MAXIMUM VON MISES STRESS IN THE LOADING ENVIRONMENT OF MASS ACCELERATION CURVE

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Acceleration Loading in Any Direction

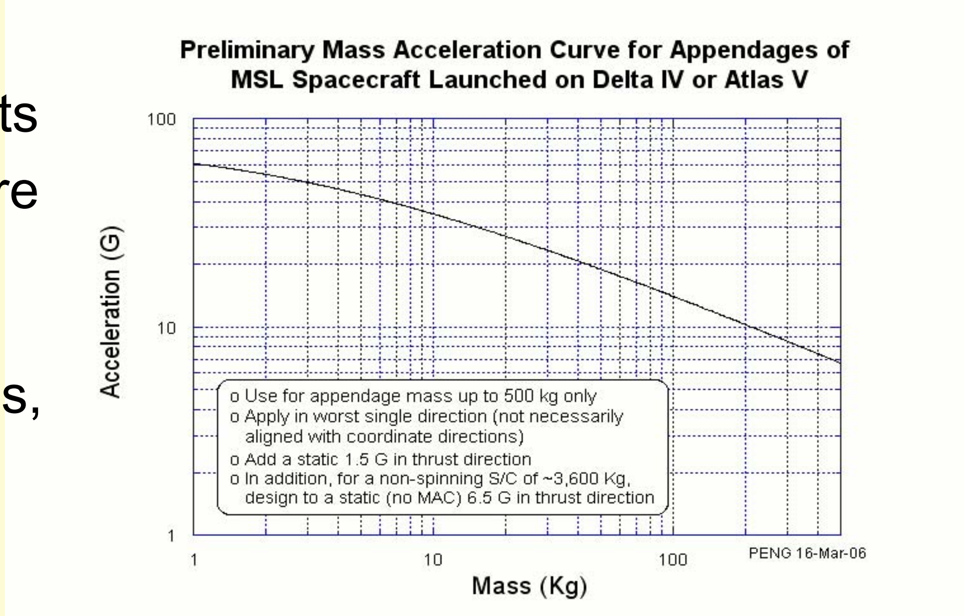
DESIGN REQUIREMENT: 300 g in any direction

"What are the stress margins in the structure?"

- Method for calculating stress due to acceleration loading
  - Part has been designed by FEA and hand calculation in one critical loading direction judged by the analyst
  - Maximum stress can be due to loading in another direction
  - Analysis procedure to be presented determines:
    - The maximum Mises stress at any point
    - The direction of maximum loading associated with the "stress"

Mass Acceleration Curve

- Concept of Mass Acceleration Curves (MAC)
  - Developed by JPL to perform preliminary structural sizing
    - Mariners, Voyager, Galileo, Pathfinder, MER, … MSL
  - Acceleration of physical masses are bounded by a curve
  - G-levels of vibro-acoustic and transient environments
  - Convergent process before the couple loads cycle
  - Semi-empirical method to effectively bound the loads, not a simulation of the actual response



MAC Application

- Current MAC operations and limitations
  - Quasi-static loads
  - Applied in one direction at a time for three orthogonal axes
  - Root-sum-square component loads to produce peak loads
    - Peak loads from the most critical MAC spatial directions
    - Good for a single direction force, moment, and stress
  - Combined stresses terms can not be root-sum-squared
    - Peak principal stresses not calculated
    - Peak Hencky-Von Mises stresses not calculated
    - Hencky-Von Mises stress (Mises stress)
      - Equivalent stress level evaluation against yield value
      - May design certain stable components

Critical MAC Mises stress

- Procedures to calculate critical MAC Mises stress
  - Apply three orthogonal MAC loadings
  - Superimpose stress results from three loading cases
    - Stresses are weighted sum from the three directions
  - Calculate Mises stress from the stress components
  - Mises stress equation is differentiated with respect to each loading case weight and set to zero
    - Three derivatives provide three linear equations in weights
    - Simultaneously solve three equations by eigen transformation to find the extrema Mises stresses
    - Critical MAC direction can be derived from the solution vector

Solution Procedures

The six stress components follow superposition so each stress can be written as below. The superscript (capital) indicates a loading case and the subscript (small) indicates the stress component. The multipliers, A, B and C represent the weighted variable from each loading case to that stress component.

$$\sigma_x = A\sigma_x{}^A + B\sigma_x{}^B + C\sigma_x{}^C \qquad \tau_{xy} = A\tau_{xy}{}^A + B\tau_{xy}{}^B + C\tau_{xy}{}^C$$

$$\sigma_y = A\sigma_y{}^A + B\sigma_y{}^B + C\sigma_y{}^C \quad \text{and} \quad \tau_{yz} = A\tau_{yz}{}^A + B\tau_{yz}{}^B + C\tau_{yz}{}^C$$

$$\sigma_z = A\sigma_z{}^A + B\sigma_z{}^B + C\sigma_z{}^C \qquad \tau_{zx} = A\tau_{zx}{}^A + B\tau_{zx}{}^B + C\tau_{zx}{}^C$$

Von Mises stress is determined from the stress state as:

$$\sigma_V = \sqrt{\frac{1}{2}\left((\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2\right) + 3\left(\tau_{xy}{}^2 + \tau_{yz}{}^2 + \tau_{zx}{}^2\right)}$$

Solution Procedures

Differentiate the Mises stress with respect to each loading case weight and set to zero.

$$\frac{\partial \sigma_V}{\partial A} = 0 \qquad \frac{\partial \sigma_V}{\partial B} = 0 \qquad \frac{\partial \sigma_V}{\partial C} = 0$$

The results can be written in a matrix form and be solved by the eigen transformation. The determinant is the characteristic equation for the matrix. The root λ of the equation is an eigenvalue of the matrix and the corresponding values for *A, B* and *C* is the eigenvector associated with the λ eigenvalue.

$$\begin{bmatrix} \left(\sigma_V{}^A\right)^2 & b & c \\ b & \left(\sigma_V{}^B\right)^2 & d \\ c & d & \left(\sigma_V{}^C\right)^2 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \begin{vmatrix} \lambda^2 - \left(\sigma_V{}^A\right)^2 & -b & -c \\ -b & \lambda^2 - \left(\sigma_V{}^B\right)^2 & -d \\ -c & -d & \lambda^2 - \left(\sigma_V{}^C\right)^2 \end{vmatrix} = 0$$

Solution Procedures

Expanding the determinant produces a cubic equation in λ^2

$$\left(\lambda^2\right)^3 + a_2\left(\lambda^2\right)^2 + a_1\left(\lambda^2\right) + a_0 = 0$$

Where :

$$a_0 = -\left(\sigma_V{}^A\right)^2\left(\sigma_V{}^B\right)^2\left(\sigma_V{}^C\right)^2 - 2bcd + b^2\left(\sigma_V{}^C\right)^2 + c^2\left(\sigma_V{}^B\right)^2 + d^2\left(\sigma_V{}^A\right)^2$$

$$a_1 = \left(\sigma_V{}^A\right)^2\left(\sigma_V{}^B\right)^2 + \left(\sigma_V{}^A\right)^2\left(\sigma_V{}^C\right)^2 + \left(\sigma_V{}^B\right)^2\left(\sigma_V{}^C\right)^2 - b^2 - c^2 - d^2$$

$$a_2 = -\left(\sigma_V{}^A\right)^2 - \left(\sigma_V{}^B\right)^2 - \left(\sigma_V{}^C\right)^2$$

$$b = \frac{1}{2}\left(\left(\sigma_x{}^B - \sigma_y{}^B\right)\left(\sigma_x{}^A - \sigma_y{}^A\right) + \left(\sigma_y{}^B - \sigma_z{}^B\right)\left(\sigma_y{}^A - \sigma_z{}^A\right) + \left(\sigma_z{}^B - \sigma_x{}^B\right)\left(\sigma_z{}^A - \sigma_x{}^A\right)\right) + 3\left(\tau_{xy}{}^B\tau_{xy}{}^A + \tau_{yz}{}^B\tau_{yz}{}^A + \tau_{zx}{}^B\tau_{zx}{}^A\right)$$

$$c = \frac{1}{2}\left(\left(\sigma_x{}^C - \sigma_y{}^C\right)\left(\sigma_x{}^A - \sigma_y{}^A\right) + \left(\sigma_y{}^C - \sigma_z{}^C\right)\left(\sigma_y{}^A - \sigma_z{}^A\right) + \left(\sigma_z{}^C - \sigma_x{}^C\right)\left(\sigma_z{}^A - \sigma_x{}^A\right)\right) + 3\left(\tau_{xy}{}^C\tau_{xy}{}^A + \tau_{yz}{}^C\tau_{yz}{}^A + \tau_{zx}{}^C\tau_{zx}{}^A\right)$$

$$d = \frac{1}{2}\left(\left(\sigma_x{}^B - \sigma_y{}^B\right)\left(\sigma_x{}^C - \sigma_y{}^C\right) + \left(\sigma_y{}^B - \sigma_z{}^B\right)\left(\sigma_y{}^C - \sigma_z{}^C\right) + \left(\sigma_z{}^B - \sigma_x{}^B\right)\left(\sigma_z{}^C - \sigma_x{}^C\right)\right) + 3\left(\tau_{xy}{}^B\tau_{xy}{}^C + \tau_{yz}{}^B\tau_{yz}{}^C + \tau_{zx}{}^B\tau_{zx}{}^C\right)$$

Extract Roots and Vectors

- Use either a closed form solution or a numerical routine to solve the cubic equation
- Roots are Mises stress ellipsoid values for different loading directions
- Maximum root is the largest Mises stress being sought
- Substitute the maximum root back into the earlier matrix equation:

$$\begin{bmatrix} \left(\sigma_V{}^A\right)^2 - \lambda^2 & b & c \\ b & \left(\sigma_V{}^B\right)^2 - \lambda^2 & d \\ c & d & \left(\sigma_V{}^C\right)^2 - \lambda^2 \end{bmatrix} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

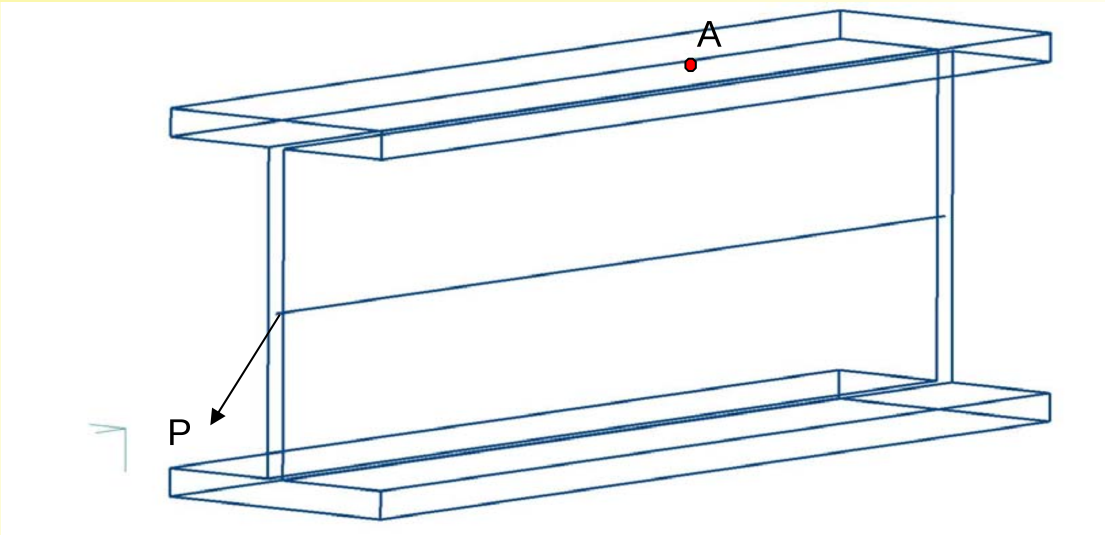
- Solve two of the three equations for the vector pointed in the direction associated with the largest root

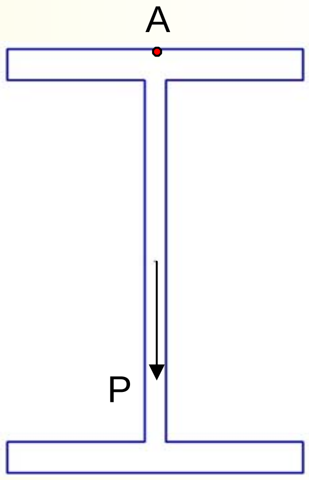
The largest root is the maximum Mises stress

Its vector points in the direction of maximum MAC loading

Verification of Solution

To demonstrate the procedural validity, a simple cantilever I-beam under 2-D MAC loads is used to compare the maximum Mises stress. The MAC loading produces 1000 lbs force at the free end in the vertical plane. The nature of the MAC load orients the force along any direction in the vertical plane. There is an unique orientation that produces maximum Mises stress at point A.





Verification of Solution

I-Beam Section properties:

A_{gross} = 2.21 in^2, A_{shear} = 0.76 in^2, MOI = 6.00 in^4

Point A is at 5" from the loading end. The closed form solution of the maximum Mises stress at point A is 2837 psi.

Using the procedure outlined earlier, the following cubic equation in terms of λ^2 for the extrema Mises stresses at point A is established:

$$\left(\left(\lambda^2\right)^2 - 8.1788\times10^6\left(\lambda^2\right) + 1.0615\times10^{12}\right)\left(\lambda^2\right) = 0$$

The solution of the quadratic equation yields two roots for λ^2 with the largest root also found to be λ = 2837 psi.

Comparison of Mises stress

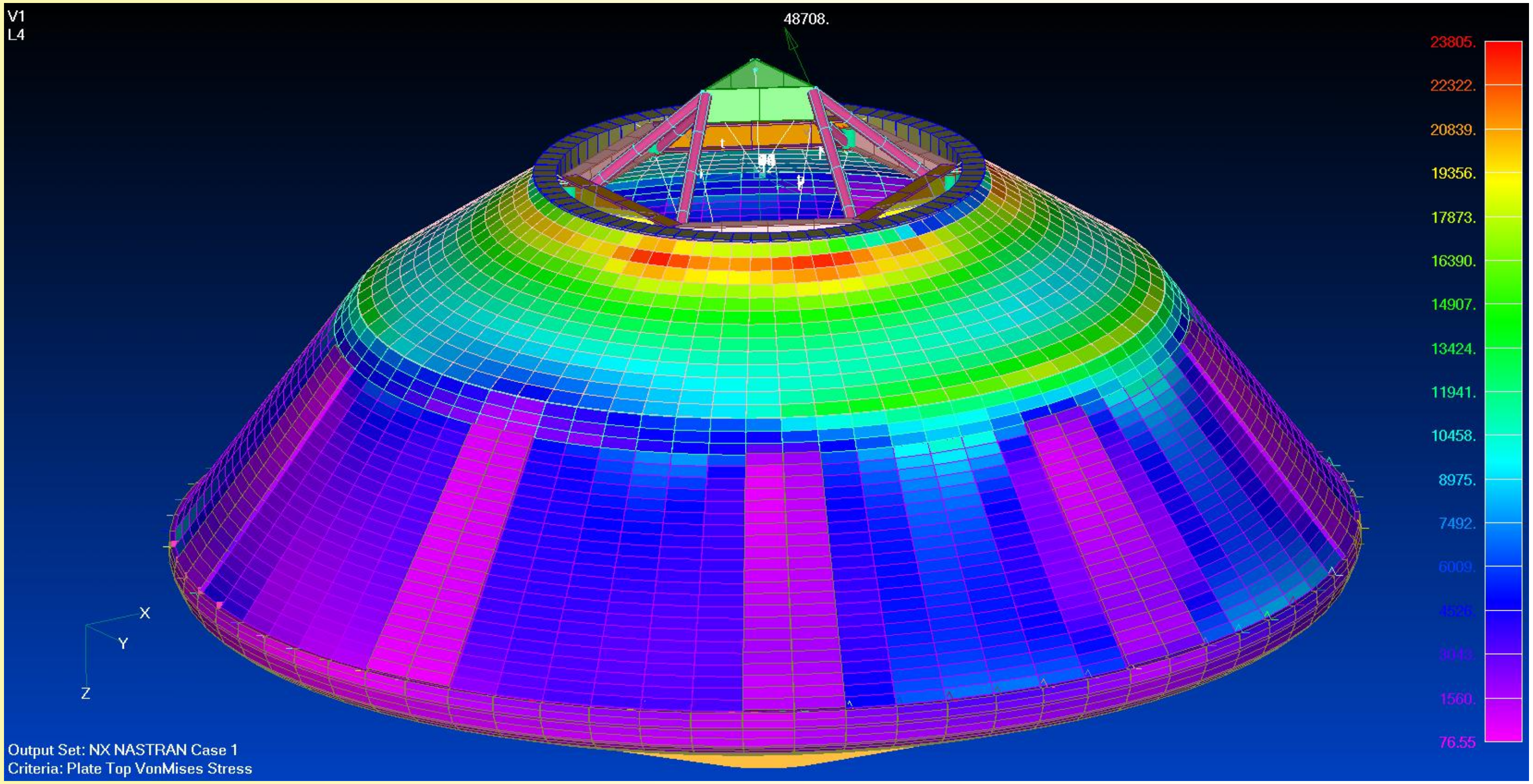
- Current MAC Mises stress
  - Loading in the MAC basic coordinate direction
  - Three orthogonal loading directions
  - Each loading case evaluated separately
  - Critical loading case based on engineering judgment
  - May need to change MAC basic coordinate to ensure more critical loadings
  - Hard to capture the most critical Mises stress
- Maximum MAC Mises stress
  - Loading in the MAC basic coordinate direction
  - Three orthogonal loading directions
  - Detects the most critical MAC loading direction at each point
  - Engineering judgment replaced by the accurate mathematical procedure
  - The most critical Mises stress captured automatically

Application of Mises Stress

- Mises stress is an equivalent stress compared to yielding
  - Based on maximum distortion energy and octahedral shearing stress
  - Widely used by the analysts to evaluate stress levels
- Correlates well with experimental results for yielding of many ductile homogeneous materials
- A generic stress for durability and fatigue analysis
- May be used to size compact structures such as certain fittings and thick wall sections
- Structures designed by stability cannot be evaluated by Mises stress alone
  - Shear panels, thin wall tubes, thin wall shapes, columns, blade flexures,…etc

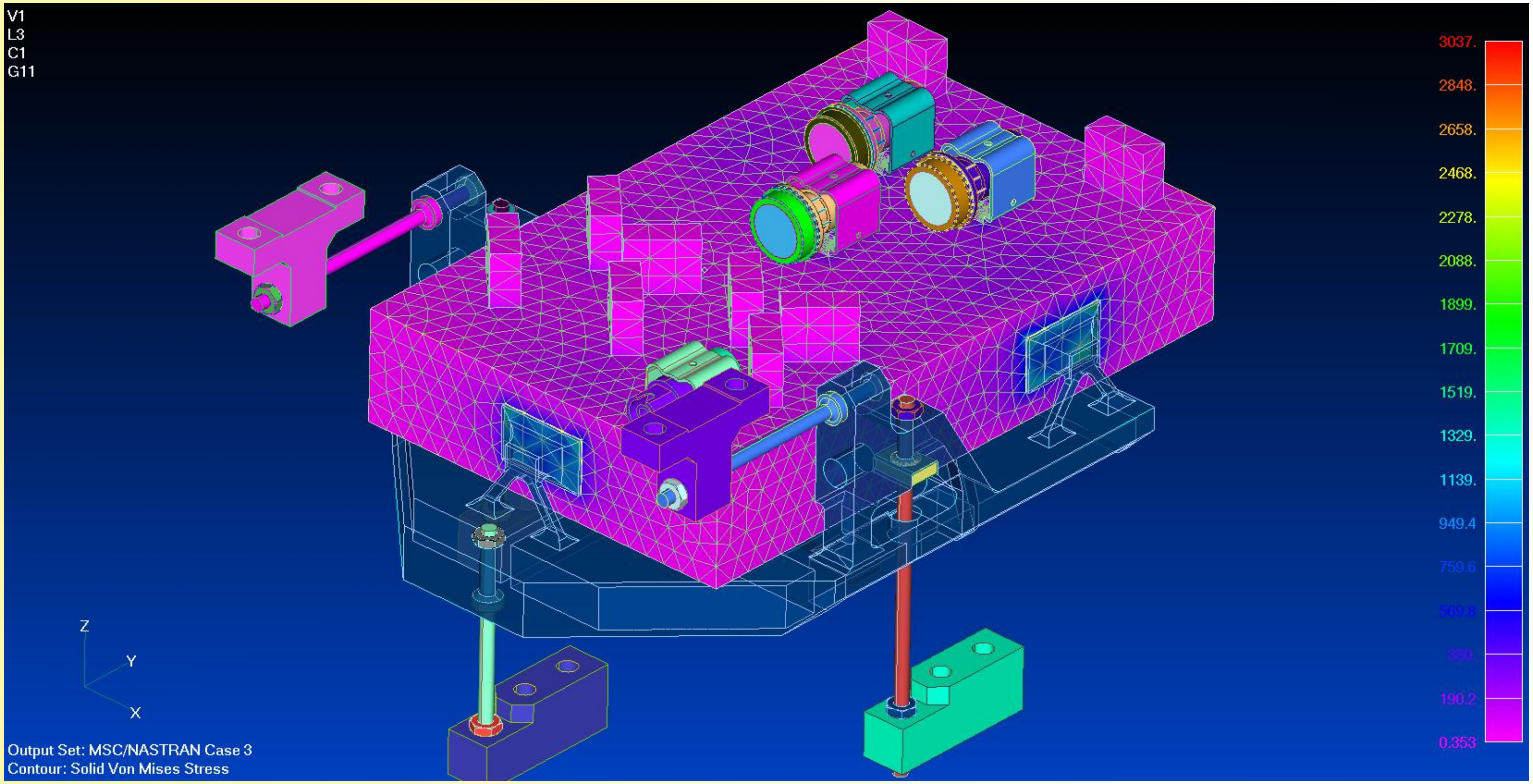
Typical Mises Stress Contour

This contour plot is for MAC loading in one axis only. The new procedure could produce similar plot in terms of the maximum Mises stress for each element from the MAC critical loading direction that varies from point to point.



Practical Mises Application

This kinematic optical bench made of ULE glass is very sensitive to the stress level. It is highly desirable to capture the maximum Mises stress.



Discussion

- MAC loading can be applied in any spatial direction
- For a given loading condition there is only one Mises stress at a point in the structure
- Mises stress is positive because it is proportional to the distortion energy at yielding
- The procedure being presented here finds the critical MAC direction to maximize the Mises stress at a point
- Since the Mises stress is always positive, the critical direction applies to both positive and negative MAC loading
- The Mises stress surface at a point from MAC loading is shaped like an ellipsoid
- The largest axis of the ellipsoid defines the direction of the maximum loading

Future Works

- Demonstration with the three dimensional loading case
  - Closed form solution if available
  - Verify the predicted critical loading direction using FEA
- Eigen solution procedure needs automation
  - External routine
  - Integrated into pre/post processors
    - Femap, Patran, Ideas
- MAC extrema principal stresses
  - Currently the critical MAC loading direction of the principal stress can not be detected
  - More complex due to three distinct principal directions
    - Mises stress is omni-directional and also a single value