Aerodynamics of Trans-Atmospheric Vehicles: A Non-Dimensional Approach

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A novel non-dimensional approach to flight dynamics of a trans-atmospheric vehicle is proposed. This approach explicitly takes into account the vertical span of the atmosphere (atmospheric scale height) as well as atmospheric mass load (atmospheric pressure) at the given flight level. As an example of application of this approach, a simple analytic model of the flight dynamics is considered for the powered boost and non-powered glide of a trans-atmospheric vehicle.

Nomenclature

\( c \) = speed of sound
\( C \) = constant of integration
\( C_D \) = aerodynamic drag coefficient
\( D \) = aerodynamic drag force, distance
\( F \) = thrust
\( g \) = gravitational acceleration
\( g_0 \) = gravitational acceleration on Earth, a conversion factor between unities of mass and weight
\( h \) = altitude
\( H \) = atmospheric scale height
\( I_p \) = specific impulse
\( i_p \) = non-dimensional specific impulse
\( L \) = aerodynamic lift force
\( M \) = Mach number
\( m \) = mass of the vehicle
\( m_A \) = mass of the atmospheric column over unit area
\( n \) = thrust-to-weight ratio
\( p \) = atmospheric (static) pressure
\( q \) = dynamic pressure
\( R \) = gas constant; planetary radius
\( S \) = reference area of the vehicle
\( T \) = atmospheric temperature
\( t_H \) = atmospheric scale time
\( V \) = airspeed, velocity
\( V_H \) = atmospheric scale airspeed
\( V_o \) = orbital velocity
\( \Delta \) = non-dimensional distance
\( \lambda \) = \((L/D)\), lift-to-drag ratio
\( \mu \) = \(\mu\) - ratio, a non-dimensional mass parameter
\( \gamma \) = \(c_p/c_v\), ratio of specific heats
\( \theta \) = flight path angle
\( \rho \) = atmospheric density
\( \tau \) = non-dimensional time
\( \nu \) = \( \nu \)-number, a non-dimensional velocity
\( \nu_o \) = \( V_o/V_H \), non-dimensional velocity

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I. Introduction

NASA and JPL, in particular, is pursuing a vigorous program of exploration of Solar System. The target planets with atmospheres pose additional problem, it the particular mission involves atmospheric flight of planetary spacecraft. Usually, the preliminary design of spacecraft involves numerous trade studies in a multidimensional space of parameters describing the atmosphere and the spacecraft, essentially a trans-atmospheric flight vehicle (TAV) of special kind, depending on the particular mission.

The future reusable launch vehicles, which are expected to substantially reduce the cost of access to space, will, very likely, use aerodynamic lift on their way to space and back to Earth. In this sense, they will also be TAVs, and their design process will also involve trade studies involving both the parameters of TAV and those of the Earth atmosphere.

It is intuitively understood that the design of a TAV, traversing the atmosphere depends on the vertical extent and mass of the atmosphere. Hence, one might speculate that there should be some explicit physical interdependencies involving the mass and extent of the atmosphere on the one hand and some parameters of the vehicle, presumably, mass and size, on the other. Further on, one can expect that, if these parameters, describing both the atmosphere and the vehicle could be gathered into some combinations, in which they could be kept in the analysis and trade studies, then the net number of dimensions in the relevant parameter space will, herewith be decreased, making the trade studies more easy and efficient.

In this paper we present a non-dimensional approach that offers an analytic insight into some of key interdependencies relevant to performance of TAV. In addition to well-known non-dimensional parameters, such as lift-to-drag and thrust-to-weight ratios, we identify two non-dimensional parameters, which, indeed, contain combinations of relevant dimensional parameters of the vehicle and of the atmosphere. These non-dimensional parameters make it possible to reformulate the equations of motion of atmospheric flight in a non-dimensional form.

II. Vertical extent and mass of the atmosphere

This short Section summarizes some reference results to be used in following Sections below. Vertical extent of a planetary atmosphere is described by its scale height \( H \). Using the hydrostatic equation in the form
\[
dp = -\rho g dh
\]
and expressing the atmospheric density \( \rho \) through the equation of state
\[
p = \rho RT
\]
we obtain
\[
\frac{dp}{p} = -\frac{g}{RT}dh = -\frac{dh}{H}
\]
This yields a textbook expression for the scale height of exponential decay of atmospheric pressure with altitude:
\[
H = \frac{RT}{g}
\]

The mass of an atmospheric column over a unit area at a given flight level is expressed through atmospheric (static) pressure:
\[
m_A = \frac{p}{g} = \rho H
\]

The validity of second equality in Eq. (II.3) is easily verified using Eqs. (II.1) and (II.2).

III. Non-dimensional velocity and non-dimensional mass ratio

We introduce the non-dimensional velocity based on consideration of the ratio of the dynamic pressure
\[
q = \frac{1}{2} \rho V^2
\]

The validity of second equality in Eq. (II.3) is easily verified using Eqs. (II.1) and (II.2).
Introducing the *atmospheric scale airspeed*:

\[ V_H = \sqrt{2gH} \quad \text{(III.3a)} \]

we see that it has a meaning of velocity of free fall from altitude equal to the atmospheric scale height. Using definition of atmospheric scale height, Eq. (II.2), we have an alternative definition of \( V_H \):

\[ V_H = \sqrt{2RT} \quad \text{(III.3b)} \]

We define the first non-dimensional parameter introduced here, the \( \nu \) - *number*, as a ratio of the airspeed of the vehicle to \( V_H \):

\[ \nu = \frac{V}{V_H} \quad \text{(III.4)} \]

Using the \( \nu \) - number, the last equality in Eq. (III.2) can be transformed to a simple general relation between dynamic and static pressure

\[ q = p\nu^2 \quad \text{(III.5)} \]

This relation Eq. (III.5) is obtained directly from the definition of dynamic pressure, Eq. (I.1). Thus, the dynamic-to-static pressure ratio is equal to square of \( \nu \) - number at any regime of atmospheric flight. The dynamic pressure directly determines the aerodynamic forces acting on the vehicle flying in the atmosphere, and, as we see below, the relation Eq. (III.5) is instrumental in non-dimensionalizing of equations of motion in atmospheric flight.

The atmospheric scale velocity \( V_H \) and \( \nu \) - number have a straightforward relation to the speed of sound and corresponding non-dimensional velocity, the Mach number. Comparing their definitions:

\[ c = \sqrt{\gamma RT}, \quad M = \frac{V}{c} \]

with definitions of \( V_H \) and \( \nu \), Eqs. (III.3) and (III.4), we have correspondingly:

\[ V_H = \sqrt{\frac{2}{\gamma} c} \quad \text{(III.6)} \]

and

\[ \nu = \sqrt{\frac{\gamma}{2}} M \quad \text{(III.7)} \]

Using Eq. (III.7), the above relation between dynamic and static pressure, Eq. (III.5) can be rewritten in the form of a textbook relation between dynamic pressure and Mach number [cf., e.g., Eq. (5.42) of I]:

\[ q = \frac{\gamma}{2} pM^2 \]

The second non-dimensional parameter, the non-dimensional mass ratio, is introduced considering accelerations due to aerodynamic forces. Normalizing the acceleration due to atmospheric drag by gravitational acceleration, and using Eq. (III.5) we have:

\[ \frac{1}{g} \left( \frac{dV}{dt} \right)_{\text{drag}} = \frac{D}{mg} = \frac{1}{mg} qC_D S \frac{p}{m/(C_D S)} \nu^2 \quad \text{(III.8)} \]

The fraction in the last equality of Eq. (III.8) represents a non-dimensional ratio of the atmospheric mass over unit area to the ballistic coefficient of the vehicle. This is the second non-dimensional parameter introduced here, the \( \mu \) - ratio:

\[ \mu = \frac{\frac{p}{g}}{m/(C_D S)} \quad \text{(III.9)} \]
We see that by its meaning, this parameter measures the mass of the atmosphere above the given flight level in units of the ballistic coefficient of the vehicle. Since $\mu -$ ratio is proportional to atmospheric pressure, it can be used to represent the (pressure) altitude.

Using $\mu -$ ratio, the last equality of Eq. (III.8) can be rewritten as

$$\frac{1}{g} \left( \frac{dV}{dt} \right)_{\text{Drag}} = \mu \nu^2$$

(Int.9)

Introducing two auxiliary parameters, the atmospheric scale time $t_H$, as a time of free fall from the altitude equal to the atmospheric scale height:

$$t_H = \sqrt{\frac{2H}{g}} = \frac{V_H}{g} = \frac{2H}{V_H}$$

(III.10)

and corresponding non-dimensional time $\tau$:

$$\tau = \frac{t}{t_H} = \frac{g}{V_H}$$

(III.11)

we can rewrite Eq. (III.9) in a non-dimensional form:

$$\left( \frac{d\nu}{d\tau} \right)_{\text{Drag}} = \mu \nu^2$$

(III.12)

It should be noted that the representation of an acceleration term, as in the right side of Eq. (III.12) is based on an implicit assumption that $V_H$ varies with time much slower than $V$. For the strictly exponential (isothermal) atmospheres, this is always true. For realistic, non-isothermal atmospheres, this assumption turns out to be quite practical, but it has to be verified for the given flight profile under consideration.

IV. Non-dimensional equations of motion

First, we consider two equations of atmospheric flight with respect to the velocity $V$ and flight path angle $\theta$ in the form (see e.g., 1)

$$m \frac{dV}{dt} = F - qC_D S - mg \sin \theta$$

$$V \frac{d\theta}{dt} = \lambda qC_D S - mg \cos \theta$$

(IV.1)  

(IV.2)

where $\lambda = (L/D)$ is the lift-to-drag ratio. Expressing the thrust $F$ through thrust-to-weight ratio $n$, as

$$F = nmg$$

(IV.3)

and using Eq. (III.12), we transform the equations of motion to a non-dimensional form:

$$\frac{d\nu}{d\tau} = n - \mu \nu^2 - \sin \theta$$

(IV.4)

$$\nu \frac{d\theta}{d\tau} = \lambda \mu \nu^2 - \cos \theta$$

(IV.5)

These non-dimensional equations will be referred to here as $\nu -$ equation, and $\theta -$ equation respectively.

The non-dimensional equations Eq. (IV.4) and (IV.5) need to be complemented by an equation describing the variation of the $\mu -$ ratio with time. Using its definition, Eq. (III.9) and neglecting possible variation of the ballistic coefficient $C_D$ due to varying airspeed, we have:

$$\frac{d\ln \mu}{d\tau} = \frac{d\ln p}{d\tau} - \frac{d\ln m}{d\tau}$$

(IV.6)

For the first term in Eq. (IV.6), using Eqs. (II.1a) and (III.10), we obtain:
\[
\frac{d \ln p}{d \tau} = -\frac{1}{H} \frac{dh}{d \tau} = -\frac{t_H}{H} \cdot \frac{dh}{dt} = 2 \left(\frac{V}{V_H}\right) \sin \theta
\]

or, in the non-dimensional form,

\[
\frac{d \ln p}{d \tau} = -2 \nu \sin \theta \quad \text{(IV.7)}
\]

The second term in Eq. (IV.6) becomes non-zero in the case of powered flight, when the mass of vehicle decreases due to depletion of propellant. Expressing the thrust through specific impulse \( I_{sp} \) and propellant mass rate \( m \) as \( F = g_0 m I_{sp} \), and using Eq. (IV.3) we have:

\[
\frac{d \ln m}{d \tau} = -\frac{t_H}{m} \cdot \frac{F}{mg_0 I_{sp}}
\]

or, in the non-dimensional form,

\[
\frac{d \ln m}{d \tau} = -\frac{g}{g_0} \cdot \frac{n}{i_{sp}} \quad \text{(IV.8)}
\]

where \( i_{sp} = I_{sp} / t_H \) is a non-dimensional specific impulse. Substituting Eqs. (IV.7) and (IV.8) in Eq. (IV.6) we obtain the resulting \( \mu - \) equation:

\[
\frac{d \ln \mu}{d \tau} = \frac{g}{g_0} \cdot \frac{n}{i_{sp}} - 2 \nu \sin \theta \quad \text{(IV.9)}
\]

Equations (IV.4), (IV.5), and (IV.5) will be used below in simple analytic models of trans-atmospheric flight.

V. Aero-surfing, an equilibrium hypersonic flight

In a trans-atmospheric flight, the airspeed envelope extends, in general, all the way to orbital velocity \( V_o \), and possibly beyond. Thus, the centrifugal correction to the acceleration of gravity cannot be ignored, like in Eqs. (IV.1) and (IV.2). For illustration purposes, below we consider the case of equilibrium between aerodynamic lift and gravity. In this case, as we see below, the flight path angle becomes small at low hypersonic velocities, so that \( \sin \theta = \theta \), and \( \cos \theta = 1 \). Corresponding equations of motion have the form:

\[
m \frac{dV}{dt} = F - q C_D S - mg \left(1 - \frac{V^2}{V_o^2}\right) \theta
\]

\[
\lambda q C_D S = mg \left(1 - \frac{V^2}{V_o^2}\right)
\]

Performing the transformations of Eqs. (V.1) and (V.2), similar to those of Eqs. (V.1) and (V.2) above, we can obtain corresponding non-dimensional \( \nu - \) and \( \theta - \) equations in the form:

\[
\frac{d\nu}{d\tau} = n - \mu \nu^2 - \left(1 - \nu^2 / \nu_o^2\right) \theta
\]

\[
\lambda \mu \nu^2 = \left(1 - \nu^2 / \nu_o^2\right)
\]

\[
\frac{d \ln \mu}{d \tau} = \frac{g}{g_0} \cdot \frac{n}{i_{sp}} - 2 \nu \theta
\]

The \( \nu - \) and \( \theta - \) equations can be combined into a single \( \nu - \theta - \) equation.
which will be used below.

Before considering the specific cases of boost and glide below, we obtain two results applicable to both these cases. The \( \theta - \) equation (which, here, does not contain \( \theta \) at all) can be resolved with respect to \( \mu \):

\[
\mu = \frac{1}{\lambda} \left( \frac{1}{\nu} - \frac{1}{\nu_0^2} \right) \tag{V.7}
\]

or with respect to \( \nu \):

\[
\nu = \left( \frac{\lambda \mu + \nu_0^2}{\nu_0^2} \right)^{-1/2} \tag{V.8}
\]

Since \( \mu \) - ratio represents the pressure altitude, Eqs. (V.7) and (V.8) provide a non-dimensional relation between velocity and altitude. A corresponding dimensional expression can be found, e.g., in 2, and it is presented in a slightly modified form, which can be directly compared to Eq. (V.8):

\[
V = g^{1/2} \frac{(L/D) \rho}{2 m/(C_p S)} + \frac{1}{R} \tag{V.9}
\]

The second result refers to the flight path angle, which becomes dependent on velocity at \( \nu \)-numbers comparable to unity or greater. For the trans-atmospheric vehicle, to keep equilibrium between lift and gravity means that its acceleration (deceleration) should match variation of velocity due to atmospheric pressure varying with altitude. In other words, we demand that the value of

\[
\frac{d\nu}{d\tau} = \frac{d\nu}{d\mu} \cdot \frac{d\mu}{d\tau} \tag{V.10}
\]

where \( d\nu/d\mu \) and \( d\mu/d\tau \) are obtained, respectively, from Eqs. (V.8) and (V.5), is matching the value of \( d\nu/d\tau \) in Eq. (V.6). From Eqs. (V.8) and (V.5), we have:

\[
\frac{d\nu}{d\mu} = -\frac{1}{2} \nu^2 \cdot \nu^2 \cdot \left( \frac{g}{g_0} \cdot \frac{n}{2i_{sp} \nu} - \theta \right)
\]

and, using underway the \( \theta - \) equation, we rewrite Eq. (V.10) as

\[
\frac{d\nu}{d\tau} = -\frac{1}{2} \lambda \mu \nu^2 \cdot \nu^2 \left( \frac{g}{g_0} \cdot \frac{n}{2i_{sp} \nu} - \theta \right) = -\left( \frac{1}{\nu_0^2} - \theta \right) \cdot \nu^2 \left( \frac{g}{g_0} \cdot \frac{n}{2i_{sp} \nu} - 2\theta \right)
\]

Comparing the obtained expression for \( d\nu/d\tau \), with the right side of Eq. (V.6) we obtain an equation

\[
-\nu^2 \left( \frac{g}{g_0} \cdot \frac{n}{2i_{sp} \nu} - \theta \right) = n \left( 1 - \frac{\nu^2}{\nu_0^2} \right) \cdot \nu^2 \left( \frac{g}{g_0} \cdot \frac{n}{2i_{sp} \nu} - 2\theta \right)
\]

which can be solved for the flight path angle \( \theta \):

\[
\theta = \left( n \left( 1 - \frac{\nu^2}{\nu_0^2} \right) + \frac{g}{g_0} \cdot \nu \right) \cdot \left( \frac{1}{\lambda} \right) \left( \nu^2 + 1 \right)
\]

Corresponding expression through dimensional parameters has the form:

\[
\theta = \left( \frac{F}{mg} \left( 1 - \frac{V^2}{V_0^2} \right) + \frac{V}{2 g_0 I_{sp}} \right) \cdot \left( \frac{1}{\lambda} \right) \left( \frac{V}{V_0} \right)^2 + 1
\]

\[
\tag{V.12}
\]

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At low $\nu$-numbers, all terms with $\nu$ in Eq. (V.11) become negligible and we have:

$$\theta = n - \frac{1}{\lambda}$$  \hspace{1cm} (V.13)

If $\nu$-numbers are comparable to, or exceeding unity, but the term with $i_p$ still can be neglected, we have:

$$\theta = \left( n - \frac{1}{\lambda} \right) \left( \nu^2 + 1 \right)^{1/4}$$  \hspace{1cm} (V.14)

So, at $\nu = 1$ the flight path angle $\theta$ is twice less than its low-speed value, and it decreases rapidly with further increase of $\nu$-number. Thus, if the velocity of the vehicle is comparable or exceeds the atmospheric scale velocity $V_H$, the finite vertical extent of the atmosphere needs to be taken into account.

In this Section we have obtained relations between altitude and velocity and between velocity and flight path angle. These results are directly applicable to the boost case. For the glide case we simply let $n = 0$.

To complete the analytic model of aero-surfing, we need to include the dependence on time. Corresponding solutions for the velocity are obtained below.

### VI. Aero-surfing: Atmospheric boost

To obtain $\nu(t)$, we use the $\nu - \theta$ equation, Eq. (V.6). It is possible to substitute the general expression for $\theta$, Eq. (V.11) in Eq. (V.6), reduce its right side to a rational fraction of $\nu$ and analytically integrate the resulting differential equation. For demonstration purposes we neglect $\theta$ as compared to $1/\lambda$, since $\theta$ is rapidly decreasing with $\nu$. Then, the resulting differential equation reduces to a simple, well-known case:

$$\lambda \frac{d\nu}{dt} = \frac{\nu^2}{\nu_o^2} + \lambda n - 1$$  \hspace{1cm} (VI.1)

Assuming $\lambda = \text{const}$, $n = \text{const}$, and $\lambda n > 1$, and using substitutions

$$x = \frac{\sqrt{\lambda n - 1}}{\lambda \nu_o} \tau, \quad y = \frac{\nu}{\nu_o \sqrt{\lambda n - 1}}$$

this equation can be transformed to the form $y' = y^2 + 1$ with the solution $y = \tan(x + C)$. Omitting the constant of integration $C$ we obtain the solution of Eq. (VI.1) in the form:

$$\nu(t) = \nu_o \sqrt{\lambda n - 1} \tan \left( \frac{\sqrt{\lambda n - 1}}{\lambda \nu_o} \tau \right) \rightarrow \left( n - \frac{1}{\lambda} \right) \tau$$  \hspace{1cm} (VI.2)

The dimensional counterparts of the differential equation, Eq. (VI.1), and of its solution, Eq. (VI.2) are:

$$\lambda \frac{dV}{dt} = \frac{V^2}{V_o^2} + \lambda n - 1$$  \hspace{1cm} (VI.3)

$$V(t) = V_o \sqrt{\lambda n - 1} \tan \left( \frac{\sqrt{\lambda n - 1}}{\lambda V_o} \frac{gt}{(V/V_o)^2 < 1} \right) \rightarrow \left( n - \frac{1}{\lambda} \right) \frac{gt}{2}$$  \hspace{1cm} (VI.4)

The dimensional solution, Eq. (VI.4) can be directly obtained from Eq. (VI.3). Apparently, first solutions of this kind were obtained by Eugen Sanger and published in his famous monograph.

Expressions for the velocity, Eqs. (VI.2) and (VI.4) can be integrated over time to obtain the non-dimensional and dimensional distances. We have:

$$\Delta(t) = \int \nu dt = -\nu_o^2 \int \cos \left( \frac{\sqrt{\lambda n - 1}}{\lambda \nu_o} \tau \right) \rightarrow \left( n - \frac{1}{\lambda} \right) \frac{\tau^2}{2}$$  \hspace{1cm} (VI.5)

$$D(t) = \int V dt = -\lambda R \int \cos \left( \frac{\sqrt{\lambda n - 1}}{\lambda V_o} \frac{gt}{(V/V_o)^2 < 1} \right) \rightarrow \left( n - \frac{1}{\lambda} \right) \frac{gt^2}{2}$$  \hspace{1cm} (VI.6)

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Also, we can obtain expressions of distance through velocity. From Eqs. (VI.2) and (VI.4) we have respectively:

\[
\frac{\sqrt{\lambda n - 1}}{\lambda V_o} \tau = \tan^{-1} \left( \frac{\nu}{\nu_x \sqrt{\lambda n - 1}} \right), \quad \frac{\sqrt{\lambda n - 1}}{\lambda V_o} g t = \tan^{-1} \left( \frac{V}{V_x \sqrt{\lambda n - 1}} \right)
\]

Substitution in Eqs. (VI.5) and (VI.6) yields, respectively:

\[
\Delta(\nu) = \frac{\lambda V_o^2}{2} \ln \left[ 1 + \left( \frac{\nu}{\nu_x \sqrt{\lambda n - 1}} \right)^2 \right] \left( \frac{\nu}{\nu_x \sqrt{\lambda n - 1}} \right) \left( \frac{n - 1}{\lambda} \right) \frac{\nu^2}{2}
\]

(VII.7)

\[
D(V) = \frac{\lambda R}{2} \ln \left[ 1 + \left( \frac{V}{V_x \sqrt{\lambda n - 1}} \right)^2 \right] \left( \frac{V}{V_x \sqrt{\lambda n - 1}} \right) \left( \frac{n - 1}{\lambda} \right) \frac{v^2}{2g}
\]

(VII.8)

**VII. Aero-surfing: Atmospheric glide**

Under same assumptions as for the boost case, the \( \nu - \theta - \) equation reduces to:

\[
\frac{\lambda}{\nu_x} \frac{d\nu}{d\tau} = \frac{\nu^2}{\nu_x^2} - 1
\]

(VII.1)

With substitutions

\[
x = \frac{\tau}{\lambda \nu_o}, \quad y = \frac{\nu}{\nu_o}
\]

it takes the form \( y_x = y^2 - 1 \) with the solution \( y = \tanh \left( - x + C \right) \). Omitting the constant of integration \( C \) we obtain the solution of Eq. (VI.1) in the form:

\[
\nu(\tau) = \nu_o \tanh \left( - \frac{\tau}{\lambda \nu_o} \right) \left( \frac{\nu}{\nu_x} \right) \left( \frac{n - 1}{\lambda} \right) \frac{\nu^2}{2}
\]

(VII.2)

The “minus” sign accounts for the decrease of velocity with increasing time.

The dimensional counterparts of the differential equation, Eq. (VI.1), and of its solution, Eq. (VI.2) are:

\[
\frac{\lambda}{\nu_x} \frac{dV}{dt} = \frac{V^2}{\nu_x^2} - 1
\]

(VII.3)

\[
V(t) = V_o \tanh \left( - \frac{gt}{\lambda V_o} \right) \left( \frac{V}{V_x} \right) \left( \frac{n - 1}{\lambda} \right) \frac{g t^2}{2}
\]

(VII.4)

The solution of this kind were also obtained by Sänger.

Expressions for the velocity, Eqs. (VII.2) and (VII.4) can be integrated over time to obtain the non-dimensional and dimensional distances. We have:

\[
\Delta(\tau) = -\lambda V_o^2 \ln \cosh \left( \frac{1}{\lambda \nu_o} \tau \right) \left( \frac{\nu}{\nu_x} \right) \left( \frac{n - 1}{\lambda} \right) \frac{\tau^2}{2}
\]

(VII.5)

\[
D(t) = -\lambda R \ln \cosh \left( \frac{1}{\lambda V_o} g t \right) \left( \frac{V}{V_x} \right) \left( \frac{n - 1}{\lambda} \right) \frac{g t^2}{2}
\]

(VII.6)

The “minus” sign accounts for the decrease of the distance to the destination with increasing time.

As for the boost case, we can obtain expressions of distance through velocity. From Eqs. (VII.2) and (VII.4) we have respectively:

\[
\frac{\tau}{\lambda \nu_o} = \tanh^{-1} \left( \frac{\nu}{\nu_o} \right), \quad \frac{g t}{\lambda V_o} = \tanh^{-1} \left( \frac{V}{V_o} \right)
\]
Substitution in Eqs. (VI.5) and (VI.6) yields, respectively:

\[
\Delta(v) = \frac{\lambda v^2}{2} \ln \left[ 1 - \left( \frac{v}{v_o} \right)^2 \right] \xrightarrow{(v/v_o) \ll 1} \frac{1}{\lambda} \frac{v^2}{2}
\]

(VII.7)

\[
D(V) = \frac{\lambda R}{2} \ln \left[ 1 - \left( \frac{V}{V_o} \right)^2 \right] \xrightarrow{(V/V_o) \ll 1} -\frac{1}{\lambda} \frac{V^2}{2g}
\]

(VII.8)

VIII. Discussion

An idea to use non-dimensional parameters combining dimensional parameters of the vehicle and of the atmosphere is not novel per se. In late 1950's Chapman has developed an analytical method for studying entry in planetary atmospheres (see and references to earlier work in this area therein), based on two non-dimensional parameters. Later on, Vinh (see his monograph and references to his earlier work therein) used these parameters, in a slightly modified form, to conduct studies of a wider range of problems of atmospheric flight. Using Vinh's modification of Chapman parameters and notations of this paper, these parameters have the form:

\[
Z = \frac{\rho}{2m/(C_D S)} \sqrt{RH}, \quad u = \frac{V^2}{gR}
\]

The non-dimensional parameters Z and u can be expressed through \( \nu \) - number and \( \mu \) - ratio as follows:

\[
Z = \frac{1}{\sqrt{2}} \frac{\mu \nu_o}{\nu^2}, \quad u = \frac{\nu^2}{\nu_o^2}
\]

Below we compare some of results obtained by Vinh, with corresponding results obtained in this paper. For better comparison, we use our notations of parameters other than Z and u. Equation numbers in brackets refer to Vinh's monograph and to this paper respectively.

Relation between dynamic and static pressure [(11.15) and (III.5)]:

\[
\frac{q}{\nu} = m \frac{\sqrt{RH}}{C_D S} \frac{\nu^2}{Z}, \quad q = \rho \nu^2
\]

Non-dimensional acceleration due to atmospheric drag [(11.11) and (III.9)]:

\[
\frac{1}{g} \frac{(dV)}{dr}_{\text{drag}} = \frac{R}{H} Z \nu, \quad \frac{1}{g} \frac{(dV)}{dr}_{\text{drag}} = \mu \nu^2
\]

Altitude-velocity ratio [(11.30) and (V.7)]:

\[
Z = \frac{1}{\lambda} \sqrt{\frac{R}{H} \frac{1-u}{u}}, \quad \mu = \frac{1}{\lambda} \left( \nu^{-2} - \nu_o^{-2} \right)
\]

Glide flight path angle [(11.35) and (V.14) with \( n = 0 \)]:

\[
\theta = -\frac{2(1-u)}{\lambda \left[ \frac{R}{H} (1-u) + (2-u) \right]}, \quad \theta = -\frac{1}{\lambda} \left( \nu^2 + 1 \right)^{1/2}
\]

The results for the flight path angle are irreducible to each other. Vinh's result can be rewritten, using the our notations, as

\footnote{It should be pointed out that in his earlier monograph, in Chapter 6, Vinh, in addition to Chapman's variables has also used non-dimensional variables similar to \( \nu \) - number and \( \mu \) - ratio. For some reason, use of these variables was dropped in his later monograph, and their direct relevance to vertical extent and mass of the atmosphere was never stated explicitly.}

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\[ \theta = -\frac{1}{\lambda} \left[ \frac{u^2}{2} + \frac{1}{2} \left( 1 - \frac{v^2}{v_0^2} \right)^{-1} \right]^{-1} \]

A quick check shows that the relative difference between these results stays below 2% for \( u < 0.98v_0 \) (\( u < 0.95 \)). The cause of this difference is associated with the assumption \( g = \text{const} \) in our analysis, while Vinh's analysis allows for slight decrease of gravity acceleration with altitude.

The applications of the non-dimensional parameterization proposed here are not limited to analytic studies illustrated above on examples of equilibrium boost and glide. The solution of the system of non-dimensional equations of motion can be implemented in numerical integration schemes, where more realistic, variable parameters, such as lift-to-drag ratio, or thrust-to-weight ratio can be used.

**IX. Conclusion**

Summing up, we have introduced two non-dimensional parameters, a \( \nu \)-number and a \( \mu \)-ratio, which directly relate the vertical extent and mass of the atmosphere to velocity and ballistic coefficient of a TAV. Use of this parameters results in a straightforward non-dimensionalization of equations of motion, which were studied analytically for cases of equilibrium boost and glide of a TAV.

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**References**