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Stochastic representation of chaos using terminal attractors

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Abstract

A nonlinear version of the Liouville equation based upon terminal attractors is proposed for describing post-instability motions of dynamical systems with exponential divergence of trajectories such as those leading to chaos and turbulence. As a result, the post-instability motions are represented by expectations, variances, and higher moments of the state variables as functions of time. The proposed approach can be applied to conservative chaos, and in particular, to n-bodies problem, as well as to dissipative systems, and in particular, to chaotic attractors and turbulence.

1. Introduction

Mathematical formalism of nonlinear dynamics does not discriminate between stable and unstable motions, and therefore, an additional stability analysis is required for that. However, such an analysis is not constructive: in case of instability, it does not suggest any model modifications to efficiently describe post-instability motions. The most important type of instability, that necessitates post-instability description, is associated with positive Lyapunov exponents leading to exponential growth of small errors in initial conditions (chaos, turbulence). The approach proposed in this paper is based upon the removal of positive Luapunov exponents by introducing special control forces represented by terminal attractors. The role of these forces is to suppress the divergence of the trajectories corresponding to initial conditions that are different from the prescribed ones without affecting the "target" trajectory that starts with the prescribed initial conditions. Since the terminal attractors include expected values of the state variables as new unknowns, the corresponding Liouville equation should be invoked for the closure. This equation is different from its classical version by additional nonlinear sinks of the probability represented by terminal attractors.

2. Representation of expected values of state variables

Consider a system of n first order ordinary differential equations with n unknowns $x_1(t), \dots, x_n(t)$:

$$\dot{x}_i = f_i[\{x(t)\}, t], \quad \{x\} = x_1, \dots x_n, \quad i = 1, 2, \dots, n. \tag{1}$$

subject to initial conditions

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$$x_i(0) = x_i^0. \tag{2}$$

Due to finite precision, the values (2) are not known exactly, and we assume that the error possesses some joint distribution

$$\mathrm{Err}(X_i^0) = P(X_1^0, \ldots, X_n^0) = P_0. \tag{3}$$

It is reasonable to assume that the initial conditions (2) coincide with the initial expectations i.e. that P_0 has a maximum at $X_i^0 = x_i^0, \; i = 1, 2, \ldots, n$. This means that

$$\frac{\partial P_0}{\partial X_0} = 0, \quad \frac{\partial^2 P_0}{\partial X_i \partial X_j} < 0, \quad i = 1, 2, \ldots, n. \tag{4}$$

This is true for any symmetric initial density (for instance, the normal distribution) when the expected values have the highest probability to occur. The Liouville equation describing the evolution of the joint density P is [1]

$$\frac{\partial P}{\partial t} + \nabla \bullet (Pf) = 0, \quad f = f_1, \ldots, f_n, f_i = f_i(\{X\}, t), \quad P = P(\{X\}, t). \tag{5}$$

Its formal solution

$$P = P_0 \exp\left(-\int_0^t \nabla \bullet f \, \mathrm{d}\tau\right) \tag{6}$$

suggests that the flattening of the error distribution is caused by the divergence of the trajectories of the governing Eqs. (1) from the target trajectory that starts with the prescribed initial conditions (2), Fig. 1a. Let us introduce the following control forces into Eqs. (1)

$$F_{i_i} = \gamma_i(\langle X_i \rangle - x_i)^{1/3}, \quad \gamma_i = \mathrm{const} > 0. \tag{7}$$

Here

$$\langle X_i \rangle = \int_{-\infty}^{\infty} X_i P \, \mathrm{d}X_1, \ldots, \mathrm{d}X_n \tag{8}$$

is the expected value of x_i, and γ_i are scaling parameters that have dimensionalities matching those of f_i. Adding the terminal attractor (7) to Eqs. (1), one arrives at the governing equations of the controlled dynamics

$$\dot{x}_i = f_i(\{x\}, t) + \gamma_i(\langle X_i \rangle - x_i)^{1/3}, \quad i = 1, 2, \ldots, n. \tag{9}$$

As follows from Eqs. (7) and (8), the control forces depend upon the density functionals rather than the density function. The control forces possess two important properties. Firstly, they vanish at $x_i\langle x_i \rangle$, and therefore, they do not affect the target trajectory. Secondly, their derivatives become unbounded at the target trajectory:

$$\left|\frac{\partial F_i}{\partial x_i}\right| = \left|\frac{1}{3}(< X_i > - x_i)^{-2/3}\right| \to \infty \quad \text{at } x_i \to \langle x_i \rangle \tag{10}$$

and that makes the target trajectory infinitely stable since

$$\frac{\partial \dot{x}_i}{\partial x_i} \to \infty \quad \text{at } x_i \to \langle x_i \rangle, \quad i = 1, 2, \ldots, n \tag{11}$$

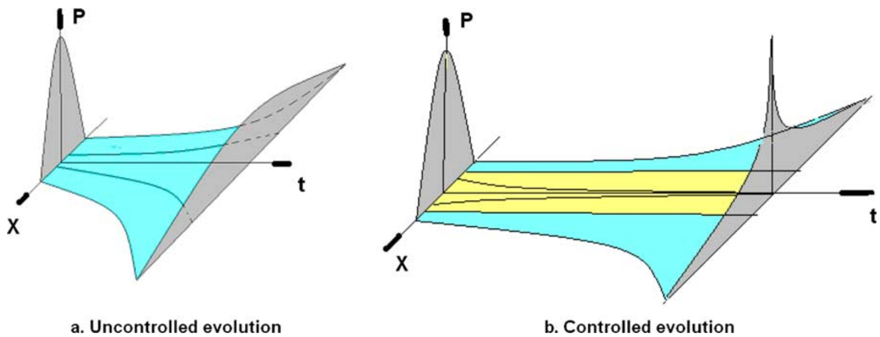

Fig. 1. Divergence of trajectories.

for any functions f_i as long as they satisfy the Lipschitz conditions. In other words, the target trajectory $x = \langle x(t)\rangle$ becomes a terminal attractor. The terminal attractor was introduced and investigated in our earlier works [2,3]. As shown there, this attractor is not only infinitely stable, but it is intersected by all the attracted transients, and therefore, it provides an infinitely large compression of the family of the trajectories in the neighborhood of the target trajectory. All these properties result from the relaxing the Lipschitz conditions (11) due to a special form of the control forces (7).

Obviously the system (9) is not closed yet since it includes expectations $\langle x_i\rangle$ as additional unknowns. For their closure one has to invoke the corresponding Liouville equation that includes the control forces (7)

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^{n}\frac{\partial}{\partial X_i}\{P[f_i + \gamma_i(\langle X_i\rangle - X_i)^{1/3}]\}. \tag{12}$$

At the limit $x_i \rightarrow \langle X_i\rangle$, one can ignore the functions f_i due to terminal properties of the control forces (7) and (12) is decomposed into n independent equations:

$$\frac{\partial P_i'}{\partial t} = -\gamma_i\frac{\partial P_i'}{\partial X_i}(\langle X_i\rangle - X_i)^{1/3} \quad \text{at} \quad X_i \rightarrow \langle x_i\rangle, \tag{13}$$

where $P_i'(X_i)$ are the independent densities, and

$$P(X_1,\ldots,X_n) = \prod_{i=1}^{n}P_i'(X_i) \quad \text{at} \quad X_i \rightarrow \langle x_i\rangle. \tag{14}$$

Although Eq. (14) are nonlinear with respect to the densities P_i', one can write their formal solutions as

$$P_i' = P_{i0}'(X_i \rightarrow \langle X_i\rangle)\exp\left(\frac{\gamma_i}{3}\int_0^t(\langle X_i\rangle - X_i)^{-2/3}\,\mathrm{d}t\right) \rightarrow \delta(X_i - \langle X_i\rangle), \tag{15}$$

where

$$P_{i0}' = P_i'(X_i = X_{i0}). \tag{16}$$

Hence with the probability one the solution to Eq. (12) approaches the target trajectory

$$\langle X_i\rangle = \langle X_i(t)\rangle \tag{17}$$

regardless of the fact that the original dynamical system may be chaotic. This suggests that if our objective would be to control the chaos, then the Liouville equation (12) should be implemented as an additional control device that is coupled with the original dynamical system (9) via the control force (7), and the target trajectory will be preserved, Fig. 1b. However, our objective is more general, namely, to obtain the complete stochastic structure of chaos, and therefore, the evolution of the expected values of the state variables is the first step in our approach.

It should be recalled that the target trajectory has been identified with the expected trajectory, and the latter is usually unknown in advance in chaotic systems: it can be found only as a result of the solution to Eq. (12) (but not of the solution to the non-controlled version (6) which is chaotic). In order to clarify that and to illuminate the mechanism of stabilization by a terminal attractor, let us consider a very simple exponentially unstable linear equation

$$\dot{x} = x. \tag{18}$$

In this particular case, the expected trajectory is known in advance:

$$\langle X\rangle = 0. \tag{19}$$

However, any small error in initial conditions leads to a different trajectory that diverge exponentially from those in Eq. (19):

$$x = x_0\exp t, \tag{20}$$

Similar result follows from the corresponding Liouville equation:

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial X}(PX), \quad \langle X\rangle = \langle X_0\rangle\exp t. \tag{21}$$

Let us now introduce the control force (7) that, in view of Eq. (19) should be

$$F = -\gamma x^{1/3}, \quad \gamma \ll 1. \tag{22}$$

Then the controlled version of the original dynamical system (28) is

$$\dot{x} = x - \gamma x^{1/3}. \tag{23}$$

For the values of x of the order of 1, the control force can be ignored. But when x is close to the target trajectory, $x \ll 1$, the control force is dominating, and the solution to Eq. (23) consists of a regular solution

$$x = \left[x_o^{2/3} - \frac{2}{3}\gamma t\right]^{3/2} \tag{24}$$

and a singular solution

$$x \equiv 0. \tag{25}$$

Regardless of the value of the initial conditions $x_0 \ll 1$, the regular solution (24) approaches the singular solution (25), i.e. the target trajectory, in a finite time

$$T = \frac{3}{2\gamma} x_0^{2/3} \tag{26}$$

and both solutions coexist for $t > T$. Therefore, an exponential growth of initial errors is totally eliminated. It should be noticed that the coexistence of solutions with different initial conditions as well as a finite time of approaching an equilibrium are special properties of the terminal attractors reported in [2,3]. One has to recall again that although the example we just discussed is much simpler than chaos, the stabilization mechanism performed by the terminal-attractor-based control forces is the same.

3. Representation of higher moments

Although expected values of the state variables play an important role in description of post-instability motions, they do not expose the full dynamical picture, and the behavior of the higher moments is required. For that purpose, let us turn to Eq. (1) and introduce new variables

$$x_{ij} = x_i x_j. \tag{27}$$

After trivial transformations, the system (1) can be rewritten in an equivalent form being expressed via new variables

$$\dot{x}_{ij} = f_{ij}(x_{11}, \ldots, x_{nn}) \tag{28}$$

in which

$$f_{ij} = \sqrt{x_{jj}} f_i(\sqrt{x_{11}}, \ldots, \sqrt{x_{nn}}) + \sqrt{x_{ii}} f_j(\sqrt{x}_{11}, \ldots, x_{nn}). \tag{29}$$

Let us now augment Eq. (28) with the control forces similar to (7)

$$\dot{x}_{ij} = f_{ij}(x_{11}, \ldots, x_{nn}) + \gamma_i(x_{ij} - \langle X_{ij}\rangle)^{\frac{1}{3}}. \tag{30}$$

Then the corresponding Liouville equation will be similar to (12)

$$\frac{\partial P'}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial X_i}\left\{P'[fj_i + \gamma_{ij}(\langle X_{ij}\rangle - X_{ij})^{1/3}]\right\}. \tag{31}$$

Solving Eqs. (30) and (31) simultaneously (compare to the system (9) and (12)), one obtains the evolution of the expectations of the new state variables that are equivalent to the second moments of the old variables (see Eq. (27))

$$\langle X_{ij}\rangle = \langle X_i X_j\rangle. \tag{32}$$

It should be noticed that P and P' are different: for instance, if initially P is normally distributed, P' must be recalculated by applying the rules for the change of variables (27); that is why the expectations and the second moments must be found from different equations.

The higher moments can be found in a similar way by introducing new variables x_{ijk}, x_{ijkl}, etc.

Based upon the expectation and higher moments, one can reconstruct the joint probability distribution of state variables, and therefore, to obtain a complete information about dynamics of the underlying physical process in a stable form.

4. Computational strategy

In order to solve Eq. (12), we will apply a small-time-steps strategy. Linearizing Eq. (9) with respect to the initial values of the state variables x_i during a small period of time t, one obtains

$$\dot{x}_i = f_i(x_1^0, \ldots, x_n^0) + \gamma_i(\langle X_i^0 \rangle - x_i^0)^{\frac{1}{3}} + \sum_j \frac{\partial f_i}{\partial x_j}|_{x_i = x_i^0} x_j - \frac{1}{3}\gamma_i(\langle X_i^0 \rangle - x_i^0)^{-\frac{2}{3}} x_i. \tag{33}$$

During the same period Δt, the corresponding Liouville equation (12) reads

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^{n} \left\{ \left[f_i(x_1^0, \ldots x_i^0) + \gamma_i(\langle X_i^0 \rangle - x_i^0)^{\frac{1}{3}} \right] \frac{\partial}{\partial X_i}(P) + \left[\sum_j \frac{\partial f_i}{\partial x_j}\bigg|_{x_j = x_j^0} - \frac{1}{3}\gamma_i(\langle X_i^0 \rangle - x_i^0)^{-\frac{2}{3}} \right] \frac{\partial}{\partial X_i}(PX_i) \right\}. \tag{34}$$

This equation can be reduced to the system of ODE. For that purpose, let us multiply Eq. (34) by X_i and integrate over the whole space assuming that

$$P \to 0, \quad PX_i \to 0 \quad \text{at} \quad |X_i| \to \infty. \tag{35}$$

Then Eq. (34) reduces to the following system of ODE with respect to expectations $\langle X_i \rangle$

$$\langle \dot{X}_i \rangle = A_{ij} \langle X_j \rangle + B_i \tag{36}$$

in which the vector

$$B_i = f_i(x_1^0, \ldots x_n^0) + \gamma_i(\langle X_i^0 \rangle - x_i^0)^{\frac{1}{3}} \tag{37}$$

and the matrix

$$A_{ij} = \frac{\partial f_i}{\partial x_j}\bigg|_{x_j = x_j^0} - \frac{1}{3}\gamma_i(\langle X_i^0 \rangle - x_i^0)^{-\frac{2}{3}} \delta_{ij}, \quad \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \tag{38}$$

The solution to Eq. (36) for the selected small interval Δt describing the evolution of expectations of the state variables can be written in the following closed form:

$$\langle X \rangle = (\langle X^0 \rangle - A^{-1}B)\exp(At) + A^{-1}B. \tag{39}$$

Substituting $t = \delta t$ into the solution (39), one finds the values of the expectations at the end of the initial interval. Taking these values as the initial conditions for the next small interval, one continues the computational process in a similar way.

Let us discuss the role of the control forces (7). It is reasonable to assume that initial error, i.e. the difference between the initial value of a state variable and its initial expected value is small

$$|\langle X_i \rangle - x_i| \ll 1. \tag{40}$$

This means that the contribution of these forces into the vector (37) can be ignored. However, the contribution of the same forces into the diagonal components of the matrix (38) is dominating since the gradient of these forces is inversely proportional to the error. As a result, fore a very small error, all the diagonal terms become negative with large absolute values, and that suppresses all the positive characteristic roots of the matrix (38). This means that small errors do not grow, and therefore, the trajectories that are sufficiently close to the target trajectory do not diverge. The boundary between the area of diverging and converging trajectories is controlled by the weights γ_i of the control forces whose optimal choice is problem-specific.

The same computational strategy is applicable for representation of the higher moments (see Eqs. (30) and (31)).

5. Discussion and conclusion

Thus, a nonlinear version of the Liouville equation is proposed for describing post-instability motions of dynamical systems with exponential divergence trajectories such as those leading to chaos and turbulence. The approach is based upon introduction of stabilizing control forces that couple equations of motion and the evolution of the probability density of errors in initial conditions. These stabilizing forces create a powerful terminal attractor in the probability space that corresponds to occurrence of the target trajectory with the probability one. In configuration space, this effect suppresses exponential divergence of the close-neighboring trajectories without affecting the target trajectory. As a result, the post-instability motion is represented by a set of functions describing the evolution of the statistical invariants such as expectations and higher moments, while this representation is stable. General analytical proof as well as a computational strategy has been introduced. Since the proposed approach is not restricted by any special assumptions about the original dynamical system, it can be applied to both conservative and dissipative systems. For conservative systems the control forces must be introduced in the form of the potential

$$\Pi = \sum_i \gamma(\langle X_i \rangle - x_i)^{\frac{4}{3}}. \tag{41}$$

In this case the controlled version of the system remains conservative, and it can be applied to problems displaying conservative chaos such as in celestial mechanics (for instance, many-body problems). The broad class of dissipative systems to which the proposed approach can be applied includes chaotic attractors and turbulence. In the case of turbulence, prior to application of the proposed methodology, the Navier–Stocks equation must be approximated by a system of ODE. Such an approximation can be performed using finite differences, finite elements, or the Galerkin method.

It should be noticed that the proposed approach combines several departures from the classical methods. Firstly, it introduces a nonlinear version of the Liouville equation that is coupled with the equation of motion (in Newtonian dynamics they are uncoupled). General properties of such systems were discussed in [4]. Secondly, it introduces terminal attractors characterized by violation of the Lipschitz conditions (in Newtonian dynamics as well as in theory of differential equations these conditions are preserved). The theory of non-Lipschitz dynamics was discussed in [2,3]. Finally, the idea of a forced stabilization of unstable equations follows from the stabilization principle introduced in [5].

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