White Light Modeling, Algorithm Development, and Validation on the Micro–Arcsecond Metrology Testbed

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Abstract

The Space Interferometry Mission (SIM) scheduled for launch in early 2010, is an optical interferometer that will perform narrow angle and global wide angle astrometry with unprecedented accuracy, providing differential position accuracies of 1uas, and 4uas global accuracies in position, proper motion and parallax. The astrometric observations of the SIM instrument are performed via delay measurements provided by three Michelson-type, white light interferometers. Two "guide" interferometers acquire fringes on bright guide stars in order to make highly precise measurements of variations in spacecraft attitude, while the third interferometer performs the science measurement. SIM derives its performance from a combination of precise fringe measurements of the interfered starlight (a few ten–thousandths of a wave) and very precise (tens of picometers) relative distance measurements made between a set of fiducials. The focus of the present paper is on the development and analysis of algorithms for accurate white light estimation, and on validating some of these algorithms on the MicroArcsecond Testbed.

1 Introduction.

The Space Interferometry Mission (SIM) scheduled for launch in early 2010, is an optical interferometer that will perform narrow angle and global wide angle astrometry with unprecedented accuracy, providing differential position accuracies of approximately 1uas, and a 4uas global capability in position, proper motion and parallax. The astrometric observations of the SIM instrument are performed via delay measurements provided by three Michelson-type, white light interferometers. Two "guide" interferometers acquire fringes on bright guide stars in order to make highly precise measurements of variations in spacecraft attitude, while the third interferometer performs the science measurement. SIM derives its performance from a combination of precise fringe measurements of the interfered starlight (a few ten–thousandths of a wave) and very precise (tens of picometers) relative distance measurements made between a set of fiducials. Testbeds demonstrating many of these challenging technology requirements of the instrument are currently operational. The focus of the present paper is the determination of accurate white light measurements, particularly those required by the guide interferometers.

SIM white light interferometry is based on dispersing the fringes across the detectors and measuring the phases at a number of wavelengths by modulating the pathlength in a known way to determine the unknown pathlength difference between the two arms of the inteferometer. This general technique is known as phase shifting interferometry (PSI). The guide interferometers offer particular challenges because there is not only an accuracy requirement on the order of a few tens of picometers when averaged over 30 seconds, but the interferometers are also required for real time pathlength control and must furnish delay estimates at a 1Khz update rate. The resulting challenges include low SNR for the guide interferometers (because they operate at 1Khz), vibrations of the optical train or imperfections in the phase modulation so that the phase is actually changing while being measured, and the necessary use of relatively wide passbands to improve the low SNR causing a loss of accuracy of monochromatic models from which PSI algorithms are typically fashioned.

Several approaches for mitigating these problems are developed and some of these are tested on the MicroArcsecond Testbed. The low SNR difficulties are approached with bias correction and phasor averaging methods. In the latter case an analysis is performed that shows the validity of the approach so long as the visibilities remain stable over the course of the averaging. Undesired changes in the optical path of the interferometers are partially observed by the internal metrology subsystem of the instrument. The mesurements are used via a perturbation formula to correct the initially computed white light fringe estimate. A demonstration of this approach on MAM testbed data is given.

The algorithms and methods above are all based on a primary monochromatic model that uses the two outputs of the beam combiner imaged onto the detector. Assuming a lossless beam combiner, the two outputs ensures that all of the available signal is used to form the phase estimates. The two image model has some nice symmetries that enable use to derive analytical formula for the delay estimate variances for several estimators in a general way as a function of the defining parameters of the model, such as modulator stroke length, wavelength, dtector noise, photon flux, visibility, and the number of frame reads per fringe estimate. This is the topic of the next section. In Section 3 the various methods for removing the effects of bias and vibration/modulator error are developed. In the final section some preliminary work on mitigating the errors due to the violation of the non–monochromatic assumption are introduced.

2 The Guide Interferometer Function and the Two Image Monochromatic Interferometric Model

The operation of the SIM instrument has been previously described in [1,2]. Here we will recount the very basics. The role of each of the three SIM interferometers is to determine the "external" pathlength delay, which by definition is the quantity

$$d = \langle s, B \rangle \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner–product on R^N, d is the delay, s is the unit star direction vector (normal to the planar wavefront), and B is the interferometer baseline vector. The baseline vector is defined via the positions of fiducials from which direct metrology measurements that follow the same path as the starlight are made. The difference in the distance between the fiducials of each arm of the interferometer to the beam combiner is termed the internal pathlength difference and is directly measured by metrology. The total pathlength difference is measured by white light interferometry, and the difference between the two is the measurement of the external pathlength difference, the sought after quantity. We begin by discussing the issues related to making the white light interferometry measurement, but will consider in Section 3 the use of using the metrology to improve this estimate when the modulator is non–ideal, etc.

Consider an interferometer in which the total photon flux at the two outputs of the combiner is such that it generates a current $2I_0$ in the detector. This total photon flux is split between the two combiner outputs in a manner which depends on the total delay, so that in a time interval τ

the number of photoelectrons on the two sides of the combiner is

$$N_{\pm} = N_0(1 \pm V \cos\theta) \tag{2}$$

where $\theta = \frac{2\pi D}{\lambda}$ is the phase corresponding to the total delay D, V is the visibility, λ is the wavelength,and $N_0 \equiv I_0\tau$. Now suppose we apply a triangular phase modulation, so that $\theta(t) = u(t) + \phi$, where ϕ is the (astrometric) phase being measured and $u(t)$ is the phase modulation. (For now we assume ϕ is constant during this measurement.) Then [1] [2], the number of electrons detected during interval i, i.e., during the i^{th} dither step of n dither steps, is[1]

$$N_{\pm i} = \frac{N_0}{n}(1 \pm V_m \cos(u_i + \phi)) \tag{3}$$

where $u_k = (k-1)\Delta + \Delta/2 - n\Delta/2$, the effective visibility V_m is

$$V_m \equiv V \text{sinc}(\frac{\Delta}{2}), \tag{4}$$

$\text{sinc}(x) \equiv \frac{\sin x}{x}$, $\Delta \equiv \frac{2\pi s}{\lambda n}$, and s is the length of the modulation stroke. The unknown quantities in (3) are N_0, V_m, and ϕ.

Let v denote the 3–vector with components $v = (N_0,\ N_0 V_m,\ \phi)$, and define the mapping $X : R^3 \to R^3$ by

$$X(v) = \frac{1}{n}(v_1, v_2\cos(v_3), v_2\sin(v_3)). \tag{5}$$

Next define the $n \times 3$ matrices A_+ and A_-,

$$A_+ = \begin{bmatrix} 1 & \cos(u_1) & -\sin(u_1) \\ \vdots & \vdots & \vdots \\ 1 & \cos(u_n) & -\sin(u_n) \end{bmatrix}, \qquad A_- = \begin{bmatrix} 1 & -\cos(u_1) & \sin(u_1) \\ \vdots & \vdots & \vdots \\ 1 & -\cos(u_n) & \sin(u_n) \end{bmatrix}. \tag{6}$$

The n–dither-step photon counts from the "bright" and "dark" fringes are, respectively,

$$N_{\pm} = A_{\pm}X(v), \tag{7}$$

corresponding to the respective intensity models in each dither step

$$N_{\pm i} = \frac{N_0}{n}\{1 \pm V_m \cos(u_i + \phi)\}. \tag{8}$$

The measurement error covariance matrices due to shot noise for the two fringes are $D(N_+)$ and $D(N_-)$, respectively, where $D(N_{\pm})$ is the diagonal matrix with $N_{\pm}$ on the diagonal. Let σ_n^2 denote the variance in the measurement due to read noise and dark current in each dither step at each side of the combiner. Then the total measurement error covariance matrix is $D(Y) = \text{diag}(D(Y_+),\ D(Y_-))$ where $D(Y_{\pm}) = D(N_{\pm}) + \sigma_n^2 I_{n\times n}$ and $I_{n\times n}$ denotes the $n \times n$ identity matrix.

Define A as the concatenation of the matrices A_+ and A_-:

$$A = \begin{bmatrix} A_+ \\ A_- \end{bmatrix}, \tag{9}$$

and let N denote the concatenation of the observed photoelectron counts $N_{\pm}$. Then the nominal monochromatic model is

$$y = AX(v) + \eta, \tag{10}$$

where y is the observed vector of photoelectron counts and η is a zero mean noise vector with covariance matrix $D(Y) = \text{diag}(D(Y_+),\ D(Y_-))$.

[1] We assume that the mean value of the dark current is removed by calibration, either in hardware or in software.

3 Phase estimators.

Since X is invertible, a broad class of estimators for the unknown parameter vector v has the general form

$$v = X^{-1}(Ky), \tag{11}$$

where K is any $3 \times n$ matrix with $KA = I$. For example the (unweighted) nonlinear least squares problem

$$\min_v |y - AX(v)|^2 \tag{12}$$

leads to an estimator in this class with $K = A^\dagger$ (the pseudoinverse of A).

The presence of the noise vector η in (10) leads to an error in the estimate of v. (Typically, especially for the application of the guide interferometers we are most intersted in the third component of v, v_3, the unknown phase of the delay.) To investigate this error let v^0 denote the true solution, and let $\hat{v}$ denote the estimate based on the noisy observations. Write $v^0 = \hat{v} + h$, and note that

$$X(v^0) - X(\hat{v}) = K\eta. \tag{13}$$

Expanding $X(v^0)$ around $\hat{v}$ to second order

$$X(v^0) = X(\hat{v}) + X'(\hat{v})h + \frac{1}{2}X''(\hat{v})[h, h], \tag{14}$$

and solving for h, yields the second order expression for the error

$$h = X'(\hat{v})^{-1}K\eta - \frac{1}{2}X'(\hat{v})^{-1}X''(\hat{v})[X'(\hat{v})^{-1}K\eta, X'(\hat{v})^{-1}K\eta]. \tag{15}$$

The bias in the solution is obtained by taking expected values of this expression,

$$E(h) = E(\frac{1}{2}X'(\hat{v})^{-1}X''(\hat{v})[X'(\hat{v})^{-1}K\eta, X'(\hat{v})^{-1}K\eta]), \tag{16}$$

and the error covariance matrix is

$$E(hh^T) = X'(\hat{v})^{-1}QX'(\hat{v})^{-T}, \tag{17}$$

where

$$Q = K\mathrm{diag}(D(Y_+), D(Y_-))K^T. \tag{18}$$

Next we develop useful formulas for the errors in the least squares and minimum variance solutions. To compute these we shall need expressions for $X'(\hat{v})^{-1}$ and $(A^TA)^{-1}A^T$. A simple calculation shows that

$$[X'(\hat{v})]^{-1} = \frac{1}{n}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(v_3) & \sin(\hat{v}_3) \\ 0 & \frac{-\sin(\hat{v}_3)}{\hat{v}_2} & \frac{\cos(\hat{v}_3)}{\hat{v}_2} \end{bmatrix}. \tag{19}$$

The two port model makes the calculation of $(A^TA)^{-1}A^T$ very easy since A^TA is diagonal. After some algebraic manipulations the variance in the least squares delay estimate has the final form

$$\sigma_{ls} = \frac{\lambda}{2\pi}\frac{1}{V_m\sqrt{SN_0}}\sqrt{\frac{1 + f\cos(2\phi)/n}{1 - f^2/n^2}}, \tag{20}$$

where S is the ratio of the shot noise to the total noise

$$S \equiv \frac{N_0}{N_0 + n\sigma_n^2}, \tag{21}$$

and f is the parameter derived from the the number of temporal bins and the ratio of the stroke length to the wavelength defined as

$$f = \frac{\sin(2\pi\gamma)}{\sin(2\pi\gamma/n)}; \quad \gamma \equiv s/\lambda. \tag{22}$$

The ratio under the radical in (20) illuminates the delay error as a function of the number of temporal bins and the ratio between the stroke length and wavelength. For example, if $\gamma = 1$, corresponding to equal wavelength and stroke length, $f = 0$ so that the ratio is one and the error is independent of the delay. In general this is not the case. Another interesting case is when $n = 4$ (corresponding to a 4–bin algorithm). In this case $\gamma = 2$ (stroke length is twice the wavelength) leads to a singularity. To see this how this happens observe that $f/n = -1$ when $n = 4$ and $\gamma = 2$. Now if $\phi \neq 0$ it follows that $1 - f^2/n^2 = 0$, but $1 + f\cos(2\phi)/n \neq 0$. Thus the ratio is infinite, and the variance blows up. One simple conclusion from (20) is to choose a stroke length such that $1 - f/n$ is maximized over the operating wavelengths of the interferometer.

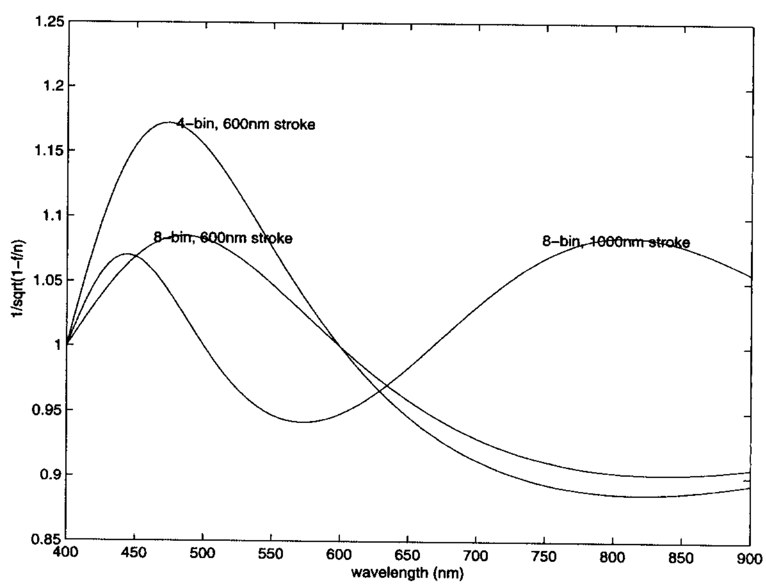

Figure 1. Comparison of performance for three estimator design parameters

In Figure 1 we plot the function $1/\sqrt{1 - f/n}$ for n=4, n=8, from wavelengths of 400nm–900nm and two different stroke lengths – 1000nm and 600nm (although the case with 4 dither steps and a 1000nm stroke is excluded because of the singularity at the 500nm wavelength). From the figure it is clear that there is an advantage to use an 8 dither step algorithm coupled with a 600nm stroke length. The picture becomes a little more complicated when the effect of the parameter S is included, since as the number of pixels that must be read increases, the value of S will decrease; and thus an increase the variance of the error occurs. In Figure 2 we make the same three comparisons as in Figure 1, but this time we use representative values for S and N_0.

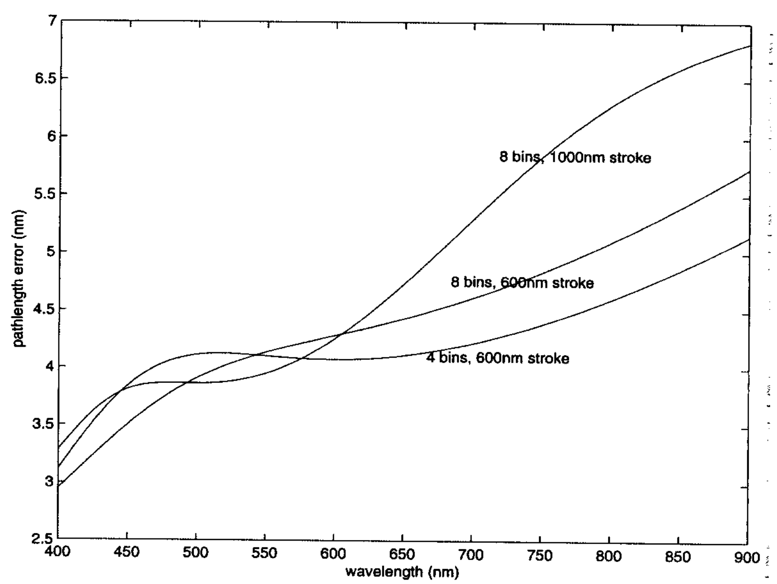

Figure 2. Comparison of performance for three estimator design parameters

A similar computation for the error variance can be made for minimum variance estimator. This estimate is obtained from the least squares problem (12) weighted by the inverse of measurement covariance matrix. The resulting variance is

$$\sigma_{mve} = \frac{\lambda}{2\pi} \frac{1}{V_m\sqrt{2N_0S}} \sqrt{\left[\frac{1}{n}\sum_{i=1}^{n} \frac{\sin^2 u_i}{1 - S^2V_m^2\cos^2 u_i}\right]^{-1}} \tag{23}$$

The ratio of the variance of the least squares and minimum variance estimators at zero delay error is given by

$$\frac{\mathrm{Var}_{ls}}{\mathrm{Var}_{mve}} = \sum \{ \frac{\sin^2(u_i)}{1 - S^2 V_m^2 \cos^2(u_i)} \} [\sum \sin^2(u_i)]^{-1}. \tag{24}$$

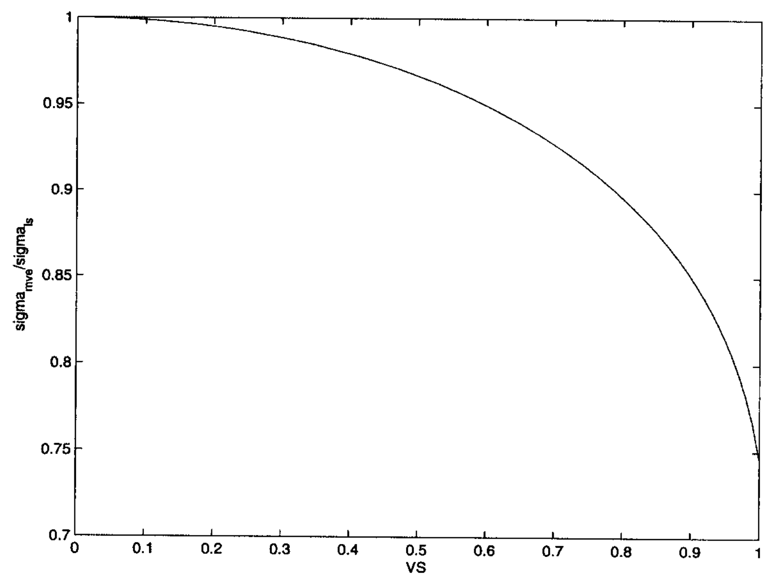

Figure 3. Ratio of $\sqrt{Var_{ls}/Var_{mve}}$ *as a funciton of* VS

Figure 3 plots the square root of this ratio as a function of the product VS for an 8 dither step design. Observe that as $V_m S \to 0$, this ratio goes to 1, while as $VS \to 1$ and $n \to \infty$, the ratio tends to $2/(1 - \mathrm{sinc}(2\pi\gamma))$. For nominal values of visibility, shot noise, and read noise for the SIM guide interferometers ($SV_m = .38$), there is a small (2%) advantage in using the minimum variance solution versus the least squares solution. Because of this relatively small advantage, for the remainder of the paper we will focus primarily on the unweighted least squares solution.

4 Improving estimates.

Several factors can degrade the performace of monochromatic estimators. For SIM applications these include low SNR for the guide interferometers (because they operate at 1Khz), vibration or imperfections in the modulation, and the necessary use of relatively wide passbands to improve the low SNR. This latter difficulty, especially when a monochromatic model becomes inadequate, is treated in a preliminary fashion in Section xx. In this section we address the first two of these problems.

The form for the general bias error for the class of estimators under consideration was given in (16). Previous work [2] has shown that the least squares algorithm is biased when the stroke length and wavelength do not match. A bias correction algorithms was worked out in [2] for the minimum variance estimator. Below we compute the bias in the least squares solution for the two port model and then discuss ways to mitigate it.

For an arbitrary 3–vector z let

$$r = \frac{1}{2} X'(\hat{v})^{-1} X''(\hat{v})[z, z]. \tag{25}$$

The bias in the phase is the expected value of the third component of r, r_3, computed using $z = X'(\hat{v})^{-1} K\eta$ above, where $K = (A^T A)^{-1} A^T$. In general it is readily established that

$$r_3 = \frac{z_2 z_3}{\hat{v}_2}. \tag{26}$$

Working through the algebra and noting that $\langle K_i, \eta \rangle^2 = Q_{ii}$ (the variance of the error in the estimate $X_i(\hat{v})$), it can be deduced that

$$E(r_3) = \frac{\sin(2\hat{v}_3)}{2\hat{2}v^2}[Q_{33} - Q_{22}] \tag{27}$$

is the bias in the least squares estimate of the two port model and has magnitude proportional to the product of the square of the reciprocal of the SNR and the phase offset because of the nonlinearity of the problem.

This bias due to the low SNR can be reduced in several ways: by direct compensation based on (29), integrating for a longer period, or use phasor averaging. But because the guide interferometers must operate at a 1Khz update rate, the second option is not viable. For precision astrometric purpsoses the primary quantity of interest is an accurate value of the average delay over an integration period of tens of milliseconds for the guide interferometers and is not required at the 1Khz rate. The bias correcting approach requires the correction with each phase estimate and is thus more computationally intensive than the phasor averaging approach. For this reason phasor averaging is perhaphs the most desirable way for treating the low SNR problem.

The issue that arises for phasor averaging is whether the average value of the computed phasors when transformed to a delay is equal to the average value of the delay. To address this question consider a sequence of observation vectors $\{y^k\}_{k=1}^M$produced by the states v^k. (In the noiseless case we would have $y^k = AX(v^k)$, identically.) Let

$$\bar{v} = \frac{1}{M} \sum v^k, \tag{28}$$

and set $\delta v^k = v^k - \bar{v}$. We recover the average state from the phasors via the estimate

$$\bar{w} \equiv X^{-1}(\sum X(v^k)/M). \tag{29}$$

To determine the error in this, note that to second order (using $\sum \delta v^k = 0$)

$$\begin{aligned} X^{-1}(\sum X(v^k)/M) &= X^{-1}(\frac{1}{M} \sum X(\bar{v}) + X'(\bar{v})\delta v^k + \frac{1}{2} X''(\bar{v})[\delta v^k, \delta v^k]) \\ &= \bar{v} + (\frac{1}{M} \sum X'(\bar{v})^{-1} X''(\bar{v})[\delta v^k, \delta v^k]. \end{aligned} \tag{30}$$

Let $\bar{r} = \bar{w} - \bar{v}$. Then from (28),

$$\bar{r}_3 = \frac{1}{M} \sum (\delta v_2^k \delta v_3^k / v_2^k). \tag{31}$$

Thus if $N_0 V$ is constant over the averaging period so that $\delta v^k = 0$, there is no error through second order in computing the average phase by first averaging the phasors.

Another difficulty encountered during phase estimation are factors that lead to a changing phase while the phase measurement is being made, e.g. the presence of vibrations, modulator performance deviating from nominal, etc. Mechanical vibrations have a strong adverse effect on MAM white light performance. The SIM equivalent of this error are large external path changes while a white light measurement is being made. Because the dominant external path error for SIM will be the relative low frequency ACS error this should not be a major problem for SIM. (The external path changes are not observable by any of the SIM instruments, other than the white light interferometers.) The internal pathlength can change due to vibration of the optical train and modulator non–idealities. These changes are monitored by the internal metrology system. Here we will discuss a correction of the phasors based on the metrology measurements.

Let k denote the wavenumber of the monochromatic light, s is the stroke length of the modulator, N denotes the number of bins, and x_i $(i = 1, ..., N)$ are the midpoints in each temporal bin. I_0 is the intensity of the signal and V is its visibility. The photo–electron counts per bin is given by

$$N_{\pm i} = \int_{x_i - \Delta/2}^{x_i + \Delta/2} I_0 \{1 \pm V \cos(kx + kr(x))\} dx, \tag{32}$$

where $\Delta = s/n$ and $r(x)$ is the change in pathlength during the fringe estimation that we are trying to compensate for using metrology measurements. The quantity we wish to estimate is $\bar{r}$, the average value of r over the fringe estimation period (or the phasor quantities associated with this mean pathlength difference). To this end let $\delta(x) = r(x) - \bar{r}$ and expand the integrand above about $kx + k\bar{r}$. Retaining only terms that are linear in $\delta(x)$ leads to the model

$$N_\pm = A_\pm X(v) - B_\pm X(v), \tag{33}$$

where the i^{th} row of $B_\pm$ is

$$B_{\pm i} = \pm \begin{bmatrix} 0 & \int_{x_i-\Delta/2}^{x_i+\Delta/2} \sin(kx)\delta(x) & \int_{x_i-\Delta/2}^{x_i+\Delta/2} \cos(kx)\delta(x) \end{bmatrix}. \tag{34}$$

Concatenating the $\pm$ quantities as before yields

$$N = AX(v) - BX(v). \tag{35}$$

The matrix A is constant and independent of the variation $\delta(x)$, while B is a function of the variation. In general we would expect B to be time–varying because of vibrations, changes in the modulator stroke, etc. Let K be any matrix such that $KA = I$, i.e. K is an unbiased estimator of $X(v)$ (assuming $B = 0$.) Then multiplying by K we get

$$\hat{X}(v) = X(v) - KBX(v). \tag{36}$$

Therefore,

$$X(v) = (I - KB)^{-1} \hat{X}(v). \tag{37}$$

And since $|KB| << 1$, the approximation $(I - KB)^{-1} \approx I + KB$ is valid, and consequently to first order in $|\delta(x)|$

$$X(v) = \hat{X}(v) + KB\hat{X}(v). \tag{38}$$

Hence the sought after perturbation is simply $KB\hat{X}(v)$.

Implementing this update depends on how B is computed. The matrix K is *fixed.* For the guide interferometers (and also on MAM), $\delta(x)$ is partially observed by metrology. Below we will indicate the computation assuming the metrology is sampled at the camera rate. With this assumption the metrology measurement is

$$\delta_i = \frac{1}{\Delta} \int_{x_i - \Delta/2}^{x_i + \Delta/2} \delta(x) dx. \tag{39}$$

Thus we can think of the n measurements made by metrology during the period of a single fringe measurement as a mapping $m : L_2 \to R^n$ with

$$m(\delta(x)) = \frac{1}{\Delta} (\int_{x_1 - \Delta/2}^{x_1 + \Delta/2} \delta(x) dx, ..., \int_{x_n - \Delta/2}^{x_n + \Delta/2} \delta(x) dx). \tag{40}$$

Given the measurement vector there is no unique way of reconstructing the function that produced it. One simple approach is to assume the function is a step function. In this case given a measurement vector $\delta \in R^n$, we define the inverse of m as the function $\tilde{\delta}$ that has the value δ_i on the interval $x_i - \Delta/2 \le x \le x_i + \Delta/2$. In this case we obtain the implementation

$$B_{\pm i} = \pm [\, 0 \quad k\delta_i \sin(kx) \quad k\delta_i \cos(kx) \,]\,. \tag{41}$$

5 Concluding remarks.

Precision white light interferometry is a cornerstone technology for the success of the Space Interferometry Mission. SIM presents many unique challenges in this area becasue of the various constraints and of the accuracies required. Although the project has many of the individual technologies well in hand, the system level aspects still provide rich motivation and fertile ground for further innovation.

Acknowledgment

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