(t_s) \in \mathbf{X}_f \end{cases}$$

where  $x(t_s)$  is the state of the real system dynamics (1).

FHC is a typical problem solved in model predictive control except for the last constraint given by

$$x(t_s) - z(t_s) \in \mathbf{X}_f, \quad (6)$$

which is a relaxation on the initial state of FHC. This relaxation together with a feedback control  $u_f$  will explicitly guarantee resolvability of FHC and lead to a robustly stabilizing controller.  $\square$

The following assumptions are instrumental for the stability proof. In Section 3 we present conditions for a class of uncertain nonlinear systems that ensure the satisfaction of these assumptions.

**Assumption 1.** *Function  $h$  in FHC satisfies the following:*

$$h(z, u) \geq a\|z\|^p + b\|u\|^r, \quad \forall z, u, \quad (7)$$

with  $p \geq 1$ ,  $r \geq 0$ , and  $a$  and  $b$  both positive constants, and  $h(0, 0) = 0$ .  $\square$

**Assumption 2.** *Function  $V$  in FHC is positive definite [6] and there exists a feedback control law  $u = \mathcal{L}(x)$  and  $u_o = \mathcal{L}(z)$  such that  $V$  defines a Lyapunov function for (1) and (5) satisfying*

$$\nabla V(x)f(x, \mathcal{L}(x), t) + h(x, \mathcal{L}(x)) \leq 0, \quad \forall x \in \Omega_o, \quad (8)$$

$$\nabla V(z)F(x, \mathcal{L}(z), t) + h(z, \mathcal{L}(z)) \leq 0, \quad \forall z \in \Omega_o, \quad (9)$$

where  $\Omega_o \subset \mathbf{X}_o$  contains the origin. Additionally, feedback law  $\mathcal{L}$  renders  $\Omega_o \subset \mathbb{R}^n$  invariant for dynamics (1) and (5), i.e., if  $x(t_0) \in \Omega_o$  ( $z(t_0) \in \Omega_o$ ) for some  $t_0$ , then  $x(t) \in \Omega_o \quad \forall t \geq t_0$  ( $z(t) \in \Omega_o \quad \forall t \geq t_0$ ). It is also assumed that

$$\mathcal{L}(x) \in \mathbf{U}_o, \quad \forall x \in \Omega_o. \quad (10)$$

$\square$

**Assumption 3.** *There exist closed balls<sup>3</sup> around the origin  $\mathbf{B}_R$  and  $\mathbf{B}_r$  in  $\mathbb{R}^n$  with radii  $R > r$  such that set  $\Omega_o$  in FHC satisfies the following*

$$\mathbf{X}_f \subseteq \mathbf{B}_r \subset \mathbf{B}_R \subseteq \Omega_o. \quad (11)$$

□

**Assumption 4.** *There exists a feedback control law  $u_f = \mathcal{K}(x, z)$  in (3) that renders the set  $\mathbf{X}_f$  invariant for  $\eta \triangleq x - z$  with dynamics (1) for  $x$  and (5) for  $z$ , that is, if  $\eta(t_0) \in \mathbf{X}_f$  for some  $t_0 \geq 0$ , then  $\eta(t) \in \mathbf{X}_f \quad \forall t \geq t_0$  and for all  $u_o(\cdot)$ . Additionally,  $u_f = \mathcal{K}(x, z) \in \mathbf{U}_f$  if  $\eta(t) \in \mathbf{X}_f$ . □*

The following algorithm defines the MPC approach used in this paper:

### MPC Algorithm

Begin with  $k = 0$  and iterate the following steps over computation times  $t_k$  for  $k \in \mathbb{Z}^+$ .

1. Obtain/measure the state  $x(t_k)$  of the real dynamics (1).
2. Solve FHC at time  $t_s = t_k$  with  $T = T_k$  and obtain  $u_{o,k}$  with  $u_o(t) = u_{o,k}(t)$  on  $t \in [t_k, t_k + T_k]$ .
3. Apply  $u = u_{o,k} + u_f$  to the real system (1) and  $u_o$  to nominal system (5) to obtain  $x$  and  $z_k$ , with  $z(t) = z_k(t)$  on  $[t_k, t_{k+1}]$ , where  $u_f = \mathcal{K}(x, z)$ .
4. If  $z(\bar{t}) \in \Omega_o$  for some  $\bar{t} \geq 0$ , then  $u_o = \mathcal{L}(z), \forall t \geq \bar{t}$ .
5. If  $x(\bar{t}) \in \Omega_o$  for some  $\bar{t} \geq 0$ , then  $u = \mathcal{L}(x), \forall t \geq \bar{t}$ .

**Lemma 1** (Resolvability of FHC). *Suppose that FHC is feasible at  $t_0$  with  $T_0$ , and let  $t_k$  for  $k \in \mathbb{Z}^+$  be the times that a solution of FHC is computed. Then, the feasibility of FHC is guaranteed at  $t_k$  with  $T_k \geq T_{k-1} - \delta_k, \forall k \in \mathbb{Z}^+, \delta_k = t_k - t_{k-1}, 0 \leq \delta_k < T_{k-1}$  provided assumptions 2, and 4 hold. □*

*Proof.* Suppose at  $t_{k-1}$  FHC is feasible with  $T_{k-1}$  and  $u_{o,k-1}(t)$  for  $t \in [t_{k-1}, t_{k-1} + T_{k-1}]$ , and  $z_{k-1}(\cdot)$  is the corresponding state trajectory of (5). Let  $t_k = t_{k-1} + \delta_k$  and resolve FHC. Note,  $u_{o,k}(t) = u_{o,k-1}(t)$  for  $t \in [t_k, t_{k-1} + T_{k-1}]$  is one feasible solution of FHC with  $T_k = T_{k-1} - \delta_k$  since  $x(t_k) - z_{k-1}(t_k) \in \mathbf{X}_f$  is invariant due to Assumption 4 (i.e.  $z_k(t_k) = z_{k-1}(t_k)$  is the initial state of a feasible trajectory).

Now, we show that we can extend this feasible trajectory on  $[t_k, t_{k-1} + T_{k-1}]$  to  $[t_k, t_k + T_k]$  for any given  $T_k \geq T_{k-1} - \delta_k$  by considering the following control input,

$$u_{o,k}(t) = \begin{cases} u_{o,k-1}(t), & t \in [t_k, t_{k-1} + T_{k-1}]; \\ u_{o,k}(t) = \mathcal{L}(z(t)), & t \in [t_{k-1} + T_{k-1}, t_k + T_k]. \end{cases} \quad (12)$$

which follows from Assumption 2 by noting that  $z_k(t_{k-1} + T_{k-1}) = z_{k-1}(t_{k-1} + T_{k-1}) \in \Omega_o$ . Consequently (12) defines a feasible trajectory on  $[t_k, t_k + T_k]$ , implying that FHC has a feasible solution at  $t_k$  with time horizon  $T_k$ , once it is feasible at  $t_{k-1}$  with  $T_{k-1}$ . Now, we can conclude the proof by using induction. □

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<sup>3</sup> $\mathbf{B}_\rho \triangleq \{v : \|v\| \leq \rho\}$ .

**Lemma 2** (Shrinking Optimal Cost with Receding Horizon). *Let  $t_k, k \in \mathbb{Z}^+$  be the computation times of FHC satisfying  $\inf_k(t_k - t_{k-1}) \geq \epsilon$  for some  $\epsilon > 0$ . Suppose that FHC is feasible for some  $t_{k-1}$  with  $T_{k-1}$  and an optimal cost of  $J_{k-1}^*$ , and assumptions 1, 2, 3, and 4 hold. Then, FHC is feasible for  $t_k$  with  $T_k \in [T_{k-1} - \delta_k, T_{k-1}]$ , and if  $z_{k-1}(t_{k-1}) \notin \Omega_o$  and  $z_{k-1}(t_k) \notin \Omega_o$  the optimal cost satisfies*

$$J_k^* - J_{k-1}^* \leq -\beta, \quad \text{for some } \beta > 0. \quad (13)$$

□

*Proof.* Since FHC is feasible at  $t_{k-1}$  with  $T_{k-1}$  and  $u_{o,k-1}(\cdot)$  provides the optimal cost  $J_{k-1}^*$ ,  $u_{o,k-1}(\cdot)$  can also be used to provide a feasible solution for FHC at  $t_k$  with  $T_k \in [T_{k-1} - \delta_k, T_{k-1}]$  by using (12) as in the proof of Lemma 1. So,  $z_k(t) = z_{k-1}(t)$  is a feasible trajectory for  $t \in [t_k, t_{k-1} + T_{k-1}]$ . We will show that (13) is satisfied with  $T_k = T_{k-1}$  which will directly imply that (13) is satisfied with  $T_k \in [T_{k-1} - \delta_k, T_{k-1}]$  from the construction of the proof.

The cost with control input (12) and  $T_k = T_{k-1}$  is

$$J_k = \int_{t_k}^{t_{k-1}+T_{k-1}} h(z_{k-1}(\tau), u_{o,k-1}(\tau))d\tau + \int_{t_{k-1}+T_{k-1}}^{t_k+T_{k-1}} h(z_k(\tau), u_{o,k}(\tau))d\tau + V(z_k(t_k + T_{k-1})). \quad (14)$$

Since

$$J_{k-1}^* = \int_{t_{k-1}}^{t_k} h(z_{k-1}(\tau), u_{o,k-1}(\tau))d\tau + \int_{t_k}^{t_{k-1}+T_{k-1}} h(z_{k-1}(\tau), u_{o,k-1}(\tau))d\tau + V(z_{k-1}(t_{k-1} + T_{k-1})), \quad (15)$$

we have

$$\begin{aligned} J_k - J_{k-1}^* &= \int_{t_{k-1}+T_{k-1}}^{t_k+T_{k-1}} h(z_k(\tau), u_{o,k}(\tau))d\tau - \int_{t_{k-1}}^{t_k} h(z_{k-1}(\tau), u_{o,k-1}(\tau))d\tau \\ &\quad + V(z_k(t_k + T_{k-1})) - \underbrace{V(z_{k-1}(t_{k-1} + T_{k-1}))}_{=z_k(t_{k-1}+T_{k-1})}. \end{aligned} \quad (16)$$

Assumption 2 implies the following with  $u_{o,k}(t) = \mathcal{L}(z_k(t))$  on  $t \in [t_{k-1} + T_{k-1}, t_k + T_{k-1}]$

$$\int_{t_{k-1}+T_{k-1}}^{t_k+T_{k-1}} \dot{V}(z_k(\tau))d\tau + \int_{t_{k-1}+T_{k-1}}^{t_k+T_{k-1}} h(z_k(\tau), u_{o,k}(\tau))d\tau \leq 0$$

implies

$$V(z_k(t_k + T_{k-1})) - V(z_k(t_{k-1} + T_{k-1})) + \int_{t_{k-1}+T_{k-1}}^{t_k+T_{k-1}} h(z_k(\tau), u_{o,k}(\tau))d\tau \leq 0 \quad (17)$$

Given assumptions 1 and 3, if  $\|z\| \geq R$ , then  $h(z, u_o) \geq a\|z\|^p \geq aR^p$  since  $p \geq 1$ . Since  $\delta_k \geq \epsilon > 0$ , then for  $t \in [t_{k-1}, t_k]$ ,

$$\int_{t_{k-1}}^{t_k} h(z_{k-1}(\tau), u_{o,k-1}(\tau))d\tau \geq \underbrace{aR^p \epsilon}_{\beta} > 0 \quad (18)$$

Combining equations (16)-(18) shows  $J_k - J_{k-1}^* \leq -\beta < 0$ , and since  $J_k^* \leq J_k$ , then

$$J_k^* - J_{k-1}^* \leq -\beta < 0. \quad (19)$$

□

The following theorem is the main result of this section.

**Theorem 1.** *Consider system (1) with a control input described by the MPC Algorithm. Suppose that assumptions 1-4 are satisfied. Then, the resulting closed loop system is asymptotically stable with a region of attraction  $\mathcal{R}_a$ ,*

$$\mathcal{R}_a = \{\xi \in \mathbb{R}^n : \text{FHC is feasible with } x(t_s) = \xi\}. \quad (20)$$

□

*Proof.* Given the MPC Algorithm, and  $x(t_0)$  such that FHC is feasible with some  $T_0$ . Suppose that  $z_{k-1}(t_{k-1}) \notin \Omega_o$  and  $z_{k-1}(t_k) \notin \Omega_o$  for some  $k \in \mathbb{Z}^+$ . This implies that  $z_{k-1}(t) \notin \Omega_o$  and (13) holds true. Since the cost is reduced by at least a finite positive amount,  $J_k^* \rightarrow -\infty$  as  $k \rightarrow \infty$  if nominal trajectory  $z$  does not enter into  $\Omega_o$ . This implies that there exists a finite time  $\tilde{t} \geq 0$  for each trajectory such that  $z(t) \in \Omega_o, \forall t \geq \tilde{t}$ . Consequently, using Assumption 2, the closed loop nominal system (5) is asymptotically to the origin when  $x(t_0) \in \mathcal{R}_a$ . Therefore, there exists some time  $\bar{t} \geq \tilde{t} \geq 0$  such that  $\|z(t)\| \leq R - r$  where  $R > r > 0$  are as defined in Assumption 3. This leads to

$$\|x(t)\| \leq \|x(t) - z(t)\| + \|z(t)\| \leq r + (R - r) = R, \quad \forall t \geq \bar{t},$$

which implies that

$$x(t) \in \Omega_o, \quad \forall t \geq \bar{t}.$$

Since we apply Step 5 in MPC Algorithm for  $t \geq \bar{t}$ , using Assumption 2,

$$\lim_{t \rightarrow \infty} \|x(t)\| = 0.$$

This proves the convergence of  $x(t)$  to the origin. The stability of the origin is a direct implication of assumptions 2 and 3, which completes the proof. □

### 3 MPC for A Class of Systems with Derivatives in Convex Sets

In this section, we will specialize MPC results to the following class of systems,

$$\begin{aligned} \dot{x} &= Ax + Bu + E\phi(t, q) \\ q &= C_q x + D_q u, \end{aligned} \quad (21)$$

where  $\phi : \mathbb{R} \times \mathbb{R}^{n_q} \rightarrow \mathbb{R}^{n_p}$  is a continuously differentiable function representing uncertain nonlinear part of the dynamics, i.e.  $f(x, u, t) = Ax + Bu + E\phi(t, q)$  in (1). Since we develop explicit solutions to the existence assumptions 1-4 for (21), this is a particularly important class of systems. Here we present the following assumption on system in (21) that will be instrumental in our control design.

**Assumption 5.** *There exists a closed and convex set of matrices  $\Theta \in \mathbb{R}^{n_p \times n_q}$  such that*

$$\frac{\partial \phi}{\partial q}(t, q) \in \Theta, \quad \forall q, t. \quad (22)$$

□

The nominal system dynamics is assumed to have the following form,

$$\begin{aligned} \dot{z} &= Az + Bu_o + E\psi(t, q_o) \\ q_o &= C_q z + D_q u_o, \end{aligned} \quad (23)$$

where  $\psi : \mathbb{R} \times \mathbb{R}^{n_q} \rightarrow \mathbb{R}^{n_p}$  is an approximation for  $\phi$  in the real system (21), i.e.  $F(z, u_o, t) = Az + Bu_o + \psi(t, q_o)$  in (5). Here, we assume the following:

**Assumption 6.** *There exists a scalar  $\gamma > 0$  such that the following holds for all  $z \in \mathbf{X}_o$  and  $u_o \in \mathbf{U}_o$*

$$\|\phi(t, C_q z + D_q u_o) - \psi(t, C_q z + D_q u_o)\| \leq \gamma, \quad \forall t. \quad (24)$$

□

**Remark 1.** *Assumption 6 is satisfied when  $\mathbf{X}_o$  and  $\mathbf{U}_o$  are compact sets, and  $\phi$  is continuous in its arguments and it has no dependence on  $t$ .* □

This class of systems has a subclass that is particularly interesting for MPC applications, namely when  $\psi(t, z) = \psi(t)$ . In this case, the nominal system above is an LTI system. Since the dynamics of the nominal system give a set of equality constraints for the finite-horizon optimal control problem, having an LTI nominal system is useful when all other state and control constraints define a convex feasible domain.<sup>4</sup> Then, the optimal control problem becomes a convex optimization problem, which can be numerically solved reliably and autonomously in real-time by using interior point methods [7]. Therefore, analysis of systems of the form (21) with LTI nominal plants is useful for real-time autonomous control. This class of systems is also motivated by the research on guidance and control of a spacecraft in the proximity of small celestial bodies (such as asteroids and comets) [5].

The dynamics between the real state and the nominal state,  $\eta \triangleq x - z$ , are called “*error dynamics*”, and they are given by

$$\dot{\eta} = A\eta + Bu_f + E[\phi(t, q) - \psi(t, q_o)]. \quad (25)$$

This equation is then rewritten as,

$$\dot{\eta} = A\eta + Bu_f + E[\phi(t, q) - \phi(t, q_o)] + E[\phi(t, q_o) - \psi(t, q_o)]. \quad (26)$$

The following lemma (see [4] for a proof), which is a generalization of mean value theorem, is used to obtain a linear differential inclusion (LDI) [2] for the error dynamics in (26).

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<sup>4</sup>All equality constraints in a convex optimization problem must be linear equalities.

**Lemma 3.** Consider a continuously differentiable function  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^m$  with its Jacobian given by  $\partial\varphi/\partial q(q)$ . Suppose that there exists a closed convex set  $\Lambda \in \mathbb{R}^{n \times m}$  such that

$$\frac{\partial\varphi}{\partial q}(q) \in \Lambda, \quad \forall q.$$

Then, for every  $q_1$  and  $q_2$  there exists  $\Delta \in \Lambda$  such that

$$\varphi(q_2) - \varphi(q_1) = \Delta(q_2 - q_1).$$

□

Based on Lemma 3, (26) can be written as

$$\dot{\eta} = A\eta + Bu_f + E[\pi(t, \eta, u_f) + w(t, z, u_o)], \quad (27)$$

where  $\pi(t, \eta, u_f) = \phi(t, C_q x + D_q u) - \phi(t, C_q z + D_q u_o)$ . Again using Lemma 3,

$$\pi(t, \eta, u_f) = \theta(t)(C_q \eta + D_q u_f), \quad \text{where } \theta(t) \in \Theta, \forall t, \quad (28)$$

and  $w(t, z, u_o) = \phi(t, C_q z + D_q u_o) - \psi(t, C_q z + D_q u_o)$ , where  $w$  is assumed to satisfy

$$\|w(t, z, u_o)\| \leq \gamma, \quad \forall z \in \mathbf{x}_o, u_o \in \mathbf{U}_o, t \geq 0. \quad (29)$$

This description of the error dynamics is particularly useful to obtain feedback laws that satisfy Assumption 4 for a class of uncertain nonlinear systems [1]. Here, we only consider two well known classes where Jacobian matrices are either norm bounded or exist in polytopes, and we give the corresponding feedback results.

An additional assumption is an explicit characterization of constraint sets in the FHC in terms of ellipsoids. Note that, more general convex characterizations are also possible and can easily be integrated into the design framework.

**Assumption 7.** The following hold for the constraint sets in FHC,

$$\begin{aligned} \mathbf{X}_o &= \{x \in \mathbb{R}^n : a_i^T x \leq 1, i = 1, \dots, m_o\}, \\ \mathbf{X}_f &= \{x \in \mathbb{R}^n : b_i^T x \leq 1, i = 1, \dots, m_f\}, \\ \mathbf{U}_o &= \{u \in \mathbb{R}^m : u^T \Pi_o u \leq 1\}, \\ \mathbf{U}_f &= \{u \in \mathbb{R}^m : u^T \Pi_f u \leq 1\}, \end{aligned} \quad (30)$$

where  $\Sigma_o, \Sigma_f, \Pi_o, \Pi_f$  are symmetric positive-definite matrices. □

**Remark 2.** If there exist sets  $\Sigma_o \subseteq \mathbf{X}_o$  and  $\Sigma_f \subseteq \mathbf{X}_f$  satisfying assumptions 2, 3, and 4, then we can replace  $\mathbf{X}_o$  and  $\mathbf{X}_f$  with  $\Sigma_o$  and  $\Sigma_f$  in the MPC Algorithm, and the results of Theorem 1 will still be valid. □

The following is corollary of Theorem 1 that describes a design procedure for system with norm-bounded derivatives.

**Corollary 1.** Consider a uncertain nonlinear system (21) with a nominal model given by (23) satisfying assumptions 5, 6, and 7 with

$$\Theta = \{\theta \in \mathbb{R}^{n_p \times n_q} : \|\theta\| \leq 1\}. \quad (31)$$



Suppose that there exist matrices  $S = S^T > 0$ ,  $Q = Q^T > 0$ ,  $L$ ,  $Y$  and positive scalars  $\lambda$ ,  $\beta$ ,  $\mu$ ,  $c_1$ , and  $c_2$  satisfying the following matrix inequalities,

$$\begin{bmatrix} SA^T + AS + BL + L^T B^T + S/\lambda & SC_q^T + L^T D_q^T \\ +(\beta + \lambda\gamma^2)EE^T & \\ C_q S + D_q L & -\beta I \end{bmatrix} \leq 0 \quad (32)$$

$$\begin{bmatrix} QA^T + AQ + BY + Y^T B^T & QC^T + Y^T D^T & QC_q^T + Y^T D_q^T \\ +\mu EE^T & & \\ CQ + DY & -I & 0 \\ C_q Q + D_q Y & 0 & -\mu I \end{bmatrix} \leq 0 \quad (33)$$

$$\begin{bmatrix} S & L^T \\ L & \Pi_f^{-1} \end{bmatrix} \geq 0, \quad \begin{bmatrix} Q & Y^T \\ Y & \Pi_o^{-1} \end{bmatrix} \geq 0, \quad (34)$$

$$a_i^T Q a_i \leq 1, \quad i = 1, \dots, m_o, \quad (35)$$

$$b_i^T S b_i \leq 1, \quad i = 1, \dots, m_f, \quad (36)$$

$$Q \geq c_1 I > c_2 I \geq S, \quad (37)$$

where  $C$  and  $D$  are matrices satisfying

$$C^T D = 0.$$

Then, the ellipsoids  $\varepsilon_Q = \{x : x^T Q^{-1} x \leq 1\}$  and  $\varepsilon_S = \{x : x^T S^{-1} x \leq 1\}$  satisfy  $\varepsilon_Q \subseteq \mathbf{X}_o$  and  $\varepsilon_S \subseteq \mathbf{X}_f$ . And, MPC Algorithm with

$$\begin{aligned} h(x, u) &= \|Cx\|^2 + \|Du\|^2 \\ V(x) &= x^T Q^{-1} x \end{aligned} \quad (38)$$

$$\begin{aligned} \mathcal{L}(x) &= Kx, & K &= YQ^{-1}, \\ \mathcal{K}(x, z) &= K_f(x - z), & K_f &= LS^{-1}, \end{aligned} \quad (39)$$

and  $\varepsilon_Q$  and  $\varepsilon_S$  replacing  $\Omega_o$  and  $\mathbf{X}_f$  results in an asymptotically stable closed loop system for (21) with a region of attraction  $\mathcal{R}_a$  given by (20).  $\square$

The following corollary establishes the results of Corollary 1 to systems with uncertain nonlinear terms having derivatives in polytopes.

**Corollary 2.** Consider a uncertain nonlinear system (21) with a nominal model given by (23) satisfying all the assumptions of Corollary 1 with the following modification: For (31), assume that there exists a set of matrices  $\Sigma_1, \dots, \Sigma_N$  such that

$$\Theta = \{\theta \in \mathbb{R}^{n_p \times n_q} : \theta \in \text{Co}\{\Sigma_1, \dots, \Sigma_N\}\}. \quad (40)$$

Then, all the conclusions of Corollary 1 hold if inequalities (32) and (33) are replaced by the following, for  $i = 1, \dots, N$

$$A_i S + SA_i^T + B_i L + L^T B_i^T + S/\lambda + (\beta + \lambda\gamma^2)EE^T \leq 0, \quad (41)$$

$$\begin{bmatrix} A_i Q + QA_i^T + B_i Y + Y^T B_i^T & QC^T + Y^T D^T \\ CQ + DY & -I \end{bmatrix} \leq 0, \quad (42)$$

where

$$A_i = A + E\Sigma_i C_q, \quad B_i = B + E\Sigma_i D_q.$$

$\square$

**Remark 3.** All the matrix inequalities given in Corollary 1 and 2 are LMIs (linear matrix inequalities) except (32), (33), (41), and (42). But, these are also LMIs for a given  $\lambda > 0$ . Therefore, a simple line search on  $\lambda$  can be applied to be able to solve the whole system of matrix inequalities in both corollaries for the design of feedback laws.  $\square$

### 3.1 Computation of Feed-forward Control for Nominally LTI Systems

In this section, we describe a methodology to calculate the feed-forward control for nominally LTI systems, that is (23) is an LTI system. It is assumed that  $\mathbf{X}_o$  and  $\mathbf{U}_o$  are convex sets. The feed-forward control can be parameterized by a zero-order-hold approach, i.e.,  $u_o$  is piecewise constant on time intervals of fixed length  $\delta t$ . Then, the resulting sampled-data system can then be written as,

$$z_{k+1} = A_d z_k + B_d u_{o,k} + \xi_k, \quad (43)$$

where

$$A_d = e^{A\delta t}, \quad B_d = \int_0^{\delta t} e^{A(\delta t - \tau)} B d\tau,$$

and

$$\xi_k = \int_{t_k}^{t_{k+1}} e^{A(\delta t - \tau)} E \psi(t) d\tau.$$

This allows us to approximate the integral part of the cost  $J$  in the FHC with a finite sum,

$$\int_{t_s}^{t_s+T} h(z(\tau), u_o(\tau)) d\tau \approx \delta t \sum_{k=1}^N e_k h(z_k, u_{o,k}),$$

where  $e_k$  results from the specific numerical integration technique used, and  $N = T/\delta t$ . Then, the state and control constraints are only imposed at temporal nodes, i.e.

$$z_k \in \mathbf{X}_o, \quad u_k \in \mathbf{U}_o, \quad k = 1, \dots, N.$$

If  $\mathbf{X}_o$  and  $\mathbf{U}_o$  are sets that can be described by semidefinite constraints, such as linear, quadratic, or conic inequalities [3], then the FHC is approximated by a finite-dimensional parameter optimization problem. Specifically, it becomes a semidefinite programming problem (SDP). SDP problems can be solved in polynomial time. There exist algorithms and software [7, 9, 8] that compute the global optimum with a deterministic stopping criteria, and with prescribed level of accuracy. Therefore, they are very well-suited for real-time, onboard computations.

## 4 An Illustrative Example

In this section, we present an example illustrating the MPC algorithm. In this example, a “standard” approach to MPC without the relaxation of the initial state constraint in the feed-forward problem (6) and without the feedback is shown to fail. The same problem is then solved successfully by the MPC algorithm. The dynamics of the system are given by,

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ -0.1 \end{bmatrix} \omega \sin(C_q x)^2 \\ C_q &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \\ \omega &\in [0, 0.5]. \end{aligned} \quad (44)$$

Here  $\omega$  is an uncertain parameter. We assume that the nominal system is the linear part of (44), that is  $\psi(t, q) = 0$  and  $\phi(t, q) = \omega \sin q^2$  in (23) and (21). This implies that  $\|\partial\phi/\partial q\| \leq 1$ , and Assumption 5 is satisfied with  $\Theta$  as in (31). The state constraints are given by,

$$\begin{aligned} -0.25 &\leq x_1 \leq 5 \\ -1 &\leq x_2 \leq 2 \end{aligned}$$

and the control constraint is  $|u| \leq 1.4$ . For the MPC algorithm, we partition the control constraint into

$$|u_o| \leq 1.2, \quad \text{and} \quad |u_f| \leq 0.2.$$

The integral cost function  $h(z, u) = \|Cx\|^2 + \|Du\|^2$  is determined by,

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Other design parameters needed in Corollary 1 are  $\gamma = 0.05$ ,  $\Pi_o = (1/1.2^2)I$ ,  $\Pi_f = (1/0.2^2)I$ , and

$$\begin{aligned} a_1 &= [0.2 \ 0], & b_1 &= [4 \ 0] \\ a_2 &= [-4 \ 0], & b_2 &= [-4 \ 0] \\ a_3 &= [0 \ 0.5], & b_3 &= [0 \ 10] \\ a_4 &= [0 \ -1], & b_4 &= [0 \ -10]. \end{aligned}$$

Then, the values of the solution variables obtained by solving the LMIs [9] in Corollary 1 are

$$\begin{aligned} K_o &= \begin{bmatrix} -4.9491 & -2.1129 \end{bmatrix}, & K_f &= \begin{bmatrix} -4.0868 & -3.4005 \end{bmatrix} \\ Q &= \begin{bmatrix} 0.0625 & -0.0776 \\ -0.0776 & 0.3431 \end{bmatrix}, & S &= \begin{bmatrix} 0.0018 & -0.0018 \\ -0.0018 & 0.005 \end{bmatrix}. \end{aligned}$$

The invariant ellipsoids  $\varepsilon_Q$  and  $\varepsilon_S$  are given in Figure 1. A simulation for the standard MPC implementation is given in Figure 2, where  $\omega = 0.3$ . In this simulation, the finite horizon optimal control problem is solved by applying a zero-order-hold discretization with a time increment  $\delta t = 0.5$  seconds, and constraints are guaranteed at the temporal nodes. The feed-forward solution is recomputed at every 10 seconds with a finite time horizon of 30 seconds. The trajectory converged to set  $\Omega_o$  (the ellipsoid around origin in the plot), and the trajectory asymptotically converges to the origin. However, the state constraint on  $x_1$  is violated as depicted by a solid line at  $x_1 = -0.25$  that is crossed in Figure 2. The same simulation is repeated with the MPC algorithm of this paper, and the results are given in Figure 3, which shows that the trajectory asymptotically converges to the origin without violating any state or control constraints. Figure 4 shows the error between the nominal and real trajectories also goes to zero.

Actually, one can reduce the time interval between two computations in order to make a standard MPC solution to give feasible state trajectories, and we observed that reducing the computation interval from 10 to 5 seconds would have worked for this example. However, this reduction can not always be pre-computed analytically, and the required reduction may as well be computationally too demanding for a real-time implementation for a complex system. Therefore, using the MPC algorithm of this paper can especially be useful when explicit guarantees are needed for feasibility once an initial feasible solution is computed.

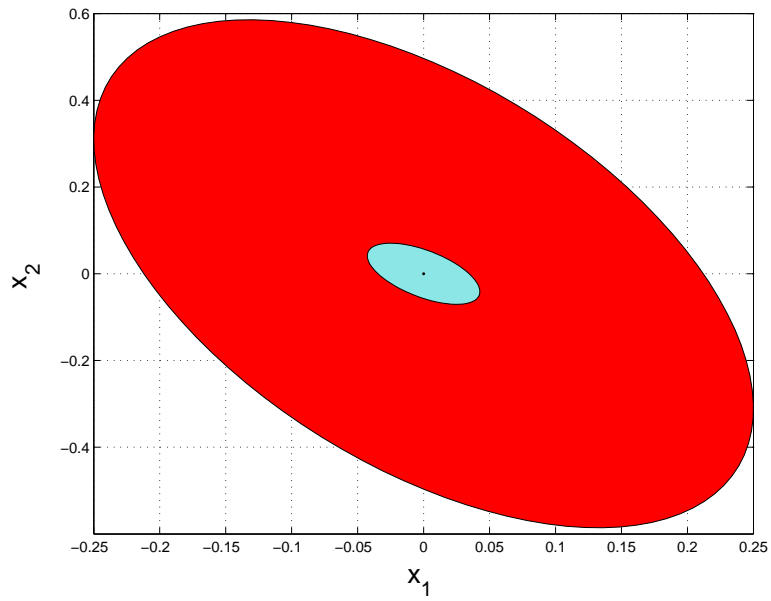


Figure 1: Invariant Ellipsoids  $\varepsilon_S \subset \varepsilon_Q$  for the MPC Algorithm

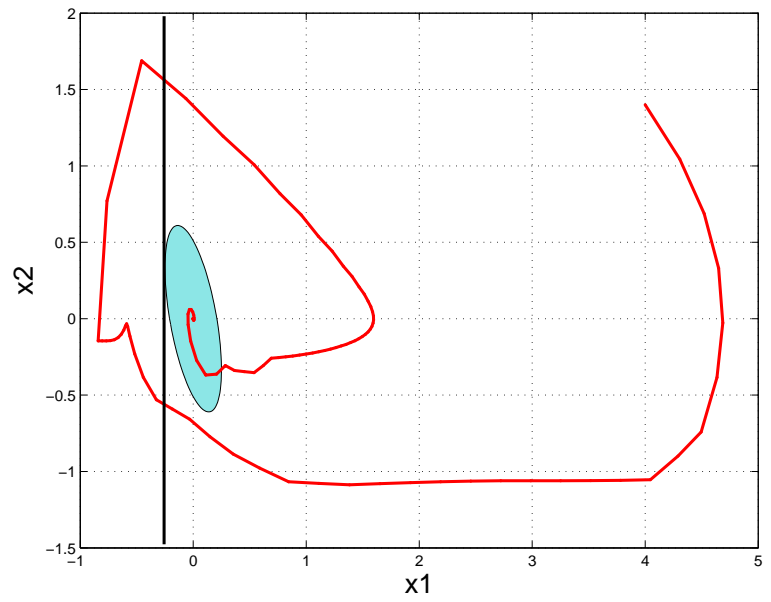


Figure 2: State Trajectory with Standard MPC

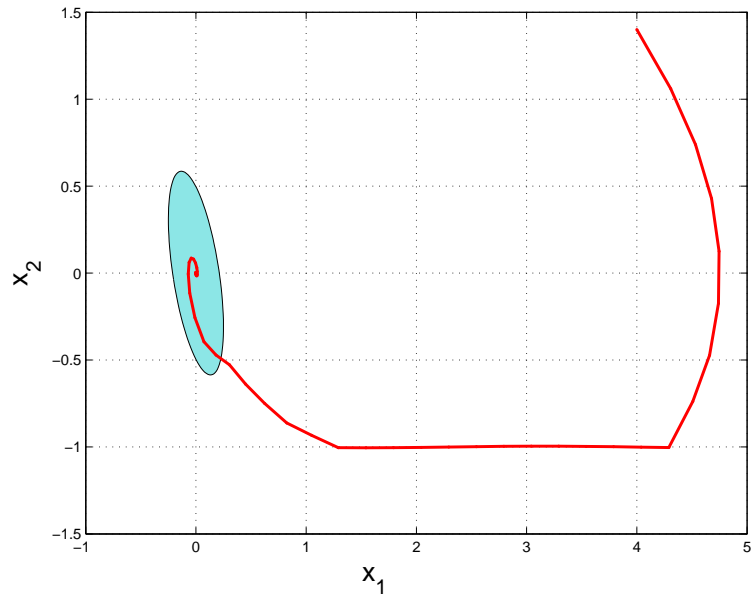


Figure 3: State Trajectory with the MPC Algorithm

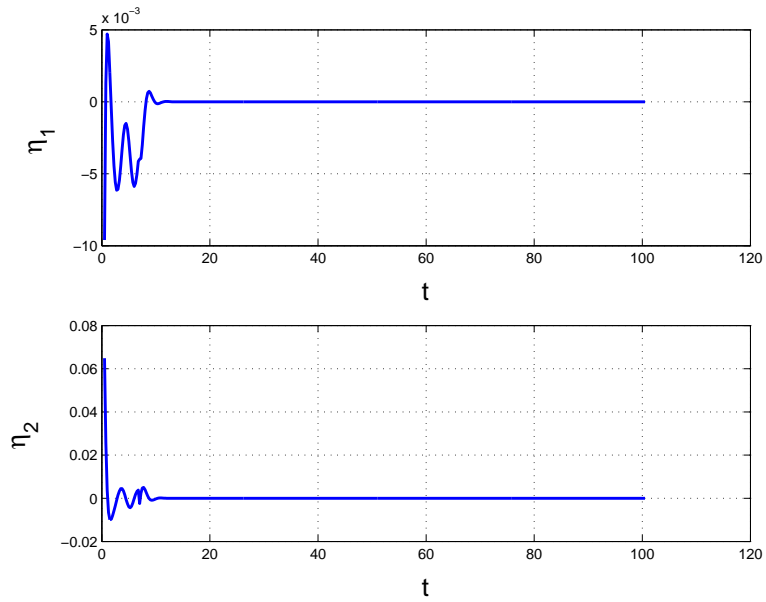


Figure 4:  $\eta$  as a function of time

## 5 Conclusions

In this paper, we presented a generic MPC algorithm that differs from earlier counterparts in terms of using a feed-forward and feedback control components, and relaxing the initial state constraint for the solution of the finite-horizon optimal control problem at each computation. This relaxation makes it possible to design the feedback control policy off-line, and guarantee the resolvability and asymptotic stability once an initial feasible solution is obtained by solving the finite-horizon optimal control problem at the start of a maneuver. This MPC algorithm is robust to system uncertainties that are explicitly accounted for in the design of feedback portion of the control input. This explicit characterization of the robustness to the uncertainties (which can easily be extended to external disturbances) is particularly desirable for real-time autonomous control applications. Design procedures for a particular class of uncertain nonlinear systems is also given, as well as an illustrative example demonstrating the approach.

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