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Low-Thrust Orbit Transfers Using Candidate Lyapunov Functions with a Mechanism for Coasting

Anastassios E. Petropoulos*
Jet Propulsion Laboratory, California Institute of Technology
Pasadena, California 91109-8099

We consider low-thrust orbit transfers around a central body, where specified changes are sought in orbit elements except true anomaly. The desired changes in the remaining five elements can be arbitrarily large. Candidate Lyapunov functions are created based on analytic expressions for maximum rates of change of the orbit elements and the desired changes in the elements. These functions may be thought of as proximity quotients because they provide a measure of the proximity to the target orbit. The direction of thrust needed for steepest descent to the target orbit is also available analytically. The thrust is shut-off if the effectivity of the thrust at the current location on the osculating orbit is below some threshhold value. Thus, the equations of motion can be numerically integrated to obtain quickly and simply a transfer to the target orbit. A series of transfers can be easily computed to assess the trade-off between propellant mass and flight time. Preliminary comparisons to optimal solutions show that the method, while sub-optimal, performs well.

1 Introduction

The problem of computing many-revolution, low-thrust orbit transfers around a central body is a difficult one; its study began at least as early as the 1950s[1,2] and continues today. Much of the work has focused on finding propellant-optimal trajectories using either indirect or direct techniques or mixtures of the two, as recently exemplified by Refs. [3–6], [7,8], and [9,10], respectively. Given the dearth of analytic solutions to the optimisation problem, and the difficulty of computing optimal solutions, some attention has also been focused on heuristic control laws. The advantage of the heuristics lies in the speed of computation, which can be orders of magnitude greater than that for optimisation, while the drawback is that the solutions are non-optimal. One category of heuristics[11–13] involves "blending" the instantaneously optimal thrust directions for changing each of the orbit elements during each of several phases of the orbit transfer. The precise nature of the blending and the delineation of the phases is guided by experience of the mission designer and perhaps by optimisation of the parameters in the control scheme. A second category of heuristics[13–15] is based on Lyapunov feedback control, where a suitable Lyapunov function must be defined by the mission designer.

In this paper, we extend the Lyapunov feedback approach of Ref. [13], with the aim of providing both simple approximations to, and good initial guesses for, propellant-optimal, low-thrust orbit transfers which involve specified changes in all orbit elements except true anomaly. In Ref. [13], the candidate Lyapunov function, termed the "proximity quotient," Q, exhibited divergence for some orbit transfers. Here, the problem of divergence is addressed, and, in addition, a penalty function method is used to enforce minimum-periapsis-radius constraints. The new proximity quotient, Q, is still termed a candidate Lyapunov function, because, although convergence is seen in all examples studied thus far, it has not been rigorously proved, due to the complexity of the function. As in Ref. [13], the proximity quotient takes into account the analytically-known optimal locations on the osculating orbit for changing each of the orbit elements, as well as the corresponding optimal rates of change that can be obtained for each element. The decision of whether to coast or thrust is based on the effectiveness of thrust in changing the proximity quotient on the osculating orbit. The Lyapunov feedback control law based on the proximity quotient (the "Q-law") has but few

*Senior Member of the Engineering Staff, Navigation and Mission Design Section, Mail-Stop 301-140L. Member AIAA. Member AAS. Email: Anastassios.Petropoulos@jpl.nasa.gov

input parameters, yet captures the complexity of a wide variety of orbit transfers, including those involving multiple coast arcs.

We present a number of orbit transfers computed using the Q-law and compare some of these to optimal orbit transfers. As an approximation, the control law provides mission designers with rapid estimates of propellant requirements and times of flight, as well as the trade-offs between the two. In providing initial guesses for optimisation, the control law would be particularly useful for the case where large numbers of revolutions are required. Both continuous and intermittent thrusting is permitted for the transfer, but no constraints are placed on when thrusting can occur. When non-zero, the thrust is assumed to be constant, and the specific impulse is similarly constant. The Q-law, as currently formulated, does not attempt to capitalise on the increasing thrust acceleration — the increase will often be of small utility for high specific-impulse missions. The current Q-law logic is oblivious to thruster characteristics and simply provides a thrust direction on the osculating orbit, and an indication of whether to thrust or not. The central body is modelled as a point mass, and the initial and final orbits are assumed closed. No perturbing forces are considered.

2 The Q-Law Algorithm

Definition of the proximity quotient, Q

The proximity quotient, Q, which serves as a candidate Lyapunov function, attempts to judiciously quantify the proximity of the osculating orbit to the target orbit. Q is defined as follows:

$$Q = (1 + W_P P) \sum_{\text{æ}} W_{\text{æ}} S_{\text{æ}} \left(\frac{d(\text{æ}, \text{æ}_T)}{\dot{\text{æ}}_{\text{xx}}} \right)^2, \qquad \text{for } \text{æ} = a, e, i, \omega, \Omega \tag{1}$$

where the five orbital elements (æ) are the semimajor axis (a), eccentricity (e), inclination (i), argument of periapsis (ω), and longitude of the ascending node (Ω); W_P and the $W_{\text{æ}}$ are scalar weights greater or equal to zero; the subscript T denotes the target orbit element value (without subscript, the osculating value is indicated); $\dot{\text{æ}}_{\text{xx}}$ denotes the maximum over thrust angle and over true anomaly on the osculating orbit of the rate of change of the orbit element (due to thrust); P is a penalty function; $S_{\text{æ}}$ is a scaling function; and $d(\text{æ}, \text{æ}_T)$ is a distance function. The penalty function is used in the present paper to enforce minimum-periapsis-radius constraints and takes the form

$$P = \exp\left[k \left(1 - \frac{r_p}{r_{p\min}} \right) \right] \tag{2}$$

where k is a scalar, r_p is the osculating periapsis radius, and $r_{p\min}$ is near or equal to the lowest permissible value of r_p. The scaling function is used primarily to prevent non-convergence to the target orbit and takes the form

$$S_{\text{æ}} = \begin{cases} \left[1 + \left(\frac{a - a_T}{m a_T} \right)^n \right]^{\frac{1}{r}} & \text{for } \text{æ} = a \\ 1 & \text{for } \text{æ} = e, i, \omega, \Omega \end{cases} \tag{3}$$

where m, n, and r are scalars with nominal values of 3, 4, and 2, respectively. The distance function is defined as

$$d(\text{æ}, \text{æ}_T) = \begin{cases} \text{æ} - oem_T & \text{for } \text{æ} = a, e, i \\ \cos^{-1}\left[\cos(\text{æ} - \text{æ}_T)\right] & \text{for } \text{æ} = \omega, \Omega \end{cases} \tag{4}$$

where the principal value, namely $[0, \pi]$ is used for the arc cosine. The peculiar form of the distance function for ω and Ω is used because it provides an angular measure of the distance between two positions on a circle using the "short way round" the circle, because it is differentiable with respect to æ [except when $d(\text{æ}, \text{æ}_T) = \pi$], and because the sign of the derivative indicates whether æ leads or lags æ_T based on the short way round.

Analytic expressions for the $\dot{\text{æ}}_{\text{xx}}$

Analytic expressions are available[13] for the maximum rates of change achievable for each of the orbit elements both over the true anomaly on the osculating orbit and over the thrust direction. For convenience, a summary of the derivations is presented here. We commence with Gauss's form of the variational equations for the orbit:[16]

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{r\sin(\theta+\omega)}{h\sin i} f_h \tag{5}$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{r\cos(\theta+\omega)}{h} f_h \tag{6}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{1}{eh}\left[-p\cos\theta f_r + (p+r)\sin\theta f_\theta\right] - \frac{r\sin(\theta+\omega)\cos i}{h\sin i} f_h \tag{7}$$

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2a^2}{h}\left(e\sin\theta f_r + \frac{p}{r} f_\theta\right) \tag{8}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{1}{h}\left\{p\sin\theta f_r + \left[(p+r)\cos\theta + re\right] f_\theta\right\} \tag{9}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{h}{r^2} + \frac{1}{eh}\left[p\cos\theta f_r - (p+r)\sin\theta f_\theta\right] \tag{10}$$

where t is time; θ is true anomaly; p is the semilatus rectum; h is the specific orbital angular momentum; r is the radius from the central body, related to the osculating elements through the conic equation $r = p/(1+e\cos\theta)$; and f_r, f_θ, and f_h are the components of the thrust acceleration in the radial, circumferential and angular momentum directions, repsectively. Using the thrust angles α (measured in the orbit plane off of the circumferential direction, positive away from the gravitational centre) and β (measured off of the orbit plane and perpendicular to it, positive in the direction of the angular momentum), the thrust acceleration components are given as:

$$\begin{aligned} f_r &= f\cos\beta\sin\alpha \\ f_\theta &= f\cos\beta\cos\alpha \\ f_h &= f\sin\beta \end{aligned}$$

The following definition is used for $\dot{\text{æ}}_{\text{xx}}$,

$$\dot{\text{æ}}_{\text{xx}} = \begin{cases} \max\limits_{\alpha,\beta,\theta}(\dot{\text{æ}}) & \text{for } \text{æ} = a, e, i, \Omega \\ \max\limits_{\alpha,\theta}\left(\dot{\text{æ}}|_{\beta=0}\right) & \text{for } \text{æ} = \omega \end{cases} \tag{11}$$

Then, using f for the thrust-acceleration magnitude and μ for the gravitational parameter of the central body, there arises for the semimajor axis:

$$\dot{a}_{\text{xx}} = 2f\sqrt{\frac{a^3(1+e)}{\mu(1-e)}} \tag{12}$$

and for the eccentricity:

$$\dot{e}_{\text{xx}} = \frac{2pf}{h} \tag{13}$$

and for the inclination:

$$\dot{i}_{\text{xx}} = \frac{pf}{h\left(\sqrt{1-e^2\sin^2\omega} - e|\cos\omega|\right)} \tag{14}$$

and for the argument of periapsis:

$$\dot{\omega}_{\text{x}} = \frac{f}{eh}\sqrt{p^2\cos^2\theta_{\text{xx}} + (p+r_{\text{xx}})^2\sin^2\theta_{\text{xx}}} \tag{15}$$

where

$$\cos\theta_{\rm xx} = \left[\frac{1-e^2}{2e^3}+\sqrt{\frac{1}{4}\left(\frac{1-e^2}{e^3}\right)^2+\frac{1}{27}}\right]^{\frac{1}{3}} - \left[-\frac{1-e^2}{2e^3}+\sqrt{\frac{1}{4}\left(\frac{1-e^2}{e^3}\right)^2+\frac{1}{27}}\right]^{\frac{1}{3}} - \frac{1}{e} \tag{16}$$

$$r_{\rm xx} = \frac{p}{1+e\cos\theta_{\rm xx}} \tag{17}$$

and for the longitude of the ascending node:

$$\dot{\Omega}_{\rm xx} = \frac{pf}{h\sin i\,\left(\sqrt{1-e^2\cos^2\omega}-e|\sin\omega|\right)} \tag{18}$$

(We note that Eq. 16 is a corrected version of Eq. 27 in Ref. [13], which contained a typographical error.)

Discussion of the Q-law and effectivity

It is clear from the definition of the proximity quotient in Eq. 1 that Q is zero at the target orbit and positive elsewhere. Thus, our goal in the orbit transfer is to drive Q to zero. Q may be thought of as a "best-case quadratic time-to-go," in that it captures the best possible rate of change for each of the orbit elements over the osculating orbit — the ratio $d(\text{æ}, \text{æ}_T)/\dot{\text{æ}}_{\rm xx}$ is the time it would take to reach the target value for that æ if this best possible rate of change could be sustained throughout the transfer. We note that Q is a function only of the five orbit elements, and not of true anomaly or the thrust angles. The summation in Eq. (1) is available analytically since analytic expressions have been derived for each of the $\dot{\text{æ}}_{\rm xx}$. Now, the time rate of change of Q is simply

$$\frac{{\rm d}Q}{{\rm d}t} = \sum_{\text{æ}} \frac{\partial Q}{\partial \text{æ}}\dot{\text{æ}} \tag{19}$$

where each of the $\dot{\text{æ}}$ are available explicitly from the variational equations (5)–(9). Thus, unlike Q, $\dot{Q}$ depends on the thrust angles. At any point on the transfer, we choose the thrust angles, $\alpha_{\rm n}$ and $\beta_{\rm n}$, which make $\dot{Q}$ most negative:

$$\dot{Q}_{\rm n} = \min_{\alpha,\beta} \dot{Q} \tag{20}$$

$\dot{Q}_{\rm n}$ is always less zero. The angles $\alpha_{\rm n}$ and $\beta_{\rm n}$ that minimise $\dot{Q}$ are available analytically. The Q-law uses these thrust angles, thereby ensuring that Q is being sent towards zero as quickly as possible at each instant. From the functional form of Q, we see that reducing Q might involve not only reducing $d(\text{æ}, \text{æ}_T)$, but also increasing $\dot{\text{æ}}_{\rm xx}$. Sacrificial changes in one orbit element can thus be made [increasing $d(\text{æ}_1, \text{æ}_{1T})$], if other elements can then be changed more easily (increasing $\dot{\text{æ}}_{2xx}$). This sort of balancing between orbit elements is akin to the classic example of a large plane change for a circular orbit: The propellant-optimal way to accomplish this is to enlarge the orbit, making the plane change easier, and then to shrink the orbit back to its original size.

One complication that is difficult to address analytically is that of convergence. Although we can always apply thrust so as to reduce Q, since $\dot{Q}_{\rm n} < 0$, we have not proved that doing so will always drive the orbit elements to their target values. For example, if we replace the scaling coefficient of Eq. 3 with $S_a = 1$, we see that Q becomes zero not only at the target orbit, but also at $a = \infty$, which would prevent some initial orbits from converging to the target orbit (converging instead to $a = \infty$). However, for the nominal Q of Eq. 1, convergence has been seen over all of the wide range of orbit transfers studied numerically so far.

While the thrust angles α_n and β_n ensure the optimal rate of reduction of Q at the current true anomaly, they do not provide any information about how effective the thrust is, as compared with other locations on the osculating orbit. Thus, it is natural to define the effectivity of the thrust at the current true anomaly as

$$\eta = \frac{\dot{Q}_{\rm n}}{\dot{Q}_{\rm nn}} \tag{21}$$

where

$$\dot{Q}_{\rm nn} = \min_{\theta} \dot{Q}_{\rm n} \tag{22}$$

A mission designer may then chose to prevent the spacecraft from thrusting if the effectivity is below some cut-off value, η_{cut}. Broadly speaking, the greater the cut-off, the greater the expected propellant savings and the longer the expected flight time. An analytic expression is not available for $\dot{Q}_{\text{nn}}$, and so this value must be computed numerically — an approximate value is normally sufficient, and so the computational burden is very slight.

Using the Q-law as a feedback algorithm

In this paper, the orbit transfers are computed by numerically integrating the variational equations 5–9 and the mass-flow-rate equation, where Eq. 8 is replaced by the variational equation for $\dot{p}$, where the thrust angles are determined by the Q-law, and where the decision of whether to apply thrust or not is based on the Q-law effectivity cut-off. A mission designer specifies the thrust, the specific impulse, initial values for $(a, e, i, \omega, \Omega, \theta)$, and final values for the orbit elements of interest (except θ, of course). For any element, $æ$, whose final value is free, the corresponding weight, $W_{æ}$, in Eq. 1 is set to zero. The remaining $W_{æ}$ are set to non-zero values, nominally unity. A minimum periapsis radius constraint is imposed by setting the penalty-function weight, W_P, to be non-zero, nominally unity. The associated parameters k and $r_{p\min}$ in Eq. 2 are normally set in concert with each other — the size of k determines how steeply the exponential barrier rises at $r_p = r_{p\min}$. The numerical integration is performed using a 5th-6th–order Runge-Kutta algorithm, with fixed step size in true longitude.

Due to the use of the classical orbit elements, the Q-law and the variational equaitons have singularities at zero-inclination and at zero-eccentricity. Thus, initial and target orbits are always specified to be outside of a small region surrounding the singularities. As a rather coarse approximation, in the unlikely event that during the numerical integration the inclination (in radians) or eccentricity try to drop below 10^{-4}, their values are artificially frozen at this value until their rates of change become positive.

3 Results

Transfers between five pairs of initial and final orbits are studied and in some instances compared to optimal orbit transfers. (The term orbit transfer is here sometimes used to refer to a particular trajectory joining an initial and final orbit, and sometimes to refer simply to the pair of orbits to be joined by some as-yet undetermined trajectory.) Table ?? lists the orbit transfers and their associated thrust characteristics and central body. As is evident from the table, the transfers become increasingly complex, involving larger changes in elements and more-constrained target orbits. In each case, the trade-off between propellant mass and flight time is investigated by varying the effectivity cut-off. For Earth $\mu = 398600.49\text{km}^3/\text{s}^2$, and for the asteroid Vesta, $\mu = 17.8\text{km}^3/\text{s}^2$. The standard acceleration due to gravity is taken as $9.80665\text{m}/\text{s}^2$.

Unless otherwise noted, nominal values are used for the Q-law parameters: Zero and unity are used for the $W_{æ}$ (depending on whether the target value of an element is free or fixed); m, n, r are taken as 3,4,2, respectively; and $W_{¶}$ is unity when a periapsis constraint is imposed, zero otherwise. The minimum permitted length of a thrust arc is 10° in true longitude (over-riding η_{cut}, if need be), to prevent thrust-on-off chatter around $\eta = \eta_{\text{cut}}$. Unless otherwise noted, the initial true anomaly is taken as zero.

Case A

Case A is essentially a simple coplanar, circle-to-circle orbit transfer from low Earth orbit to geostationary orbit. No periapsis constraint is imposed during the transfer, as the natural dynamics do not decrease the periapsis altitude. As expected, the $\eta_{\text{cut}} = 0$ case yields the shortest flight-time for the Q-law, as thrust is applied continuously. The trajectory, shown in Fig. ??, is roughly a circular spiral. According to Edelbaum's averaging analysis[2], the optimal ΔV for the minimum-time transfer between two coplanar circular orbits is the difference in circular orbit speeds (from which the minimum time can be computed).

At large η_{cut} values, the transfer trajectory takes a rather different form, opting to emulate a Hohmann transfer by performing multiple burns around periapsis or around apoapsis. The trajectory for the case of $\eta_{\text{cut}} = 0.968$ is shown in Fig. ??. Fig. ?? plots the osculating apoapsis radius, r_a, versus osculating periapsis radius, r_p, for both the $\eta_{\text{cut}} = 0$ and $\eta_{\text{cut}} = 0.968$ cases. In the former, the osculating orbit is seen to remain close to circular, while in the latter, the maintenance of low r_p until r_a becomes supersynchronous is dramatically evident.

#Figures to be added!

Data for the Q-law transfers are listed in Table ?? alongside corresponding numbers for the Edelbaum transfer and an impulsive Hohmann transfer (assuming circular coplanar orbits). The minimum-time Q-law transfer is seen to be close to Edelbaum's transfer, and the multi-burn Q-law transfer is seen to be close to the flight time and ΔV of the Hohmann transfer.

It is worth noting that for some η_{cut} values, the phasing of the final approach to the target orbit is disadvantageous, which causes a dip in flight-time and propellant-mass performance. The poor phasing causes the spacecraft to arrive at apoapsis when both a and e are just above their target values, in which case the effectivity is very low, and only a very narrow range of thrust directions can reduce Q. These η_{cut} values can be sreened out, or the poor performance can be rectified by adjusting other Q-law parameters such as the weights or the initial true anomaly.

Table 1 Case A Orbit transfer

Solved by	TOF (days)	ΔV (km/s)	Propellant Mass (kg)	Revs	Comments
Q-law V1.004	14.60048	4.5257	41.4953	90.38	etaqcut=0 (Min. TOF)
	152.389	3.9524	36.5739	666.02	etaqcut=0.968
$v_{c1} - v_{c2}$ Hohmann	14.4199	4.4654	40.9820		assume circ. orbits

Case B

Case B, a transfer from a slightly-inclined geostationary transfer orbit to geostationary orbit, is slightly more complex than Case A, in that three elements have target values rather than just two. Again, no periapsis constraint is imposed as none is needed. The trade-off between final mass and flight time, shown in Fig. ??, is generated by varying the effectivity cut-off upwards from zero. Two optimal trajectories are available[4] for comparison, computed using orbit averaging and the calculus of variations. One is a propellant-optimal, minimum-flight-time transfer, and the other is a propellant-optimal, fixed-time transfer. The final mass and flight time for these transfers is plotted in Fig. ?? alongside the Q-law curve. It is seen that the Q-law curve passes close to the two optimal solutions.

Case C

Case C is a transfer from a low-eccentricity ellipse to a coplanar, high-eccentricity, larger ellipse. Again no periapsis constraint is imposed. The Q-law trade-off between final mass and flight time is shown in Fig. ?? alongside several propellant-optimal, fixed-time transfers. The propellant-optimal transfers were computed using the optimisation software named Mystic[9,?], which is based on the static/dynamic control (SDC) algorithm.[9] SDC best fits into the direct method category, although, unlike other direct methods, the explicit time dependence of the optimisation problem is not removed by parametrisation. The ratcheting behaviour of the Q-law curve is due to hanging thrust arcs — very short thrust arcs that bring us to the target orbit, but which appear after a long coast arc. Hanging thrust arcs occur when the thrust is turned off (on account of the particular effectivity cut-off value being used), just befor the target orbit is reached. The spacecraft must then coast until the effectivity cut-off is high enough again to permit thrusting, whereupon a very brief burst of thrust is sufficient to reach the target orbit. The exact flight times and final masses where the ratchet steps occur are highly dependent on the initial true anomaly and on the weights used in the Q-law. Taking the ratcheting behaviour into account, we again we see that the Q-law "curve" passes close to the propellant-optimal solutions.

Case D

Case D is a roughly-circle-to-circle orbit transfer around the asteroid Vesta, involving a small plane change. The periapsis radius penalty function is used with $W_P = 1$, $k = 100$, and r_{pmin} =300km. A propellant-optimal, fixed-time solution for this transfer is presented in Ref. [?]. The Q-law trade-off between final mass and flight time is shown in Fig. ??

[½ paragraph to be added]

Case E

Case E is a transfer from a geostationary transfer orbit to a retrograde, Molniya-type orbit. [½ paragraph to be added]

4 Conclusions

Based on Gauss's form of the variational equations, and exploiting analytic expressions for the optimal thrust direction and location on the osculating orbit for changing each of the orbit elements except true anomaly, we have developed a candidate Lyapunov function for performing low-thrust orbit transfers using Lyapunov feedback control. The algorithm has been applied to a wide range of orbit transfers, and compares favourably to optimal transfers.

[½ paragraph to be added]

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1 Introduction

The problem of computing many-revolution, low-thrust orbit transfers around a central body is a difficult one; its study began at least as early as the 1950s[1,2] and continues today. Much of the work has focused on finding propellant-optimal trajectories using either indirect or direct techniques or mixtures of the two, as recently exemplified by Refs. [3–6], [7,8], and [9,10], respectively. Given the dearth of analytic solutions to the optimisation problem, and the difficulty of computing optimal solutions, some attention has also been focused on heuristic control laws. The advantage of the heuristics lies in the speed of computation, which can be orders of magnitude greater than that for optimisation, while the drawback is that the solutions are non-optimal. One category of heuristics[11–13] involves "blending" the instantaneously optimal thrust directions for changing each of the orbit elements during each of several phases of the orbit transfer. The precise nature of the blending and the delineation of the phases is guided by experience of the mission designer and perhaps by optimisation of the parameters in the control scheme. A second category of heuristics[13–15] is based on Lyapunov feedback control, where a suitable Lyapunov function must be defined by the mission designer.

In this paper, we extend the Lyapunov feedback approach of Ref. [13], with the aim of providing both simple approximations to, and good initial guesses for, propellant-optimal, low-thrust orbit transfers which involve specified changes in all orbit elements except true anomaly. In Ref. [13], the candidate Lyapunov function, termed the "proximity quotient," Q, exhibited divergence for some orbit transfers. Here, the problem of divergence is addressed, and, in addition, a penalty function method is used to enforce minimum-periapsis-radius constraints. The new proximity quotient, Q, is still termed a candidate Lyapunov function, because, although convergence is seen in all examples studied thus far, it has not been rigorously proved, due to the complexity of the function. As in Ref. [13], the proximity quotient takes into account the analytically-known optimal locations on the osculating orbit for changing each of the orbit elements, as well as the corresponding optimal rates of change that can be obtained for each element. The decision of whether to coast or thrust is based on the effectiveness of thrust in changing the proximity quotient on the osculating orbit. The Lyapunov feedback control law based on the proximity quotient (the "Q-law") has but few

*Senior Member of the Engineering Staff, Navigation and Mission Design Section, Mail-Stop 301-140L. Member AIAA. Member AAS. Email: Anastassios.Petropoulos@jpl.nasa.gov

input parameters, yet captures the complexity of a wide variety of orbit transfers, including those involving multiple coast arcs.

We present a number of orbit transfers computed using the Q-law and compare some of these to optimal orbit transfers. As an approximation, the control law provides mission designers with rapid estimates of propellant requirements and times of flight, as well as the trade-offs between the two. In providing initial guesses for optimisation, the control law would be particularly useful for the case where large numbers of revolutions are required. Both continuous and intermittent thrusting is permitted for the transfer, but no constraints are placed on when thrusting can occur. When non-zero, the thrust is assumed to be constant, and the specific impulse is similarly constant. The Q-law, as currently formulated, does not attempt to capitalise on the increasing thrust acceleration — the increase will often be of small utility for high specific-impulse missions. The current Q-law logic is oblivious to thruster characteristics and simply provides a thrust direction on the osculating orbit, and an indication of whether to thrust or not. The central body is modelled as a point mass, and the initial and final orbits are assumed closed. No perturbing forces are considered.

2 The Q-Law Algorithm

Definition of the proximity quotient, Q

The proximity quotient, Q, which serves as a candidate Lyapunov function, attempts to judiciously quantify the proximity of the osculating orbit to the target orbit. Q is defined as follows:

$$Q = (1 + W_P P) \sum_{\text{æ}} W_{\text{æ}} S_{\text{æ}} \left(\frac{d(\text{æ}, \text{æ}_T)}{\dot{\text{æ}}_{\text{xx}}} \right)^2, \qquad \text{for } \text{æ} = a, e, i, \omega, \Omega \tag{1}$$

where the five orbital elements (æ) are the semimajor axis (a), eccentricity (e), inclination (i), argument of periapsis (ω), and longitude of the ascending node (Ω); W_P and the $W_{\text{æ}}$ are scalar weights greater or equal to zero; the subscript T denotes the target orbit element value (without subscript, the osculating value is indicated); $\dot{\text{æ}}_{\text{xx}}$ denotes the maximum over thrust angle and over true anomaly on the osculating orbit of the rate of change of the orbit element (due to thrust); P is a penalty function; $S_{\text{æ}}$ is a scaling function; and $d(\text{æ}, \text{æ}_T)$ is a distance function. The penalty function is used in the present paper to enforce minimum-periapsis-radius constraints and takes the form

$$P = \exp\left[k \left(1 - \frac{r_p}{r_{p\min}} \right) \right] \tag{2}$$

where k is a scalar, r_p is the osculating periapsis radius, and $r_{p\min}$ is near or equal to the lowest permissible value of r_p. The scaling function is used primarily to prevent non-convergence to the target orbit and takes the form

$$S_{\text{æ}} = \begin{cases} \left[1 + \left(\frac{a - a_T}{m a_T} \right)^n \right]^{\frac{1}{r}} & \text{for } \text{æ} = a \\ 1 & \text{for } \text{æ} = e, i, \omega, \Omega \end{cases} \tag{3}$$

where m, n, and r are scalars with nominal values of 3, 4, and 2, respectively. The distance function is defined as

$$d(\text{æ}, \text{æ}_T) = \begin{cases} \text{æ} - oem_T & \text{for } \text{æ} = a, e, i \\ \cos^{-1}\left[\cos(\text{æ} - \text{æ}_T)\right] & \text{for } \text{æ} = \omega, \Omega \end{cases} \tag{4}$$

where the principal value, namely $[0, \pi]$ is used for the arc cosine. The peculiar form of the distance function for ω and Ω is used because it provides an angular measure of the distance between two positions on a circle using the "short way round" the circle, because it is differentiable with respect to æ [except when $d(\text{æ}, \text{æ}_T) = \pi$], and because the sign of the derivative indicates whether æ leads or lags æ_T based on the short way round.

Analytic expressions for the $\dot{\text{æ}}_{\text{xx}}$

Analytic expressions are available[13] for the maximum rates of change achievable for each of the orbit elements both over the true anomaly on the osculating orbit and over the thrust direction. For convenience, a summary of the derivations is presented here. We commence with Gauss's form of the variational equations for the orbit:[16]

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{r\sin(\theta+\omega)}{h\sin i} f_h \tag{5}$$

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{r\cos(\theta+\omega)}{h} f_h \tag{6}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{1}{eh}\left[-p\cos\theta f_r + (p+r)\sin\theta f_\theta\right] - \frac{r\sin(\theta+\omega)\cos i}{h\sin i} f_h \tag{7}$$

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{2a^2}{h}\left(e\sin\theta f_r + \frac{p}{r} f_\theta\right) \tag{8}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{1}{h}\left\{p\sin\theta f_r + \left[(p+r)\cos\theta + re\right] f_\theta\right\} \tag{9}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{h}{r^2} + \frac{1}{eh}\left[p\cos\theta f_r - (p+r)\sin\theta f_\theta\right] \tag{10}$$

where t is time; θ is true anomaly; p is the semilatus rectum; h is the specific orbital angular momentum; r is the radius from the central body, related to the osculating elements through the conic equation $r = p/(1+e\cos\theta)$; and f_r, f_θ, and f_h are the components of the thrust acceleration in the radial, circumferential and angular momentum directions, repsectively. Using the thrust angles α (measured in the orbit plane off of the circumferential direction, positive away from the gravitational centre) and β (measured off of the orbit plane and perpendicular to it, positive in the direction of the angular momentum), the thrust acceleration components are given as:

$$\begin{aligned} f_r &= f\cos\beta\sin\alpha \\ f_\theta &= f\cos\beta\cos\alpha \\ f_h &= f\sin\beta \end{aligned}$$

The following definition is used for $\dot{\text{æ}}_{\text{xx}}$,

$$\dot{\text{æ}}_{\text{xx}} = \begin{cases} \max\limits_{\alpha,\beta,\theta}(\dot{\text{æ}}) & \text{for } \text{æ} = a, e, i, \Omega \\ \max\limits_{\alpha,\theta}\left(\dot{\text{æ}}|_{\beta=0}\right) & \text{for } \text{æ} = \omega \end{cases} \tag{11}$$

Then, using f for the thrust-acceleration magnitude and μ for the gravitational parameter of the central body, there arises for the semimajor axis:

$$\dot{a}_{\text{xx}} = 2f\sqrt{\frac{a^3(1+e)}{\mu(1-e)}} \tag{12}$$

and for the eccentricity:

$$\dot{e}_{\text{xx}} = \frac{2pf}{h} \tag{13}$$

and for the inclination:

$$\dot{i}_{\text{xx}} = \frac{pf}{h\left(\sqrt{1-e^2\sin^2\omega} - e|\cos\omega|\right)} \tag{14}$$

and for the argument of periapsis:

$$\dot{\omega}_{\text{x}} = \frac{f}{eh}\sqrt{p^2\cos^2\theta_{\text{xx}} + (p+r_{\text{xx}})^2\sin^2\theta_{\text{xx}}} \tag{15}$$

where

$$\cos\theta_{\rm xx} = \left[\frac{1-e^2}{2e^3}+\sqrt{\frac{1}{4}\left(\frac{1-e^2}{e^3}\right)^2+\frac{1}{27}}\right]^{\frac{1}{3}} - \left[-\frac{1-e^2}{2e^3}+\sqrt{\frac{1}{4}\left(\frac{1-e^2}{e^3}\right)^2+\frac{1}{27}}\right]^{\frac{1}{3}} - \frac{1}{e} \tag{16}$$

$$r_{\rm xx} = \frac{p}{1+e\cos\theta_{\rm xx}} \tag{17}$$

and for the longitude of the ascending node:

$$\dot{\Omega}_{\rm xx} = \frac{pf}{h\sin i\,\left(\sqrt{1-e^2\cos^2\omega}-e|\sin\omega|\right)} \tag{18}$$

(We note that Eq. 16 is a corrected version of Eq. 27 in Ref. [13], which contained a typographical error.)

Discussion of the Q-law and effectivity

It is clear from the definition of the proximity quotient in Eq. 1 that Q is zero at the target orbit and positive elsewhere. Thus, our goal in the orbit transfer is to drive Q to zero. Q may be thought of as a "best-case quadratic time-to-go," in that it captures the best possible rate of change for each of the orbit elements over the osculating orbit — the ratio $d(\text{æ}, \text{æ}_T)/\dot{\text{æ}}_{\rm xx}$ is the time it would take to reach the target value for that æ if this best possible rate of change could be sustained throughout the transfer. We note that Q is a function only of the five orbit elements, and not of true anomaly or the thrust angles. The summation in Eq. (1) is available analytically since analytic expressions have been derived for each of the $\dot{\text{æ}}_{\rm xx}$. Now, the time rate of change of Q is simply

$$\frac{{\rm d}Q}{{\rm d}t} = \sum_{\text{æ}} \frac{\partial Q}{\partial \text{æ}}\dot{\text{æ}} \tag{19}$$

where each of the $\dot{\text{æ}}$ are available explicitly from the variational equations (5)–(9). Thus, unlike Q, $\dot{Q}$ depends on the thrust angles. At any point on the transfer, we choose the thrust angles, $\alpha_{\rm n}$ and $\beta_{\rm n}$, which make $\dot{Q}$ most negative:

$$\dot{Q}_{\rm n} = \min_{\alpha,\beta} \dot{Q} \tag{20}$$

$\dot{Q}_{\rm n}$ is always less zero. The angles $\alpha_{\rm n}$ and $\beta_{\rm n}$ that minimise $\dot{Q}$ are available analytically. The Q-law uses these thrust angles, thereby ensuring that Q is being sent towards zero as quickly as possible at each instant. From the functional form of Q, we see that reducing Q might involve not only reducing $d(\text{æ}, \text{æ}_T)$, but also increasing $\dot{\text{æ}}_{\rm xx}$. Sacrificial changes in one orbit element can thus be made [increasing $d(\text{æ}_1, \text{æ}_{1T})$], if other elements can then be changed more easily (increasing $\dot{\text{æ}}_{2xx}$). This sort of balancing between orbit elements is akin to the classic example of a large plane change for a circular orbit: The propellant-optimal way to accomplish this is to enlarge the orbit, making the plane change easier, and then to shrink the orbit back to its original size.

One complication that is difficult to address analytically is that of convergence. Although we can always apply thrust so as to reduce Q, since $\dot{Q}_{\rm n} < 0$, we have not proved that doing so will always drive the orbit elements to their target values. For example, if we replace the scaling coefficient of Eq. 3 with $S_a = 1$, we see that Q becomes zero not only at the target orbit, but also at $a = \infty$, which would prevent some initial orbits from converging to the target orbit (converging instead to $a = \infty$). However, for the nominal Q of Eq. 1, convergence has been seen over all of the wide range of orbit transfers studied numerically so far.

While the thrust angles α_n and β_n ensure the optimal rate of reduction of Q at the current true anomaly, they do not provide any information about how effective the thrust is, as compared with other locations on the osculating orbit. Thus, it is natural to define the effectivity of the thrust at the current true anomaly as

$$\eta = \frac{\dot{Q}_{\rm n}}{\dot{Q}_{\rm nn}} \tag{21}$$

where

$$\dot{Q}_{\rm nn} = \min_{\theta} \dot{Q}_{\rm n} \tag{22}$$

A mission designer may then chose to prevent the spacecraft from thrusting if the effectivity is below some cut-off value, η_{cut}. Broadly speaking, the greater the cut-off, the greater the expected propellant savings and the longer the expected flight time. An analytic expression is not available for $\dot{Q}_{\text{nn}}$, and so this value must be computed numerically — an approximate value is normally sufficient, and so the computational burden is very slight.

Using the Q-law as a feedback algorithm

In this paper, the orbit transfers are computed by numerically integrating the variational equations 5–9 and the mass-flow-rate equation, where Eq. 8 is replaced by the variational equation for $\dot{p}$, where the thrust angles are determined by the Q-law, and where the decision of whether to apply thrust or not is based on the Q-law effectivity cut-off. A mission designer specifies the thrust, the specific impulse, initial values for $(a, e, i, \omega, \Omega, \theta)$, and final values for the orbit elements of interest (except θ, of course). For any element, $æ$, whose final value is free, the corresponding weight, $W_{æ}$, in Eq. 1 is set to zero. The remaining $W_{æ}$ are set to non-zero values, nominally unity. A minimum periapsis radius constraint is imposed by setting the penalty-function weight, W_P, to be non-zero, nominally unity. The associated parameters k and $r_{p\min}$ in Eq. 2 are normally set in concert with each other — the size of k determines how steeply the exponential barrier rises at $r_p = r_{p\min}$. The numerical integration is performed using a 5th-6th–order Runge-Kutta algorithm, with fixed step size in true longitude.

Due to the use of the classical orbit elements, the Q-law and the variational equaitons have singularities at zero-inclination and at zero-eccentricity. Thus, initial and target orbits are always specified to be outside of a small region surrounding the singularities. As a rather coarse approximation, in the unlikely event that during the numerical integration the inclination (in radians) or eccentricity try to drop below 10^{-4}, their values are artificially frozen at this value until their rates of change become positive.

3 Results

Transfers between five pairs of initial and final orbits are studied and in some instances compared to optimal orbit transfers. (The term orbit transfer is here sometimes used to refer to a particular trajectory joining an initial and final orbit, and sometimes to refer simply to the pair of orbits to be joined by some as-yet undetermined trajectory.) Table ?? lists the orbit transfers and their associated thrust characteristics and central body. As is evident from the table, the transfers become increasingly complex, involving larger changes in elements and more-constrained target orbits. In each case, the trade-off between propellant mass and flight time is investigated by varying the effectivity cut-off. For Earth $\mu = 398600.49\text{km}^3/\text{s}^2$, and for the asteroid Vesta, $\mu = 17.8\text{km}^3/\text{s}^2$. The standard acceleration due to gravity is taken as $9.80665\text{m}/\text{s}^2$.

Unless otherwise noted, nominal values are used for the Q-law parameters: Zero and unity are used for the $W_{æ}$ (depending on whether the target value of an element is free or fixed); m, n, r are taken as 3,4,2, respectively; and $W_{¶}$ is unity when a periapsis constraint is imposed, zero otherwise. The minimum permitted length of a thrust arc is 10° in true longitude (over-riding η_{cut}, if need be), to prevent thrust-on-off chatter around $\eta = \eta_{\text{cut}}$. Unless otherwise noted, the initial true anomaly is taken as zero.

Case A

Case A is essentially a simple coplanar, circle-to-circle orbit transfer from low Earth orbit to geostationary orbit. No periapsis constraint is imposed during the transfer, as the natural dynamics do not decrease the periapsis altitude. As expected, the $\eta_{\text{cut}} = 0$ case yields the shortest flight-time for the Q-law, as thrust is applied continuously. The trajectory, shown in Fig. ??, is roughly a circular spiral. According to Edelbaum's averaging analysis[2], the optimal ΔV for the minimum-time transfer between two coplanar circular orbits is the difference in circular orbit speeds (from which the minimum time can be computed).

At large η_{cut} values, the transfer trajectory takes a rather different form, opting to emulate a Hohmann transfer by performing multiple burns around periapsis or around apoapsis. The trajectory for the case of $\eta_{\text{cut}} = 0.968$ is shown in Fig. ??. Fig. ?? plots the osculating apoapsis radius, r_a, versus osculating periapsis radius, r_p, for both the $\eta_{\text{cut}} = 0$ and $\eta_{\text{cut}} = 0.968$ cases. In the former, the osculating orbit is seen to remain close to circular, while in the latter, the maintenance of low r_p until r_a becomes supersynchronous is dramatically evident.

*Figures to be added!

Data for the Q-law transfers are listed in Table ?? alongside corresponding numbers for the Edelbaum transfer and an impulsive Hohmann transfer (assuming circular coplanar orbits). The minimum-time Q-law transfer is seen to be close to Edelbaum's transfer, and the multi-burn Q-law transfer is seen to be close to the flight time and ΔV of the Hohmann transfer.

It is worth noting that for some η_{cut} values, the phasing of the final approach to the target orbit is disadvantageous, which causes a dip in flight-time and propellant-mass performance. The poor phasing causes the spacecraft to arrive at apoapsis when both a and e are just above their target values, in which case the effectivity is very low, and only a very narrow range of thrust directions can reduce Q. These η_{cut} values can be sreened out, or the poor performance can be rectified by adjusting other Q-law parameters such as the weights or the initial true anomaly.

Table 1 Case A Orbit transfer

Solved by	TOF (days)	ΔV (km/s)	Propellant Mass (kg)	Revs	Comments
Q-law V1.004	14.60048	4.5257	41.4953	90.38	etaqcut=0 (Min. TOF)
	152.389	3.9524	36.5739	666.02	etaqcut=0.968
$v_{c1} - v_{c2}$ Hohmann	14.4199	4.4654	40.9820		assume circ. orbits

Case B

Case B, a transfer from a slightly-inclined geostationary transfer orbit to geostationary orbit, is slightly more complex than Case A, in that three elements have target values rather than just two. Again, no periapsis constraint is imposed as none is needed. The trade-off between final mass and flight time, shown in Fig. ??, is generated by varying the effectivity cut-off upwards from zero. Two optimal trajectories are available[4] for comparison, computed using orbit averaging and the calculus of variations. One is a propellant-optimal, minimum-flight-time transfer, and the other is a propellant-optimal, fixed-time transfer. The final mass and flight time for these transfers is plotted in Fig. ?? alongside the Q-law curve. It is seen that the Q-law curve passes close to the two optimal solutions.

Case C

Case C is a transfer from a low-eccentricity ellipse to a coplanar, high-eccentricity, larger ellipse. Again no periapsis constraint is imposed. The Q-law trade-off between final mass and flight time is shown in Fig. ?? alongside several propellant-optimal, fixed-time transfers. The propellant-optimal transfers were computed using the optimisation software named Mystic[9,?], which is based on the static/dynamic control (SDC) algorithm.[9] SDC best fits into the direct method category, although, unlike other direct methods, the explicit time dependence of the optimisation problem is not removed by parametrisation. The ratcheting behaviour of the Q-law curve is due to hanging thrust arcs — very short thrust arcs that bring us to the target orbit, but which appear after a long coast arc. Hanging thrust arcs occur when the thrust is turned off (on account of the particular effectivity cut-off value being used), just befor the target orbit is reached. The spacecraft must then coast until the effectivity cut-off is high enough again to permit thrusting, whereupon a very brief burst of thrust is sufficient to reach the target orbit. The exact flight times and final masses where the ratchet steps occur are highly dependent on the initial true anomaly and on the weights used in the Q-law. Taking the ratcheting behaviour into account, we again we see that the Q-law "curve" passes close to the propellant-optimal solutions.

Case D

Case D is a roughly-circle-to-circle orbit transfer around the asteroid Vesta, involving a small plane change. The periapsis radius penalty function is used with $W_P = 1$, $k = 100$, and r_{pmin} =300km. A propellant-optimal, fixed-time solution for this transfer is presented in Ref. ?. The Q-law trade-off between final mass and flight time is shown in Fig. ??

[½ paragraph to be added]

Case E

Case E is a transfer from a geostationary transfer orbit to a retrograde, Molniya-type orbit. [½ paragraph to be added]

4 Conclusions

Based on Gauss's form of the variational equations, and exploiting analytic expressions for the optimal thrust direction and location on the osculating orbit for changing each of the orbit elements except true anomaly, we have developed a candidate Lyapunov function for performing low-thrust orbit transfers using Lyapunov feedback control. The algorithm has been applied to a wide range of orbit transfers, and compares favourably to optimal transfers.

[½ paragraph to be added]

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