Protograph LDPC Codes over Burst Erasure Channels

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***Abstract*— In this paper we design high rate protograph based LDPC codes suitable for binary erasure channels. To simplify the encoder and decoder implementation for high data rate transmission, the structure of codes are based on protographs and circulants. These LDPC codes can improve data link and network layer protocols in support of communication networks. Two classes of codes were designed. One class is designed for large block sizes with an iterative decoding threshold that approaches capacity of binary erasure channels. The other class is designed for short block sizes based on maximizing minimum stopping set size. For high code rates and short blocks the second class outperforms the first class. A scheme is proposed to use these LDPC codes over burst erasure channels. The proposed encoding method is also applicable to cases when packets are frequency hopped over channels with partial band jamming or frequency selective fading. Various LDPC codes are compared and simulation results are provided.**

I. Introduction

Low-density parity-check (LDPC) codes were proposed by Gallager [1] in 1962. After introduction of turbo codes by Berrou et al [2] in 1993, researchers revisited LDPC codes, and extended the work of Gallager using the code graphs introduced by Tanner [3] in 1981. After 1993 there have been many contributions to the design and analysis of LDPC codes; see for example [10], [12], [4], [13], [14], [15], and references there. Recently a flurry of work has been conducted on the design of LDPC codes with imposed sub-structures, starting with the introduction of multi-edge-type codes in [9] and [11].

II. Protograph LDPC codes

For high-speed decoding, it is advantageous for an LDPC code to be constructed from a protograph [7] or projected graph [8]. A protograph is a Tanner graph with a relatively small number of nodes. A "copy-and-permute" operation [7] can be applied to the protograph to obtain larger derived graphs of various sizes. This operation consists of first making N copies of the protograph, and then permuting the endpoints of each edge among the N variable and N check nodes connected to the set of N edges copied from the same edge in the protograph. The derived graph is the graph of a code N times as large as the code corresponding to the protograph, with the same rate and the same distribution of variable and check node degrees. LDPC codes with protograph structure are a subclass of multi-edge-type LDPC codes. As a simple example, we consider the protograph shown in Fig. 1. This graph consists of 3 variable nodes and 2 check nodes, connected by 5 edges. In this example we have 5 edge types i.e. each edge in the base protograph represents an edge type. As another example for protograph based LDPC codes we consider the rate-1/3 Repeat-Accumulate (RA) code depicted in Fig. 2(a). For this code the maximum erasure probability threshold with iterative decoding is p_{it}=0.6173, where the capacity threshold $p_{cap} = 1 - R_c$=0.6667. This code has a protograph representation shown in Fig. 2(b), as long as the interleaver π is chosen to be decomposable

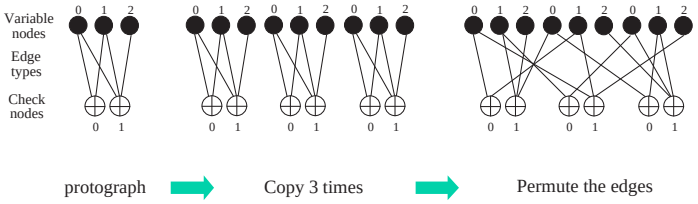

Fig. 1. Copy and Permute operation for a protograph to generate larger graphs

into permutations along each edge of the protograph. The iterative decoding threshold is unchanged despite the additional constraint imposed by the protograph. The protograph consists of 4 variable nodes and 3 check nodes, connected by 9 edges. Three variable nodes are connected to the channel (transmitted nodes) and are shown as dark filled circles. One variable node is not connected to the channel (i.e., it is punctured) and is depicted by a blank circle. The three check nodes are depicted by circles with a plus sign inside.

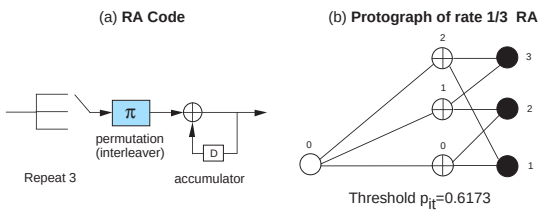

Fig. 2. (a) A rate-1/3 RA code with repetition 3, and (b) its corresponding protograph.

Repeat-Accumulate (RA) [5], Irregular Repeat-Accumulate (IRA) [6], and recently proposed Accumulate-Repeat-Accumulate (ARA) [16] codes, with suitable definitions of their interleavers, all have simple protograph representations. These codes provide capacity approaching iterative decoding thresholds but have sublinear minimum stopping set size (to be defined shortly). However for certain applications linear minimum stopping set size is required for low error floor performance.

III. Asymptotic ensemble of stopping set size enumerators

A set of variable nodes is called a stopping set if all its check node neighbors are connected to this set at least twice. Message passing iterative decoding fails whenever all the variable nodes in a stopping set are erased. Analysis of ensemble average stopping set size enumerators is useful for designing protograph LDPC codes over a BEC. In particular we are interested in protograph LDPC codes that have minimum stopping set size growing linearly with block size. To design such structured LDPC codes, we need to compute the stopping set size enumerators for protograph based LDPC code ensembles. Such computational tools have been only derived for unstructured irregular LDPC ensembles [21], [23], [27], [24], [25], [26] [29], [30], [22]. Recently a method to compute the asymptotic weight distribution for protograph LDPC codes has been proposed [20], [18], [17], and [33]. Here we propose an alternative computational method for stopping set enumerators that applies to LDPC codes with protograph structure.

We express the normalized logarithmic asymptotic stopping set size distribution of a code as $r^S(\delta) = \lim_{n\to\infty} \sup r_n^S(\delta)$ where $r_n^S(\delta) = \frac{\ln(A^S_{\delta n})}{n}$, $\delta = \frac{d}{n}$, d is stopping set size , and A_d^S is the ensemble stopping set size distribution.

If the first zero crossing of $r^S(\delta)$ (i.e. $r^S(\delta^S_{min}) = 0$ for $\delta^S_{min} > 0$) exists, and $r^S(\delta) < 0$ for all $0 < \delta < \delta^S_{min}$ then δ^S_{min} is called typical stopping set size ratio. Consider $P(d \leq \delta^S_{min} n) \leq \sum_{d=1}^{\delta^S_{min} n} A_d^S$. If the contribution of stopping sets with sizes less than $\delta^S_{min} n$ to this sum becomes negligible as n becomes large, then we can say that the normalized stopping set size is almost surely is greater than $\delta^S_{min} - \epsilon$ for any $\epsilon > 0$. In other words, with high probability the minimum stopping set size of most codes in the ensemble increases linearly with n.

Consider a protograph based LDPC code as shown in Fig. 3 with n_v variable and n_c check nodes. Let q_{v_i} represent the degree of variable node v_i, and q_{c_i} represent the degree of check node c_i. Suppose we lift (copy and permute) this protograph by a factor of N. To make the expressions simple, we first define $\tilde{r}^S(\tilde{\delta}) = \frac{\ln(A_d^S)}{N}$ where $\tilde{\delta} = \frac{d}{N}$ as $N \to \infty$. As follows the normalization will be with respect to N. At the final step, $r^S(\delta)$ can be obtained as $r^S(\delta) = \frac{1}{m}\tilde{r}^S(m\delta)$ where m represents the number of variable nodes in the protograph that are connected to the channel i.e.,transmitted nodes, and the code block size is $n = mN$. As follows, the word "weight" here will represent the "stopping set size". Assign a normalized weight δ_i to variable node v_i for $i = 1, \ldots, n_v$. Let $\mathcal{S}^S_{\tilde{\delta}}$ be the set of all possible partial normalized weights of m variable nodes that are connected to the channel, say with indices $l_1, l_2, \ldots, l_m$ such that $\delta_{l_1} + \delta_{l_2} + \cdots + \delta_{l_m} = \tilde{\delta}$. Let $\mathcal{S}^S_{\pi}$ be the set of all possible partial normalized weights of the remaining variable nodes in the protograph, namely the punctured variable nodes i.e., not transmitted nodes. Note that for stopping set calculations, the check node should satisfy the following constraint. The input weight to a single check should be zero or at least 2.

Then the asymptotic ensemble stopping set size enumerator for the protograph LDPC code can be written as [17]:

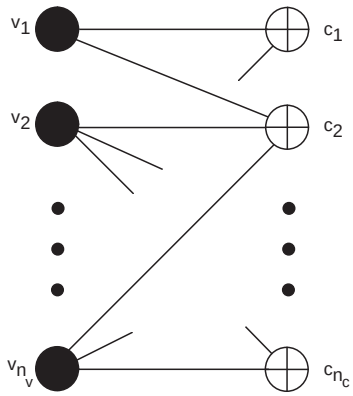

Fig. 3. An LDPC code with protograph structure

$$\tilde{r}^S(\tilde{\delta}) = \max_{\{\delta_k\}\in\mathcal{S}^S_{\tilde{\delta}}} \max_{\{\delta_j\}\in\mathcal{S}^S_{\pi}} \sum_{i=1}^{n_c} a^{S,c_i}(\boldsymbol{\delta}_i) - \sum_{i=1}^{n_v}(q_{v_i}-1)H(\delta_i) \quad (1)$$

where $H(x) = -(1-x)\ln(1-x) - x\ln x$ is the entropy function, and The $a^{S,c_i}(\boldsymbol{\delta}_i) \triangleq \lim_{N\to\infty} \sup \frac{lnA^{S,c_i}_{\mathbf{d}_i}}{N}$. The $A^{S,c_i}_{\mathbf{d}_i}$ is the stopping set size enumerator for the check node c_i. The $A^{S,c_i}_{\mathbf{d}_i}$ represents the number of sequences with partial stopping set size vector $\mathbf{d}_i = (d_{i,1}, d_{i,2}, \ldots, d_{i,q_{c_i}})$ that satisfies the check node constraint for stopping sets. Vector $\boldsymbol{\delta}_i$ represents the normalized version of the vector $\mathbf{d}_i$, where each component is normalized by N, as $N \to \infty$. The vector $\boldsymbol{\delta}_i = (\delta_{i_1}, \delta_{i_2}, \ldots, \delta_{i_{q_{c_i}}})$ represents the partial stopping set size δ_{i_k}s which correspond to the edge connections of the check node c_i to the set of variable nodes $\{v_{i_1}, v_{i_2}, \ldots, v_{i_{q_{c_i}}}\}$.

For a check with degree 3, let $\sigma = (\delta_1+\delta_2+\delta_3)/2$ such that $\max\{0, 2(\sigma-1)\} \le \eta \le 2\sigma - 2\max\{\delta_1, \delta_2, \delta_3\}$, and define

$$H_5(x_1, x_2, x_3, x_4, x_5) = -\sum_{i=1}^{5} x_i \ln x_i \quad (2)$$

then

$$a^{S,c}(\delta_1, \delta_2, \delta_3) = \max_{\eta} H_5((1-\sigma+\eta), (\sigma-\delta_1-\eta), (\sigma-\delta_2-\eta), (\sigma-\delta_3-\eta), (2\eta)) \quad (3)$$

For checks with degree 4, we have

$$a^{S,c}(\delta_1, \delta_2, \delta_3, \delta_4) = \max_{\lambda}\{a^{S,c}(\delta_1, \delta_2, \lambda) + a^{S,c}(\delta_3, \delta_4, \lambda) - H(\lambda)\} \quad (4)$$

In general, having $a^{S,c}(\delta_1, \delta_2, \delta_3)$ as a function of three partial weights $\delta_1, \delta_2, \delta_3$, we can compute recursively to obtain $a^{S,c}(\delta_1, \delta_2, \ldots, \delta_q)$ as a function of q partial weights for degree q check node for $q \ge 4$ as

$$a^{S,c}(\delta_1, .., \delta_{q-2}, \delta_{q-1}, \delta_q) = \max_{\lambda}\{a^{S,c}(\delta_1, .., \delta_{q-2}, \lambda) + a^{S,c}(\delta_{q-1}, \delta_q, \lambda) - H(\lambda)\} \quad (5)$$

Thus $a^{S,c_i}(\boldsymbol{\delta}_i)$ for all check nodes in the protograph can be computed using (3), and (5).

A. Example

Consider Repeat Jagged Accumulate (RJA) and its precoded version Accumulate Repeat Jagged Accumulate (ARJA) LDPC codes [28] as shown in Fig. 4. The RJA code has δ^S_{min}=0.011. However if we precode the RJA LDPC code with an accumulator (to create ARJA), then we obtain δ^S_{min}=0.012. The asymptotic normalized stopping set distributions and zero crossing for the RJA and ARJA LDPC codes are shown in Fig. 4.

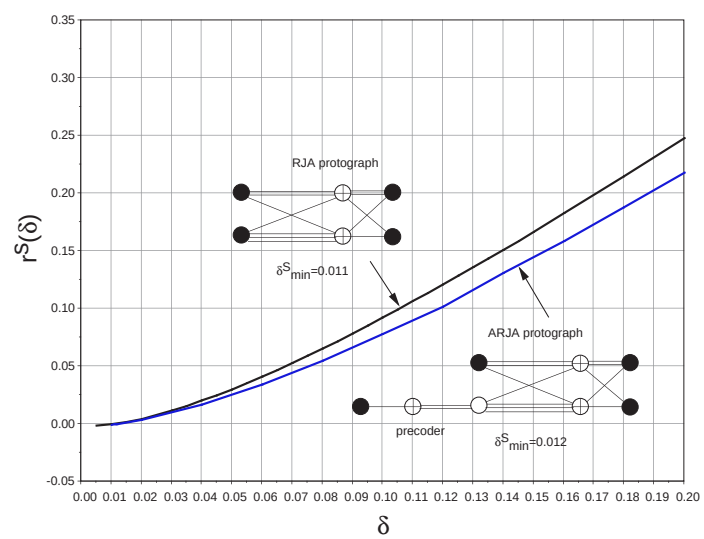

Fig. 4. Asymptotic normalized stopping set distributions for RJA LDPC code, and precoded RJA (ARJA) LDPC code

IV. Density Evolution in Protographs for the Erasure Channel

Computation of iterative decoding thresholds for the protographs in this paper is by density evolution. For the erasure channel, a single real-valued parameter, the probability of erasure p, serves as a stand-in for full density evolution. Alternatively, we can track the self-information of an erasure, $s = -\ln p$, which is additive at variable nodes. A reciprocal parameter, $r = -\ln(1-p)$, the self-information of a non-erasure, is additive at the check nodes. Since $e^{-s} + e^{-r} = 1$, we note that $r = R(s)$ and $s = R(r)$ are related to each other by the self-inverting function $R(s) \triangleq -\ln(1-e^{-s})$.

To apply density evolution to a protograph we first identify all transmitted variable nodes and select a target channel erasure probability $p_{chan} = e^{-s_{chan}}$. As shown in Fig. 5 messages $\vec{s}_e$ are passed along edges leaving variable nodes ($\vec{s}_e = s_{chan}$ from transmitted nodes and $\vec{s}_e = 0$ from punctured nodes). The transformation $R(\vec{s}_e)$ is applied and an extrinsic return message, $\overleftarrow{r}_e$, is determined by computing the sum of all incoming messages save the one along edge e. Transformation $R(\cdot)$ is then reapplied to produce $\overleftarrow{s}_e$. The process continues and the iterative decoding threshold $p_{it} = e^{-s_{chan}}$ is determined by the smallest value of s_{chan} for which unbounded growth of all messages $\vec{s}_e$ can be achieved.

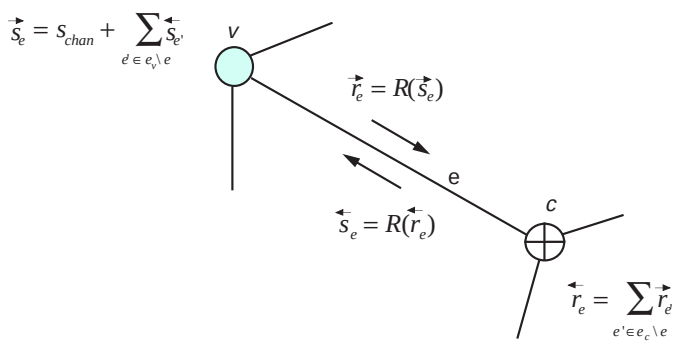

Fig. 5. Density evolution in protographs for the erasure channels.

Note that the capacity threshold p_{cap} is defined as $p_{cap} = 1 - R_c$, where R_c represents the code rate. If a code over binary erasure channel with erasure probability p_{chan} can be decoded iteratively, then $p_{chan} \leq p_{it} \leq p_{cap}$.

A. Example

Consider Repeat Jagged Accumulate (RJA) and its precoded version Accumulate Repeat Jagged Accumulate (ARJA) LDPC codes in previous example both are rate 1/2 codes. The RJA code has threshold p_{it}=0.4375. However if we precode the RJA LDPC code with an accumulator namely ARJA, then we obtain threshold p_{it}=0.4387.

V. Example of protograph codes with capacity approaching threshold

In the previous examples we considered the ARJA code with threshold p_{it}=0.4387 and with linear minimum stopping set size. Protograph codes can also be designed with thresholds approaching capacity more closely. Here one such example is shown in Fig. 6. This is a rate 1/2 Accumulate Repeat Accumulate (ARA) type protograph code with threshold p_{it}=0.4951. The capacity threshold for rate 1/2 code is p_{cap}=0.5000. Although this code has a capacity approaching threshold, it has sublinear minimum stopping set size. This type of code might be appropriate for very large block lengths.

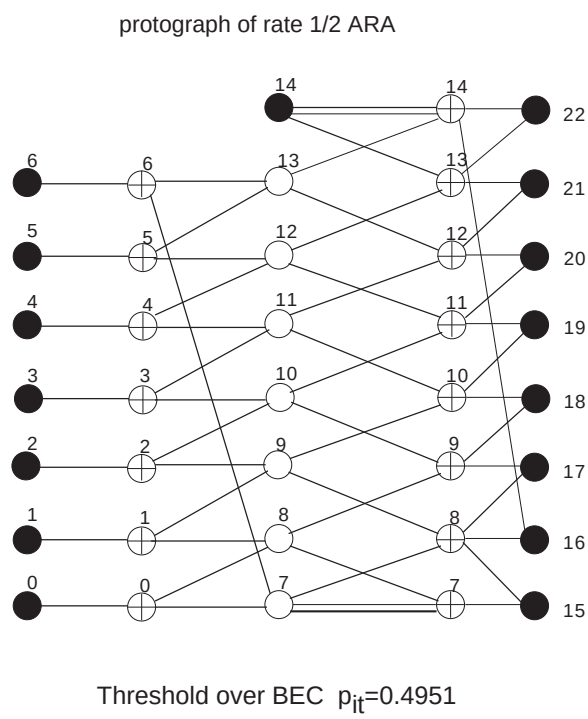

Fig. 6. Rate 1/2 ARA type protograph.

VI. An encoding method of LDPC codes over burst erasure channels

Suppose the data frames each of size n_F are transmitted over a bursty erasure channel. Here we propose a simple encoding method for channels with random burst erasures of size $B < n_F$. Suppose these bursts occur with probability p. The frames are stored as rows of a N by n_F matrix. Then each column of the matrix is

encoded with a $(N+m, N)$ LDPC code. The encoding method is shown in Fig. 7.

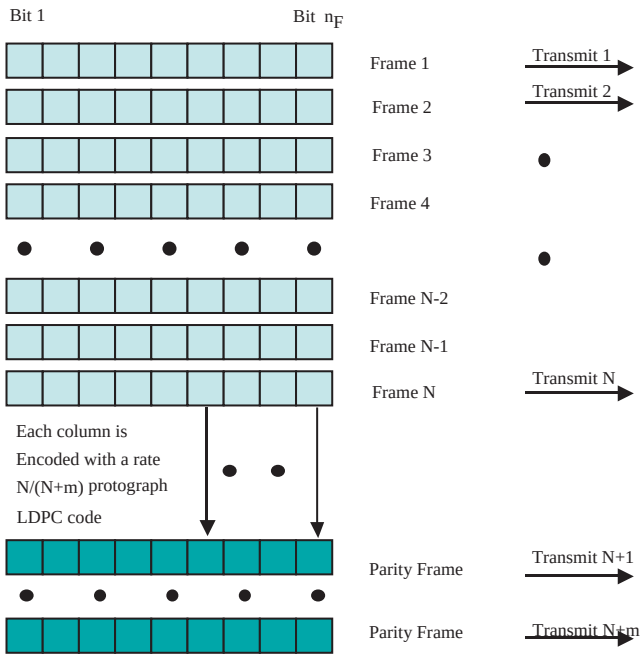

Fig. 7. An encoding method for Burst Erasure channels.

VII. Upper and Lower Bounds for BEC

For any (n, k) block code over BEC with channel erasure probability p, a lower bound on frame error rate is given by Singleton bound [31]

$$P_e > \sum_{i=n-k+1}^{n} \binom{n}{i} p^i (1-p)^{n-i} \tag{6}$$

Also for a random code (n, k), the averaged frame error rate can be upper bounded by [32] as

$$\begin{aligned} P_e \quad \leq \quad & \sum_{i=0}^{n-k} \binom{n}{i} p^i (1-p)^{n-i} 2^{-(n-k-i)} \\ & + \sum_{i=n-k+1}^{n} \binom{n}{i} p^i (1-p)^{n-i} \end{aligned} \tag{7}$$

Gallager [1] also proposed an upperbound over a discrete memoryless channel. For BEC we simplified this upperbound. The upper bound can be expressed in terms of Kullback-Leibler distance. Let a binary random variable have probability function p, and let a second random variable have probability function p^*. Then the relative entropy of p^* with respect to p, also called the Kullback-Leibler distance, is defined by

$$D(p^*, p) \triangleq p^* \ln \frac{p^*}{p} + (1-p^*) \ln \frac{1-p^*}{1-p} \tag{8}$$

Note that if we define $p^* = p_{cap} = 1 - R_c$ which is maximum erasure probability at capacity of BEC, then for a random (n, k) code with code rate $R_c = \frac{k}{n}$ we get

$$P_e \leq e^{-nD(p_{cap}, p)} \tag{9}$$

This bound is simple but it is less tight than the previous upper bound. In section IX, these bound are compared with simulation results.

VIII. High Rate Protograph Codes with Linear Stopping Set Size

In previous example we have shown that a rate 1/2 ARJA protograph LDPC code has minimum stopping set size that linearly grows with the block size. Here we design a high rate 9/10 protograph LDPC code (see Fig. 8) whose minimum stopping set size also grows linearly with the block size. We call this protograph code as R4JA3. To achieve rates higher than 9/10, we puncture the expanded version of the same protograph.

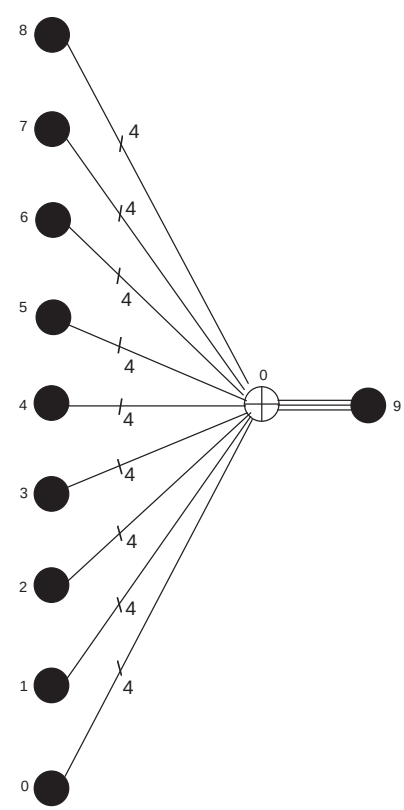

Fig. 8. Rate 9/10 R4JA3 protograph with p_{it}=0.0782.

IX. Simulation results

Simulation results in Fig. 9 compares the performance of an ARA type code (with capacity approaching iterative decoding threshold) to an ARJA code (with linear minimum stopping set size). Simulation results, show that for short blocks at low error rates, a code with linear minimum stopping set size but without capacity approaching iterative decoding threshold, outperforms a code with capacity approaching iterative decoding threshold but with sublinear minimum stopping set size. However, the ARA type code might be preferred in applications that include a retransmission protocol so that low error rates are not required.

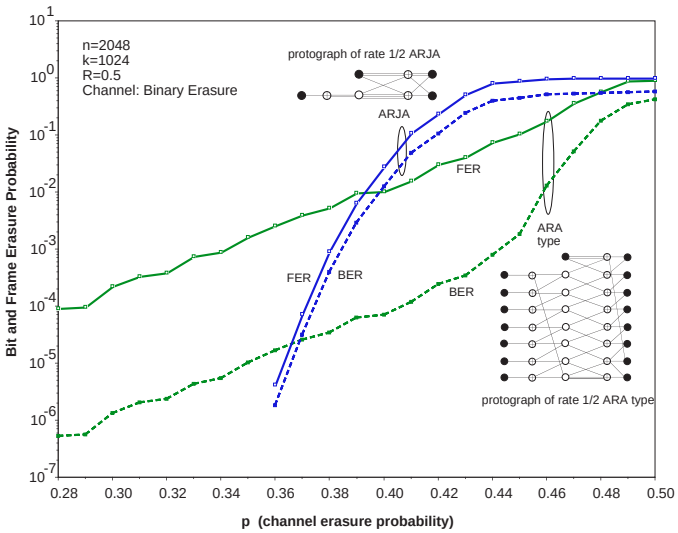

Fig. 9. Performance of ARA type code and ARJA code with input block size $k = 1024$.

In Fig. 10, frame error rate simulation results for rate 9/10 R4JA3 code are compared with bounds derived in section VII. Fig. 11 shows bit and frame error rate simulation results for LDPC codes with dimension $k = 1044$ expanded from the rate-9/10 R4JA3 protograph in Fig. 8. The rate-12/13 and rate-18/19 protographs in this case are obtained by simply puncturing the rate-9/10 protograph. Protographs were lifted using the Circulant Progressive Edge Growth (CPEG) and Approximate Cycle Extrinsic Message Degree (ACE) algorithms [19] to find circulants for each edge of the protograph.

X. Conclusion

In this paper we designed high rate protograph based LDPC codes suitable for binary erasure channels. To simplify the encoder and decoder implementation for high data rate transmission, the structure of codes are based on protographs and circulants. Two classes of codes were designed. One class is designed to have iterative decoding thresholds that approach the capacity of binary erasure channels. The other class is designed to improve minimum stopping set size. We have shown that for high code rates and short blocks the second class outperforms the first class. A scheme is proposed to use these LDPC codes over burst erasure channels. Various LDPC codes are compared and simulation results are provided.

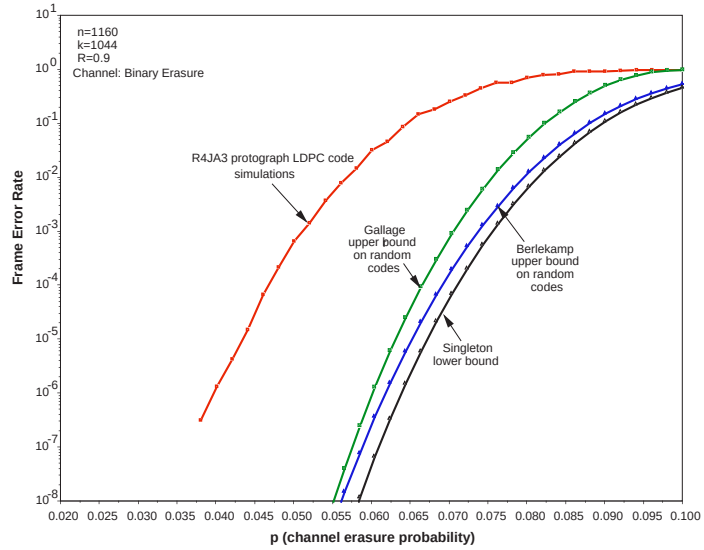

Fig. 10. Simulation results for rate 9/10 protograph LDPC code is compared to the upper and lower bounds for BEC.

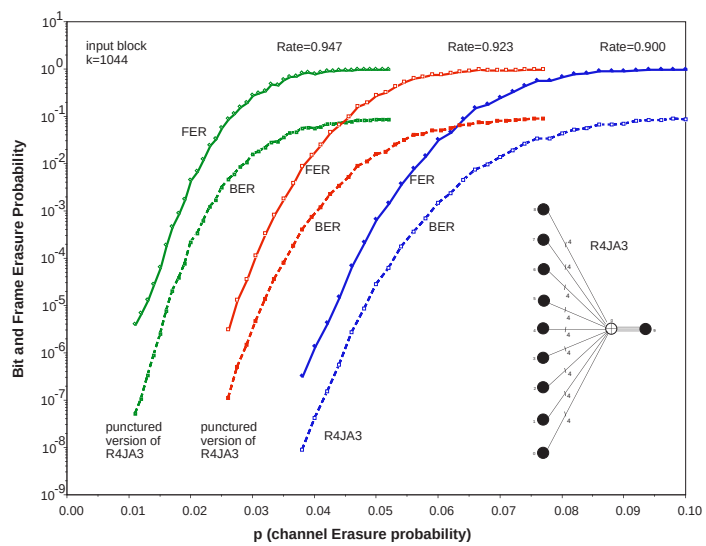

Fig. 11. Performance of R4JA3 code family with input block size $k = 1044$.

ACKNOWLEDGMENT

This research was carried out at the Jet Propulsion Laboratory, California Institute of Technology, under contract with the National Aeronautics and Space Administration.

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