

AERODYNAMICS OF TRANS-ATMOSPHERIC VEHICLES: A NON-DIMENSIONAL APPROACH

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Abstract

The non-dimensional approach to aerodynamics of trans-atmospheric flight is discussed, which explicitly takes into account the vertical span of the atmosphere as well as atmospheric mass load (atmospheric pressure) at the current flight level. As an example, a simple analytic model of the dynamics of trans-atmospheric flight is considered for both powered boost and non-powered glide of a trans-atmospheric vehicle, and the applicability to more general, numerical models of atmospheric flight is discussed.

1. Introduction

The future reusable trans-atmospheric vehicles (TAVs) will, very likely, use aerodynamic lift (aero-assist) on their way to space and back to Earth. It is understood intuitively that the cumulative effect of aerodynamic forces upon the vehicle during its ascent and descent depends on the vertical extent of the atmosphere and its mass, or rather the atmospheric pressure, which is a measure of mass of atmospheric column over a unit area. On the other hand, it is also well known that the relative effect of atmospheric forces increases with the decrease of size of the launch vehicle. This is why the smaller rockets, like *Pegasus*, need to be aircraft-launched from altitudes where most of atmospheric mass is below the launch point. This same principle applies to future aero-assisted launch vehicles. Hence, one might speculate that there should be some explicit physical interdependencies involving the mass and extent of the atmosphere on the one hand, and mass and size of the vehicle on the other that could be useful for initial sizing of TAVs and for related trade studies.

In this paper we present a non-dimensional approach that offers a method of obtaining analytic insight into some of key trades and interdependencies relevant to performance of a TAV. In addition to well-known non-dimensional parameters, such as lift-to-drag and thrust-to-weight ratios, we identify two additional relevant non-dimensional parameters, which together make it possible to reformulate the general equations of motion of atmospheric flight vehicle in a non-dimensional form. We demonstrate the application of this non-dimensional approach using a simple analytic two-dimensional (2D) model of aero-assisted trans-atmospheric flight and compare the outcome with classic results of Eugen Sänger [1] obtained using a conventional approach. The use of just four non-dimensional combinations of dimensional parameters describing the vehicle and atmosphere substantially reduces the dimension of trade space and has a potential to make the process of initial sizing of the TAVs more transparent and straightforward.

2. Vertical extent and mass of the atmosphere

It is well known that the density of the atmosphere decays exponentially with height so that the drop of atmospheric pressure p between two levels at altitudes h_1 and h_2 in the atmosphere is given by:

$$\frac{p(h_2)}{p(h_1)} = \exp\left(-\frac{h_2 - h_1}{H}\right) \quad (1)$$

where H is the atmospheric scale height. The value of H is controlled by atmospheric temperature T and gravity g , and also depends on the gas constant R specific for given atmospheric composition:

$$H = \frac{RT}{g} \quad (2)$$

Another atmospheric parameter, which is relevant here, is the total mass of the atmospheric column per unit area:

$$m_A = \frac{p}{g} \quad (3)$$

An alternate expression of m_A can be obtained through atmospheric density ρ using the equation of state of the atmospheric gas:

$$p = \rho RT \quad (4)$$

Substituting Eq.(4) into Eq.(3) and using definition of atmospheric scale height, Eq.(2), we have:

$$m_A = \rho H \quad (5)$$

In the next Section we will see that the dimensional atmospheric quantities H and m_A provide a description of the atmosphere that is sufficient for formulation of a non-dimensional approach to the analysis of atmospheric flight. As it will be seen in further Sections, this approach can be useful in performing sizing and parametric studies of trans-atmospheric vehicles.

3. Non-dimensional velocity and non-dimensional mass ratio

Obviously, velocity is an extremely important parameter for characterizing the performance of a flight vehicle. For vehicles that traverse the atmosphere, it is desirable to use a velocity unit that is directly related to the vertical extent of the atmosphere described by the scale height H . The most natural choice here is:

$$V_H = \sqrt{2gH} \quad (6)$$

Similarly to speed of sound, V_H depends on atmospheric temperature. Substituting H from Eq.(2), we obtain an alternative expression:

$$V_H = \sqrt{2RT} \quad (7)$$

This expression can be directly compared with that for the speed of sound (see, e.g., [2]):

$$c = \sqrt{\gamma RT} \quad (8)$$

where $\gamma = c_p / c_v$ is the ratio of specific heats (for air, $\gamma = 1.4$).

We will refer to V_H as *atmospheric scale velocity*. The desired non-dimensional velocity quantity is

$$v = \frac{V}{V_H} \quad (9)$$

which, by analogy to the Mach number, will be referred to here as *v-number* (“upsilon”-number). It can be shown that the *v-number* provides a very simple relation between the dynamic pressure q and static (atmospheric) pressure p . By definition of dynamic pressure we have:

$$q = \frac{1}{2} \rho V^2 \quad (10)$$

Multiplying and dividing the right-hand term of Eq.(10) by H and substituting Eqs.(5) and (3), we obtain:

$$q = \frac{1}{2H} \rho H V^2 = \frac{1}{2H} \frac{p}{g} V^2 = p \left(\frac{V}{V_H} \right)^2 \quad (11)$$

Thus, we simply have:

$$q = p v^2 \quad (12)$$

It can be seen that this relation is more straightforward than the textbook relation between q and p involving Mach number and ratio of specific heats [see, e.g., [2]]:

$$q = \frac{\gamma}{2} p M^2 \quad (13)$$

To introduce the second non-dimensional parameter to be used in this non-dimensional approach, we apply the *v-number* introduced above to the simplest model of straight-and-level flight, where the aerodynamic lift force is balanced by the weight of the vehicle:

$$L = q \lambda C_D S = mg \quad (14)$$

Here, $\lambda = (L/D)$ is the lift-to-drag ratio of the vehicle, C_D and S are its drag coefficient, and reference area respectively. Using relation between q and p , Eq.(12) we rewrite Eq.(14) in the form:

$$\lambda \frac{p}{g} v^2 = \frac{m}{C_D S} \quad (15)$$

The right-hand term of Eq.(15) is the ballistic coefficient of the vehicle. The ratio p/g in the left-hand term is the column mass of the atmosphere defined by Eqs.(3,5). Both of these quantities have a dimension of mass over area. Introducing the non-dimensional mass parameter, which we will call a *μ-ratio*:

$$\mu = \frac{\rho H}{m/(C_D S)} = \frac{p/g}{m/(C_D S)} \quad (16)$$

we can rewrite Eq.(15) in a non-dimensional form:

$$\lambda \mu v^2 = 1 \quad (17)$$

In essence, the μ – ratio reflects the importance of aerodynamic effects on the dynamics of the flight vehicle. Neither the atmospheric mass nor the ballistic coefficient are per se important, only their ratio is. On the other hand, for a given vehicle, the μ – ratio is proportional to ambient atmospheric pressure and is, therefore, directly related to the pressure altitude. Thus, together, the v – number and μ – ratio define two most important flight parameters: airspeed and altitude.

The introduced non-dimensional quantities, v – number and μ – ratio, are used below in simple analytic models of trans-atmospheric flight based on an assumption of an equilibrium between aerodynamic lift force and weight of the vehicle, which is partially balanced by the centrifugal force when velocities become comparable to the orbital velocity. As we shall see, this assumption leads to one-to-one correspondence between the airspeed of the vehicle and atmospheric density. We obtain and analyze this relation in the non-dimensional form. For trans-atmospheric non-powered glide, this concept is known since mid-20th century under the name “dynamic soaring” (hence the name of the abandoned “Dyna-Soar” Project – see, e.g., [3]). For lack of better name for this concept that combines both the equilibrium boost and equilibrium glide, we use the name “*aero-surfing*”. This reflects the notion that the vehicle merely surfs the atmosphere, all the time remaining only as deep in the atmosphere, as necessary to have adequate lift.

4. Aero-surfing, an equilibrium hypersonic flight

Starting from an equation analogous to Eq.(14), except at velocities comparable to orbital velocity, the weight of the vehicle is partially compensated by the centrifugal force. We have:

$$q\lambda C_D S = mg \left[1 - \left(\frac{V}{V_o} \right)^2 \right] \quad (18)$$

Here $V_o = \sqrt{gR}$ is the orbital velocity, where R is the radius of Earth. After substituting of the explicit expression for dynamic pressure q , Eq.(10), and performing transformations similar to those preceding Eq.(17), we obtain:

$$\mu = \frac{1}{\lambda} (v^{-2} - v_o^{-2}) \quad (19)$$

where $v_o = V_o / V_H$ is the non-dimensional orbital velocity. In the explicit non-dimensional form, we have:

$$\frac{\rho H}{(m / C_D S)} = \left(\frac{L}{D} \right)^{-1} \left[\left(\frac{V}{V_H} \right)^{-2} - \left(\frac{V_o}{V_H} \right)^{-2} \right] \quad (20)$$

Essentially, Eqs.(18,19) provide the way to compute the altitude from airspeed for the equilibrium flight. Inverting them with respect to the airspeed, we obtain correspondingly:

$$v = (\lambda\mu + v_o^{-2})^{-1/2} \quad (21)$$

and

$$\frac{V}{V_H} = \left[\left(\frac{L}{D} \right) \frac{\rho H}{(m/C_D S)} + \left(\frac{V_o}{V_H} \right)^{-2} \right]^{-1/2} \quad (22)$$

The last expression can be compared with the slightly modified textbook expression ([2], p.578):

$$V = \frac{1}{\sqrt{\left(\frac{L}{D} \right) \frac{\rho}{2g(m/C_D S)} + \frac{1}{gR}}} \quad (23)$$

This expression is analytically equivalent to Eq.(21) but its physical meaning is less straightforward. Similar expression can be found on p.302 of English translation of Sanger's monograph [1]; however, direct comparison is complicated because, on the one hand, the author directly uses numerical values of relevant parameters, and on the other hand, he accounts for variation of gravity with altitude, which is neglected in Eq.(23) and is considered negligible for purposes of this study.

The obtained relations between the airspeed and altitude of the equilibrium flight were based only on an assumption of balance between the aerodynamic lift and weight of the vehicle. The acceleration or deceleration along the flight path was not constrained so far. In the next Section, we consider the cases of powered boost and non-powered glide separately. We will consider the time histories of airspeed and travel distance, as well as the flight-path angle behavior. Altogether, these results will provide a two-dimensional (2D) description of the flight trajectory of a trans-atmospheric vehicle to and from space at hypersonic velocities.

5. Aero-surfing: Hypersonic boost and hypersonic glide

Under the equilibrium assumptions outlined above, the general 2D equations of motion of the flight vehicle can be reduced to a following non-dimensional form (see Appendix A):

$$\frac{dv}{d\tau} = -\mu v^2 + n \quad (24)$$

$$\left(\frac{v}{v_o} \right)^2 = -\lambda \mu v^2 + 1 \quad (25)$$

where

$$n = \frac{F}{mg} \quad (26)$$

is the thrust-to-weight ratio of the vehicle ($n = 0$ for the glide), and

$$\tau = \frac{t}{V_H / g} \quad (26)$$

is the suitable non-dimensional time variable. The first equation, Eq.(24), describes the acceleration/deceleration of the vehicle along the flight path, while the second equation, Eq.(25), describes the balance of accelerations across the flight path. This equation is analytically identical to Eqs.(18,20) and is therefore redundant. Using Eq.(19) we can eliminate μ from Eq.(24) to obtain:

$$\frac{dv}{d\tau} = \frac{1}{\lambda} \left[\left(\frac{v}{v_o} \right)^2 + \lambda n - 1 \right] \quad (28)$$

Depending on the sign of the term $\lambda n - 1$ in square brackets in the right side of Eq.(28), it has either ordinary trigonometric, or hyperbolic trigonometric solution. For the boost case, we have $\lambda n - 1 > 0$. It can be easily verified that, barring an arbitrary constant term, the solution of the differential equation Eq.(28) is:

$$v(\tau) = v_o \sqrt{\lambda n - 1} \tan \left(\frac{\sqrt{\lambda n - 1}}{\lambda v_o} \tau \right) \quad (29)$$

In the dimensional form we have correspondingly:

$$V(t) = V_o \sqrt{\lambda n - 1} \tan \left(\frac{\sqrt{\lambda n - 1}}{\lambda V_o / g} t \right) = \sqrt{(\lambda n - 1)gR} \tan \left(\frac{\sqrt{\lambda n - 1}}{\lambda \sqrt{R/g}} t \right) \quad (30)$$

The time variables τ and t here can be interpreted as describing the *extrapolated time from take-off* of the vehicle.

For the glide case ($n = 0$) $\lambda n - 1 < 0$, and it can be easily verified that, barring an arbitrary constant term, the solution of differential equation Eq.(28) is:

$$v(\tau) = v_o \tanh \left(-\frac{\tau}{\lambda v_o} \right) \quad (31)$$

For sake of convenience, we replace the time variables τ and t by *extrapolated time till touchdown* variables τ' and t' . In non-dimensional notations we have:

$$v(\tau') = v_o \tanh \left(\frac{\tau'}{\lambda v_o} \right) \quad (32)$$

In the dimensional form we have correspondingly:

$$V(t') = V_o \tanh \left(\frac{t'}{\lambda V_o / g} \right) = \sqrt{gR} \tanh \left(\frac{t'}{\lambda \sqrt{R/g}} \right) \quad (33)$$

The time histories of airspeed obtained for boost and glide can be integrated over time to get corresponding non-dimensional and dimensional extrapolated distances. For the *extrapolated distance from take-off* in the boost case we have:

$$\Delta(\tau) = -\lambda v_o^2 \ln \cos \left(\frac{\sqrt{\lambda n - 1}}{\lambda v_o} \tau \right) \quad (34)$$

$$D(t) = -\lambda \frac{V_o^2}{g} \ln \cos \left(\frac{\sqrt{\lambda n - 1}}{\lambda V_o / g} t \right) = -\lambda R \ln \cos \left(\frac{\sqrt{\lambda n - 1}}{\lambda \sqrt{R / g}} t \right) \quad (35)$$

For the *extrapolated distance to touchdown* in the glide case we have:

$$\Delta(\tau') = -\lambda v_o^2 \ln \cosh \left(\frac{\tau'}{\lambda v_o} \right) \quad (36)$$

$$D(t') = -\lambda \frac{V_o^2}{g} \ln \cosh \left(\frac{t'}{\lambda V_o / g} \right) = -\lambda R \ln \cosh \left(\frac{t'}{\lambda \sqrt{R / g}} \right) \quad (37)$$

To complete the 2D description of the flight trajectory of the TAV, we consider the flight-path angle. Under assumptions made above, it can be represented simply as a ratio of the vertical velocity to airspeed:

$$\theta = \frac{1}{V} \frac{dh}{dt} \quad (38)$$

It can be shown (see Appendix B) that under aero-surfing conditions there is a one-to-one relation between the flight angle and the airspeed. For hypersonic boost, in non-dimensional notation we have:

$$\theta = \frac{1}{\lambda} v^{-2} \left(\frac{\lambda n}{1 - (v/v_o)^2} - 1 \right) \quad (39)$$

For hypersonic glide ($n = 0$), from Eq.(39) directly follows an even more simple relation:

$$\theta = \frac{1}{\lambda} v^{-2} \quad (40)$$

Corresponding non-dimensional relations in the explicit form are:

$$\theta = \frac{1}{(L/D)} \left(\frac{V}{V_H} \right)^{-2} \left(\frac{(L/D)(F/mg)}{1 - (V/V_o)^2} - 1 \right) \quad (41)$$

for hypersonic boost and

$$\theta = -\frac{1}{(L/D)} \left(\frac{V}{V_H} \right)^{-2} \quad (42)$$

for hypersonic glide.

It should be reminded that, as we have seen above, for aero-surfing in general there is a one-to-one relation between airspeed and altitude, or between v - number and μ - ratio. [see Eqs.(19,21)]. From this circumstance follows a one-to-one relation between flight path angle and altitude. For example, substituting Eq.(21) into Eq.(40) we obtain for hypersonic glide:

$$\theta = -\left(\mu + \frac{1}{\lambda} v_o^{-2} \right) \quad (43)$$

or, in the explicit form:

$$\theta = - \left[\frac{\rho H}{m/(C_D S)} + \frac{1}{(L/D) \left(\frac{V_o}{V_H} \right)^{-2}} \right] \quad (44)$$

These expressions become especially simple and straightforward for lower altitudes, where the airspeed is much smaller than the orbital velocity ($V \ll V_o$):

$$\theta = -\mu \quad (45)$$

or, in the explicit form:

$$\theta = - \frac{\rho H}{m/(C_D S)} \quad (46)$$

For hypersonic boost, corresponding general expressions analogous to Eqs.(43,44) are more complicated, but for $V \ll V_o$, they reduce to much more simple relations:

$$\theta = (\lambda n - 1)\mu \quad (47)$$

$$\theta = \left[\left(\frac{L}{D} \right)^n - 1 \right] \frac{\rho H}{m/(C_D S)} \quad (48)$$

This concludes presentation of the simple analytical model of atmospheric flight of a trans-atmospheric vehicle assuming aero-surfing conditions. In addition to the airspeed—altitude relation, which is valid for aero-surfing in general, we have obtained explicit expressions for time histories of airspeed and traveled distance for both hypersonic boost and hypersonic drag, as well as expressions for the flight-path angle.

6. Discussion and conclusion

In this paper, we attempted to directly factor in the vertical extent and mass of the atmosphere into aerodynamics of TAV. For this purpose, we have introduced two non-dimensional parameters, ν – number and μ – ratio. On one hand, these parameters are directly related to the altitude and airspeed of the vehicle. On other hand, they, as desired, encapsulate the atmospheric scale height and atmospheric column mass. Using these parameters together with two other non-dimensional parameters known from the literature: lift-to-drag ratio λ and thrust-to-weight ratio n , we have analyzed the atmospheric flight of a TAV during boost to space and glide back to Earth. We have used the conventional assumption of equilibrium between aerodynamic lift, weight of the vehicle, and centrifugal force, which becomes increasingly important, at speeds close to orbital velocity. Using this non-dimensional approach, the classic results of Sanger [1] were re-derived in a form that directly shows an interplay between the parameters of the atmosphere and parameters of the vehicle.

As we have seen (Appendix A), this non-dimensional approach can be applied to initial 2D equations of motion of the vehicle before any simplifications are made. Obviously, extension to 3D description of atmospheric flight is pretty straightforward. As in the conventional approach, the resulting equations can be numerically integrated without further simplifying assumptions. Therefore, this non-dimensional approach is applicable to a wide variety of problems regarding the atmospheric flight, especially where explicit

account for vertical span and mass per unit area of the atmosphere above the flight level is required.

The simplified non-dimensional analytic model developed here illustrates the practical applications of the non-dimensional approach to aerodynamics of a flight vehicle. In the author's opinion, it can be used for practical applications to the initial sizing of TAVs based on required performance throughout the most important, hypersonic phase of the flight. Non-dimensionalization of the initial equations of atmospheric flight reduces the space for trade studies to that of the above four non-dimensional parameters, which might be helpful in development of more sophisticated, numerical models of atmospheric flight of TAVs.

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APPENDICES

Appendix A. Non-dimensional equations of motion

We start from 2D equations of motion in a standard form written for accelerations along and across the flight path (see, e.g., [2]):

$$\frac{dV}{dt} = -\frac{D}{m} - g \sin \theta + \frac{F}{m} \quad (\text{A1})$$

$$\frac{V^2}{R} = -\frac{L}{m} + g \cos \theta \quad (\text{A2})$$

where θ is the flight path angle ($\theta < 0$ for the glide), F is the thrust, and the rest of notations are described in the main text. Using the explicit expressions for the lift and drag forces:

$$L = \frac{1}{2} \rho V^2 \left(\frac{L}{D} \right) C_D S \quad (\text{A3})$$

$$D = \frac{1}{2} \rho V^2 C_D S \quad (\text{A4})$$

and introducing one more dimensional parameter associated with the atmospheric scale height, *the atmospheric scale time*

$$t_H = \frac{V_H}{g} = \sqrt{\frac{2H}{g}} \quad (\text{A5})$$

we can non-dimensionalize Eqs.(A1,A2). In the explicit non-dimensional form we have:

$$\frac{d}{d(t/t_H)} \left(\frac{V}{V_H} \right) = -\mu \left(\frac{V}{V_H} \right)^2 - \sin \theta + \frac{F}{mg} \quad (\text{A6})$$

$$\left(\frac{V_o}{V_H} \right)^{-2} \left(\frac{V}{V_H} \right)^2 = -\lambda \mu \left(\frac{V}{V_H} \right)^2 + \cos \theta \quad (\text{A7})$$

Using non-dimensional notations, we have correspondingly:

$$\frac{dv}{d\tau} = -\mu v^2 - \sin \theta + n \quad (\text{A8})$$

$$v_o^{-2} v^2 = -\lambda \mu v^2 + \cos \theta \quad (\text{A9})$$

where

$$\tau = \frac{t}{t_H} \quad (\text{A10})$$

is the non-dimensional time, and

$$n = \frac{F}{mg} \quad (\text{A11})$$

is the thrust-to-weight ratio of the vehicle.

According to the assumptions justified in the main text, the angle θ is small; hence $\sin \theta \approx 0$, and $\cos \theta \approx 1$. Then, Eqs.(A8,A9) can be rewritten in the form:

$$\frac{dv}{d\tau} = -\mu v^2 + n \quad (\text{A12})$$

$$\left(\frac{v}{v_o}\right)^2 = -\lambda\mu v^2 + 1 \quad (\text{A13})$$

Appendix B. Flight- path angle

According to our assumptions, this angle is small for typical hypersonic boost and glide trajectories and we can represent it as a ratio of the vertical velocity to airspeed:

$$\theta = \frac{1}{V} \frac{dh}{dt} \quad (\text{B1})$$

For the vertical velocity we have:

$$\frac{dh}{dt} = \frac{1}{t_H} \frac{dh}{dv} \frac{dv}{d\tau} \quad (\text{B2})$$

The factor $dv/d\tau$ is directly obtained from the equation of motion, Eq.(A12). The factor dh/dv can be evaluated using Eq.(15). From the definition of the μ – ratio, Eq.(16), we have:

$$\frac{d\mu}{\mu} = \frac{dp}{p} = -\frac{dh}{H} \quad (\text{B3})$$

Thus,

$$\frac{dh}{dv} = -\frac{H}{\mu} \frac{d\mu}{dv} \quad (\text{B4})$$

Evaluating $d\mu/dv$ from Eq.(19) we have

$$\frac{dh}{dv} = \frac{2H}{\lambda\mu} v^{-3} \quad (\text{B5})$$

Substituting Eqs.(B5,A12) into Eq.(B2) and using again Eq.(15) we have:

$$\frac{dh}{dt} = -\sqrt{\frac{g}{2H}} \frac{2H}{\lambda\mu} \frac{1}{\lambda} v^{-3} (\mu v^2 + n) = -\frac{V_H}{\lambda\mu} v^{-3} (-\mu v^2 + n) \quad (\text{B6})$$

Substituting Eq.(B6) into Eq.(B1), we obtain an initial expression for the flight path angle:

$$\theta = \frac{1}{\lambda} v^{-2} \left(\frac{n}{\mu v^2} - 1 \right) \quad (\text{B7})$$

Obtaining μv^2 from Eq.(19):

$$\mu v^2 = \frac{1}{\lambda} (1 - v_o^{-2} v^2) \quad (\text{B8})$$

and substituting this result into Eq.(B7), we obtain:

$$\theta = \frac{1}{\lambda} v^{-2} \left(\frac{\lambda n}{1 - (v/v_o)^2} - 1 \right) \quad (\text{B9})$$

We see that at hypersonic velocities, when $v^2 \gg 1$, the flight-path angle θ is small indeed, provided the ratio v/v_o does not approach unity.

It should be noted that the obtained result, Eq.(B9) is not applicable at low airspeeds, where $v \rightarrow 0$, and θ becomes unrealistically large. The reason for this is due to omission of the term containing $\sin \theta$ in the equation of motion Eq.(A8) and its derivative Eq.(A12). Indeed, if we keep this term in Eq.(A8), then Eq.(A12) is modified to the form:

$$\frac{dv}{d\tau} = -\mu v^2 - \theta + n \quad (\text{A12a})$$

Then, substituting Eqs.(B5,A12a) into Eq.(B2) and using Eq.(15) we have:

$$\frac{dh}{dt} = -\sqrt{\frac{g}{2H}} \frac{2H}{\lambda \mu} \frac{1}{\lambda} v^{-3} (\mu v^2 - \theta + n) = -\frac{V_H}{\lambda \mu} v^{-3} (-\mu v^2 - \theta + n) \quad (\text{B6a})$$

Substituting Eq.(B6a) into Eq.(B1), and solving the resulting equation for θ we obtain:

$$\theta = \frac{1}{\lambda} (v^2 + 1)^{-1} \left(\frac{n}{\mu v^2} - 1 \right) \quad (\text{B7a})$$

Substituting μv^2 from Eq.(B8) into Eq.(B7a), we obtain the desired modification of Eq.(B9), which converges to Eq.(B9) at higher v - numbers:

$$\theta = \frac{1}{\lambda} (v^2 + 1)^{-1} \left(\frac{\lambda n}{1 - (v/v_o)^2} - 1 \right) \quad (\text{B9a})$$

As a matter of fact, this derivation can be repeated without assuming that $\sin \theta \approx \theta$. Then Eq.(B1) is replaced by

$$\sin \theta = \frac{1}{V} \frac{dh}{dt} \quad (\text{B1a})$$

and the derivation above can be simply rewritten with θ replaced by $\sin \theta$. Then we obtain:

$$\sin \theta = \frac{1}{\lambda} (v^2 + 1)^{-1} \left(\frac{\lambda n}{1 - (v/v_o)^2} - 1 \right) \quad (\text{B9b})$$

For the glide case, at low airspeeds ($v^2 \ll 1$) Eq.(B9b) converges to a well-known textbook result:

$$\sin \theta = -\frac{1}{\lambda} \quad (\text{B10})$$

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