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FROM: Behçet Açıkmeşe and John M. Carson III

SUBJECT: LMI-based Generation of Feedback Laws for a Robust Model Predictive Control Algorithm

ABSTRACT

This technical note provides a mathematical proof of Corollary 1 from the paper "A Nonlinear Model Predictive Control Algorithm with Proven Robustness and Resolvability" that appeared in the 2006 *Proceedings of the American Control Conference* [1]. The proof was omitted for brevity in the publication. The paper was based on algorithms developed for the FY2005 R&TD (Research and Technology Development) project for Small-body Guidance, Navigation, and Control [2].

The framework established by the Corollary is for a robustly stabilizing MPC (model predictive control) algorithm for uncertain nonlinear systems that guarantees the resolvability of the associated finite-horizon optimal control problem in a receding-horizon implementation. Additional details of the framework are available in the publication.

[1]Cleared for U.S. and foreign release, CL#08-0025

1 Introduction

The equation numbers (aside from those with a "C1" prefix) in this document match those in the *American Control Conference* paper [1]. Some of the equations, Conditions, and a Lemma from [1] are repeated in this section for clarity in the Corollary and proof.

The MPC results are specialized to the following class of systems,

$$\begin{aligned} \dot{x} &= Ax + Bu + E\phi(t,q) \\ q &= C_q x + D_q u \,, \end{aligned} \tag{14}$$

where $\phi : \mathbb{R} \times \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$ is a continuously differentiable function representing the uncertain nonlinear part of the dynamics, i.e. $f(x,u,t) = Ax + Bu + E\phi(t,q)$ in (1). The nominal system dynamics is assumed to have the following form,

$$\begin{aligned} \dot{z} &= Az + Bu_o + E\psi(t,q_o) \\ q_o &= C_q z + D_q u_o \,, \end{aligned} \tag{15}$$

where $\psi : \mathbb{R} \times \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$ is an approximation for ϕ in the real system (14), i.e. $F(z,u_o,t) = Az + Bu_o + \psi(t,q_o)$ in (5).

The following Conditions are a part of the MPC Algorithm:

Condition 1. *Function h in the FHC satisfies*

$$h(z,u) \geq a||z||^p + b||u||^r \,, \quad \forall\, z,\, u \,,$$

with $p \geq 1$, $r \geq 0$, a and b both positive constants, and $h(0,0) = 0$. ◇

Condition 2. *Function V in the FHC is positive definite [3] and there exists a feedback control law $u = \mathcal{L}(x)$ and $u_o = \mathcal{L}(z)$ such that V defines a Lyapunov function for (1) and (5) satisfying*

$$\begin{aligned} \nabla V(x) f(x,\mathcal{L}(x),t) + h(x,\mathcal{L}(x)) &\leq 0 \,,\ \forall\, x \in \Omega_o \,, \\ \nabla V(z) F(x,\mathcal{L}(z),t) + h(z,\mathcal{L}(z)) &\leq 0 \,,\ \forall\, z \in \Omega_o \,, \end{aligned}$$

where $\Omega_o \subset \mathbf{X}_o$ contains the origin. Additionally, feedback law $\mathcal{L}$ renders $\Omega_o \subset \mathbb{R}^n$ invariant for dynamics (1) and (5), i.e., if $x(t_0) \in \Omega_o$ ($z(t_0) \in \Omega_o$) for some t_0, then $x(t) \in \Omega_o$, $\forall\, t \geq t_0$ ($z(t) \in \Omega_o$, $\forall\, t \geq t_0$). It is also assumed that

$$\mathcal{L}(x) \in \mathbf{U}_o \,, \qquad \forall\, x \in \Omega_o \,.$$

◇

Condition 3. *There exist closed balls[2] around the origin $\mathbf{B}_R$ and $\mathbf{B}_r$ in $\mathbb{R}^n$ with radii $R > r$ such that set Ω_o in the FHC satisfies the following*

$$\mathbf{X}_f \subseteq \mathbf{B}_r \subset \mathbf{B}_R \subseteq \Omega_o \,.$$

◇

Condition 4. *There exists a feedback control law $u_f = \mathcal{K}(x,z)$ in (3) that renders the set $\mathbf{X}_f$ invariant for $\eta \triangleq x - z$ with dynamics (1) for x and (5) for z, that is, if $\eta(t_0) \in \mathbf{X}_f$ for some $t_0 \geq 0$, then $\eta(t) \in \mathbf{X}_f$ $\forall\, t \geq t_0$ and for all $u_o(\cdot)$. Additionally, $u_f = \mathcal{K}(x,z) \in \mathbf{U}_f$ if $\eta(t) \in \mathbf{X}_f$.* ◇

[2] $\mathbf{B}_\rho \triangleq \{v \,:\, \|v\| \leq \rho\}$.

Condition 5. *Functions ϕ and ψ are continuously differentiable and there exists a closed and convex set of matrices $\Theta \subseteq \mathbb{R}^{n_p \times n_q}$ such that*

$$\frac{\partial \phi}{\partial q}(t,q) \in \Theta \quad \text{and} \quad \frac{\partial \psi}{\partial q}(t,q) \in \Theta\,, \qquad \forall\, q,\, t\,.$$

◇

Condition 6. *There exists a scalar $\gamma > 0$ such that*

$$\|\phi(t,q_o) - \psi(t,q_o)\| \leq \gamma\,, \quad \forall\, t\,,\ z \in \mathbf{X}_o\,,\ u_o \in \mathbf{U}_o\,,$$

where $q_o = C_q z + D_q u_o$ as in 15. ◇

Condition 7. *The following hold for the constraint sets in the FHC,*

$$\begin{array}{l}
\mathbf{X}_o = \{x \in \mathbb{R}^n :\ a_i^T x \leq 1,\ i = 1,\ldots,m_o\},\\
\mathbf{X}_f = \{x \in \mathbb{R}^n :\ b_i^T x \leq 1,\ i = 1,\ldots,m_f\},\\
\mathbf{U}_o = \{u \in \mathbb{R}^m :\ u^T \Pi_o u \leq 1\},\\
\mathbf{U}_f = \{u \in \mathbb{R}^m :\ u^T \Pi_f u \leq 1\},
\end{array}$$

where Π_o, Π_f are symmetric positive-definite matrices. ◇

Lemma 3. *Consider a continuously differentiable function $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ with its Jacobian given by $\partial\varphi(q)/\partial q$. Suppose that there exists a closed convex set $\Lambda \in \mathbb{R}^{n \times m}$ such that*

$$\frac{\partial \varphi}{\partial q}(q) \in \Lambda\,, \qquad \forall q\,.$$

Then, for every q_1 and q_2 there exists $\Delta \in \Lambda$ such that

$$\varphi(q_2) - \varphi(q_1) = \Delta(q_2 - q_1)\,.$$

◇

2 Corollary 1 with Proof

Corollary 1. *Consider an uncertain nonlinear system (14) with a nominal model given by (15) satisfying Conditions 5, 6, and 7 with*

$$\Theta = \{\theta \in \mathbb{R}^{n_p \times n_q}\ :\ \|\theta\| \leq 1\}\,. \tag{24}$$

Suppose that there exist matrices $S = S^T > 0$, $Q = Q^T > 0$, L, Y and positive scalars λ, β, μ, c_1, and c_2 satisfying the following matrix inequalities,

$$\begin{bmatrix} \begin{array}{c} SA^T + AS + BL + L^T B^T + S/\lambda \\ +(\beta + \lambda\gamma^2)EE^T \\ C_q S + D_q L \end{array} & \begin{array}{c} SC_q^T + L^T D_q^T \\ \\ -\beta I \end{array} \end{bmatrix} \leq 0 \tag{25}$$

$$\begin{bmatrix} QA^T + AQ + BY + Y^T B^T + \mu EE^T & QC^T + Y^T D^T & QC_q^T + Y^T D_q^T \\ CQ + DY & -I & 0 \\ C_q Q + D_q Y & 0 & -\mu I \end{bmatrix} \leq 0 \tag{26}$$

$$\begin{bmatrix} S & L^T \\ L & \Pi_f^{-1} \end{bmatrix} \geq 0\,, \quad \begin{bmatrix} Q & Y^T \\ Y & \Pi_o^{-1} \end{bmatrix} \geq 0\,, \tag{27}$$

$$a_i^T Q a_i \le 1\,, \quad i = 1, \ldots, m_o\,, \tag{28}$$
$$b_i^T S b_i \le 1\,, \quad i = 1, \ldots, m_f\,, \tag{29}$$

$$Q \ge c_1 I > c_2 I \ge S\,, \tag{30}$$

where C and D are matrices satisfying

$$C^T D = 0\,.$$

Then, the ellipsoids $\varepsilon_Q = \{x : x^T Q^{-1} x \le 1\}$ and $\varepsilon_S = \{x : x^T S^{-1} x \le 1\}$ satisfy $\varepsilon_Q \subseteq \mathbf{X}_o$ and $\varepsilon_S \subseteq \mathbf{X}_f$. And, the MPC Algorithm with

$$\begin{aligned} h(x,u) &= \|Cx\|^2 + \|Du\|^2 \\ V(x) &= x^T Q^{-1} x \end{aligned} \tag{31}$$

$$\begin{aligned} \mathcal{L}(x) &= Kx\,, & K &= YQ^{-1}\,, \\ \mathcal{K}(x,z) &= K_f(x-z)\,, & K_f &= LS^{-1}\,, \end{aligned} \tag{32}$$

and ε_Q and ε_S replacing Ω_o and $\mathbf{X}_f$ results in an asymptotically stable closed loop system for (14) with a region of attraction $\mathcal{R}_a$ given by (13). □

Proof.

Establish Ω_o Invariance for System (14) when $x \in \Omega_o$:

Pre- and post-multiplying (26) by $diag(Q^{-1}, I, I)$ and utilizing $K = YQ^{-1}$ from (32) gives

$$\begin{bmatrix} A^TQ^{-1}+Q^{-1}A+Q^{-1}BK+K^TB^TQ^{-1}+\mu Q^{-1}EE^TQ^{-1} & C^T+K^TD^T & C_q^T+K^TD_q^T \\ C+DK & -I & 0 \\ C_q+D_qK & 0 & -\mu I \end{bmatrix} \le 0. \tag{C1-1}$$

Utilizing multiple Schur complements [4] and $C^TD = 0$, this matrix inequality can be written as

$$\begin{bmatrix} \begin{pmatrix} A^TQ^{-1}+Q^{-1}A+Q^{-1}BK+K^TB^TQ^{-1}+C^TC \\ +K^TD^TDK+\frac{1}{\mu}(C_q+D_qK)^T(C_q+D_qK) \end{pmatrix} & Q^{-1}E \\ E^TQ^{-1} & -\frac{1}{\mu}I \end{bmatrix} \le 0. \tag{C1-2}$$

For $\zeta = (x^T, \phi^T)^T$, corresponding to the state x and nonlinearity ϕ for the actual system (14), taking $\zeta^T M \zeta$, where M is the matrix in the preceding inequality, gives

$$x^T\left(A^TQ^{-1}+Q^{-1}A\right)x + 2x^TQ^{-1}(Bu+E\phi) + \|Cx\|^2 + \|Du\|^2 + \frac{1}{\mu}\left(q^Tq - \phi^T\phi\right) \le 0, \tag{C1-3}$$

where $u = \mathcal{L}(x) = Kx$ from (32) and $q = C_qx + D_qu$ have been utilized.

Applying Lemma 3 with Condition 5, and making use of (24) and $\phi(0,t) = 0\ \forall t$, gives

$$q^Tq - \phi^T\phi \ge 0. \tag{C1-4}$$

Further, $h(x,u) = \|Cx\|^2 + \|Du\|^2 \ge 0$ (which satisfies Condition 1). These inequalities imply

$$x^T\left(A^TQ^{-1}+Q^{-1}A\right)x + 2x^TQ^{-1}(Bu+E\phi) \le 0, \tag{C1-5}$$

which gives $\dot{V}(x) \le 0$ for $V(x) = x^TQ^{-1}x$ as defined in (31). Thus, $V(x)$ is a Lyapunov function for the actual system (14) and establishes the invariance of $\Omega_o = \{x : x^TQ^{-1}x \le 1\}$ with control policy

$u = Kx$ from (32). Further, inequality (28) ensures $\Omega_o \subseteq \mathbf{X}_o$ (with $\mathbf{X}_o$ defined as in Condition 7) [4].

Establish Bounded Input $u \in \mathbf{U}$ when $x \in \Omega_o$:

Pre- and post-multiplying the second LMI in (27) by $diag(Q^{-1}, I)$ and utilizing a Schur complement gives $Q^{-1} - K^T \Pi_o K \geq 0$, which is equivalent to

$$u^T \Pi_o u \leq x^T Q^{-1} x \tag{C1-6}$$

after pre- and post-multiplying with x^T and x, respectively, and utilizing the feedback policy $u = Kx$ from (32). Thus, for $x \in \Omega_o$, $u^T \Pi_o u \leq 1$, so $u \in \mathbf{U}_o$ from Condition 7.

Establish Ω_o Invariance for System (15) and Bounded Input $u_o \in \mathbf{U}_o$ when $z \in \Omega_o$:

The above results apply identically to the nominal system (15) when $z \in \Omega_o$ through the substitution of $\zeta = (z^T, \psi^T)^T$, with ψ corresponding to the nonlinearity for the nominal system (15), $u_o = \mathcal{L}(z) = Kz$, and $q_o = C_q z + D_q u_o$. Thus, Corollary 1 establishes a framework and feedback policy satisfying Conditions 1 and 2 for systems (14) and (15).

Establish ε_S Invariance (i.e. $\mathbf{X}_f$ invariance when $\mathbf{X}_f$ restricted to ε_S):

Let positive-definite function $V(\eta) = \eta^T S^{-1} \eta$ be a Lyapunov function candidate. Pre- and post-multiplying (25) by $diag(S^{-1}, I)$ and utilizing $K_f = LS^{-1}$ from (32) gives

$$\begin{bmatrix} A^T S^{-1} + S^{-1} A + S^{-1} B K_f + K_f^T B^T S^{-1} + S^{-1}/\lambda + (\beta + \lambda\gamma^2) S^{-1} E E^T S^{-1} & C_q^T + K_f^T D_q^T \\ C_q + D_q K_f & -\beta I \end{bmatrix} \leq 0 \tag{C1-7}$$

Using multiple Schur complements, this matrix inequality can be written as

$$\begin{bmatrix} \begin{pmatrix} A^T S^{-1} + S^{-1} A + S^{-1} B K_f + K_f^T B^T S^{-1} \\ + \frac{1}{\beta}(C_q + D_q K_f)^T (C_q + D_q K_f) + S^{-1}/\lambda \end{pmatrix} & S^{-1} E & S^{-1} E \\ E^T S^{-1} & -\frac{1}{\beta} I & 0 \\ E^T S^{-1} & 0 & -\frac{1}{\lambda}\frac{1}{\gamma^2} I \end{bmatrix} \leq 0 \tag{C1-8}$$

For $\zeta = (\eta^T, \pi^T, w^T)^T$, taking $\zeta^T M \zeta$, where M is the matrix in the preceding inequality, gives

$$\begin{aligned} &\eta^T (A^T S^{-1} + S^{-1} A)\eta + 2\eta^T S^{-1}(B u_f + E\pi + E w) \\ &\qquad + \tfrac{1}{\beta}\left((C_q \eta + D_q u_f)^T (C_q \eta + D_q u_f) - \pi^T \pi\right) + \tfrac{1}{\lambda}\left(\eta^T S^{-1} \eta - \tfrac{1}{\gamma^2} w^T w\right) \leq 0, \end{aligned} \tag{C1-9}$$

where $u_f = K_f \eta$ from (32) has been utilized for the difference dynamics in system (20).

Making use of relationship (22) for π and the properties of θ in (24) gives

$$(C_q \eta + D_q u_f)^T (C_q \eta + D_q u_f) - \pi^T \pi \geq 0, \tag{C1-10}$$

thus

$$\eta^T (A^T S^{-1} + S^{-1} A)\eta + 2\eta^T S^{-1}(B u_f + E\pi + E w) + \tfrac{1}{\lambda}\left(\eta^T S^{-1} \eta - \tfrac{1}{\gamma^2} w^T w\right) \leq 0. \tag{C1-11}$$

From (21), $\frac{1}{\gamma^2} w^T w \leq 1$, thus

$$\tfrac{1}{\gamma^2} w^T w \leq \eta^T S^{-1} \eta \text{ when } \eta^T S^{-1} \eta \geq 1, \tag{C1-12}$$

which implies

$$\eta^T(A^TS^{-1}+S^{-1}A)\eta+2\eta^TS^{-1}(Bu_f+E\pi+Ew)\leq 0 \text{ when } \eta^TS^{-1}\eta\geq 1, \tag{C1-13}$$

and thus $\dot{V}(\eta)\leq 0$ when $\eta^TS^{-1}\eta\geq 1$. Thus, $\varepsilon_S=\{\eta:\ \eta^TS^{-1}\eta\leq 1\}$ is an invariant set for η [5]. Additionally, inequality (29) ensures $\varepsilon_S\subseteq\mathbf{X}_f$ (with $\mathbf{X}_f$ defined as in 7) [4].

Establish Bounded Feedback $u_f\in\mathbf{U}_f$ when $\eta\in\varepsilon_S$:

Pre- and post-multiplying the first LMI in (27) by $diag(S^{-1},I)$ and utilizing a Schur complement gives $S^{-1}-K_f^T\Pi_fK_f\geq 0$, which is equivalent to

$$u_f^T\Pi_fu_f\leq\eta^TS^{-1}\eta \tag{C1-14}$$

after pre- and post-multiplying with η^T and η, respectively, and utilizing the feedback policy $u_f=K_f\eta$ from (32). Thus, for $\eta\in\varepsilon_S$, $u_f^T\Pi_fu_f\leq 1$, so $u_f\in\mathbf{U}_f$ from Condition 7. Restricting $\mathbf{X}_f$ to ε_S provides a conservative feedback policy that renders the restricted $\mathbf{X}_f$ invariant.

Thus, Corollary 1 establishes a framework and feedback policy satisfying Condition 4 for the difference dynamics in system (20).

Establish $\mathbf{X}_f$ Contained in Ω_o:

Inequality (30) implies $S^{-1}\geq\frac{1}{c_2}I>\frac{1}{c_1}I\geq Q^{-1}$. Then, for $x\in\mathbf{X}_f$ (with $\mathbf{X}_f$ restricted to ε_S)

$$1\geq x^TS^{-1}x\geq\tfrac{1}{c_2}x^Tx>\tfrac{1}{c_1}x^Tx\geq x^TQ^{-1}x. \tag{C1-15}$$

Thus, an $x\in\mathbf{X}_f$ is also $x\in\mathbf{B}_r$, $x\in\mathbf{B}_R$, and $x\in\Omega_o$ with $r=\sqrt{c_2}$, $R=\sqrt{c_1}$, and $\Omega_o=\varepsilon_Q$, respectively. The inequality establishes Condition 3 such that $\mathbf{X}_f\subset\Omega_o$. □

References

[1] A. B. Açıkmeşe and J. M. Carson. A nonlinear model predictive control algorithm with proven robustness and resolvability. In *Proceedings of the American Control Conference*, pages 887–893, June 2006.

[2] A. B. Açıkmeşe and J. M. Carson. Small body GN&C research report: A robust model predictive control algorithm. Internal Document JPL D-32947, Jet Propulsion Laboratory, California Institute of Technology, Pasadena, CA 91109, USA, September 2005.

[3] H. K. Khalil. *Nonlinear Systems, Second Edition.* Prentice Hall, 1996.

[4] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* SIAM, 1994.

[5] A. B. Açıkmeşe and M. Corless. Robust tracking and disturbance rejection of bounded rate signals for uncertain/non-linear systems. *International Journal of Control*, 76(11):1129–1141, 2003.