Optimal estimation of clock values and trends from finite data

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Ingredients

1. Stochastic model for clock phase residual $x(t)$

   Stationary dth increments: d differencing operations give stationary noise

   Degree of $x(t)$:= minimum such d

   Spectral density $S_x(f)$; $\int_0^{f_0} f^{2d} S_x(f)\,df < \infty$

   Generalized autocovariance (GACV) $s_x(t)$

2. Data $x(t_1),\ldots,x(t_n)$
3. Estimation targets

   a) $x(t_*)$ ("prediction")

   b) Coefficient c_d of long-term trend $\dfrac{c_d t^d}{d!}$

   $d = 1,2,3$: frequency, drift rate, aging rate

Some clock noises

$$\overbrace{S_y^+(f)}^{2(2\pi f)^2 S_x(f)}$$

Name	degree	$S_y^+(f)$	$s_x(t)$
White FM	1	h_0	$-\dfrac{h_0\lvert t\rvert}{4}$
Flicker FM	1	$h_{-1}f^{-1}$	$\dfrac{h_{-1}t^2 \ln\lvert t\rvert}{2}$
Random walk FM	2	$h_{-2}f^{-2}$	$\dfrac{h_{-2}\pi^2 \lvert t\rvert^3}{6}$
Flicker walk FM	2	$h_{-3}f^{-3}$	$-\dfrac{h_{-3}\pi^2 t^4 \ln\lvert t\rvert}{6}$

The GACV $s_x(t)$ is a form of generalized Fourier transform of $S_x(f)$.

GACV of uncorrelated sum = sum of GACVs.

Degree of sum = max degree of terms

What GACV does and doesn't

Say that $a(t_1),\ldots,a(t_n)$ satisfy the moment condition for d if

$$\sum_{i=1}^{n} a(t_i)t_i^k = 0, \quad k = 0,\ldots,d-1$$

Let $x(t)$ have stationary dth increments, GACV $s_x(t)$.

Let $\{a(t_i)\}, \{b(u_j)\}$ satisfy the moment condition for d. Then

$$\mathrm{E}\left[\sum_i a(t_i)x(t_i)\right]\left[\sum_j b(t_j)x(t_j)\right] = \sum_{i,j} a(t_i)b(t_j)s_x(t_i - t_j)$$

If $x(t)$ not stationary (degree ≥ 1), then $s_x(t)$ **not** an ACV function:

$$\mathrm{E}\,x(t)x(u) \ne s_x(t-u)$$

Problem

For each target, find the best linear invariant estimator (BLIE)

$$\sum_{i=1}^{n} a(t_i)\,x(t_i)$$

Best:

Minimal mean square error (MSE)

Invariant:

Prediction: error does not change when a polynomial of degree $\le d-1$ is added to $x(t)$.

$d=2$: invariance to added constant phase and frequency.

Trend estimation: error is invariant to polynomials of degree $\le d$.

We will solve for $a(t_1),\dots,a(t_n)$ numerically from a set of linear equations.

Prediction solution

Predictor of $x(t_*)$: $\hat{x}(t_*) = \sum_{i=1}^{3} a_i x(t_i) \quad (d=2)$

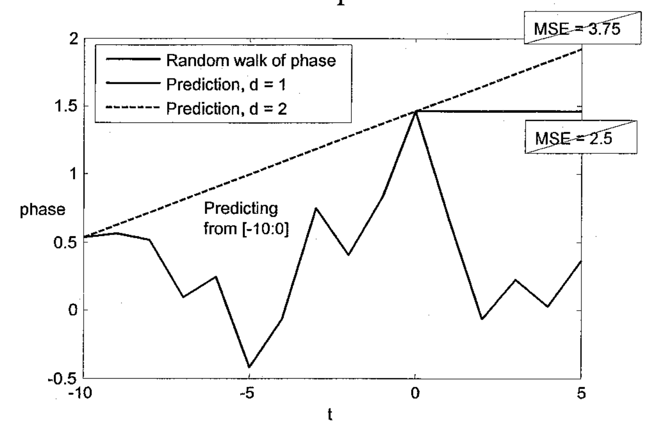
$$\left[\begin{array}{ccc|cc} s_x(0) & s_x(t_1-t_2) & s_x(t_1-t_3) & 1 & t_1 \\ s_x(t_2-t_1) & s_x(0) & s_x(t_2-t_3) & 1 & t_2 \\ s_x(t_3-t_1) & s_x(t_3-t_2) & s_x(0) & 1 & t_3 \\ \hline 1 & 1 & 1 & 0 & 0 \\ t_1 & t_2 & t_3 & 0 & 0 \end{array}\right] \left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ \hline \theta_0 \\ \theta_1 \end{array}\right] = \left[\begin{array}{c} s_x(t_1-t_*) \\ s_x(t_2-t_*) \\ s_x(t_3-t_*) \\ \hline 1 \\ t_* \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} n \\ \left.\vphantom{\begin{array}{c}1\\1\end{array}}\right\} d \end{array}$$

$$\begin{bmatrix} R & G^{\mathrm{T}} \\ G & 0 \end{bmatrix} \begin{bmatrix} a \\ \theta \end{bmatrix} = \begin{bmatrix} r \\ g \end{bmatrix}$$

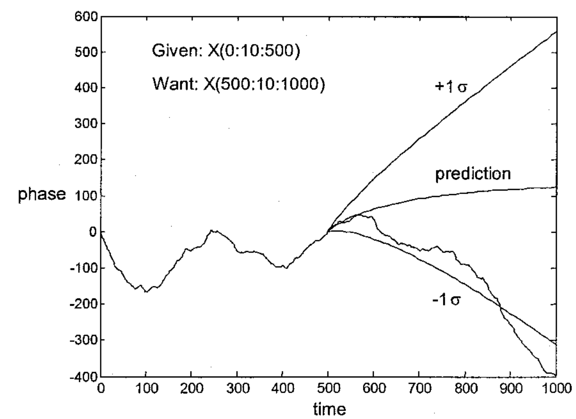
Solve for a, θ.

Mean square error: $\mathrm{E}\left[x(t_*) - \hat{x}(t_*)\right]^2 = s_x(0) - r^{\mathrm{T}} a - g^{\mathrm{T}} \theta$

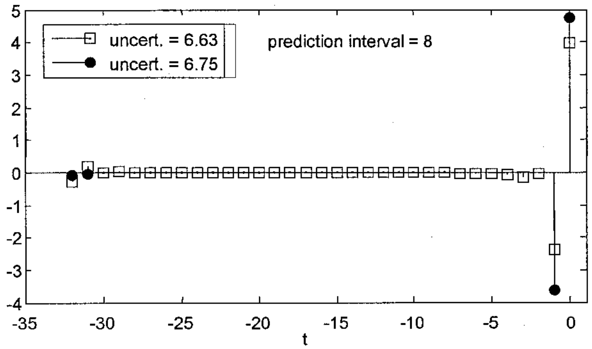
White FM prediction



Flicker FM prediction



Flicker FM: prediction coefficients



Drift estimation solution

Trend $\frac{c_2 t^2}{2}$. Estimator $\hat{c}_2 = \sum_{i=1}^{4} a_i x(t_i)$

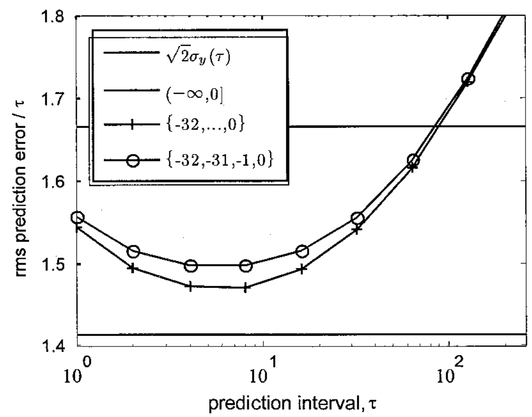
$$\left[\begin{array}{cccc|ccc} s_x(0) & s_x(t_1-t_2) & s_x(t_1-t_3) & s_x(t_1-t_4) & 1 & t_1 & t_1^2 \\ s_x(t_2-t_1) & s_x(0) & s_x(t_2-t_3) & s_x(t_2-t_4) & 1 & t_2 & t_2^2 \\ s_x(t_3-t_1) & s_x(t_3-t_2) & s_x(0) & s_x(t_3-t_4) & 1 & t_3 & t_3^2 \\ s_x(t_4-t_1) & s_x(t_4-t_2) & s_x(t_4-t_3) & s_x(0) & 1 & t_4 & t_4^2 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ t_1 & t_2 & t_3 & t_4 & 0 & 0 & 0 \\ t_1^2 & t_2^2 & t_3^2 & t_4^2 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \hline \theta_0 \\ \theta_1 \\ \theta_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 2! \end{array}\right]$$

$$\begin{bmatrix} R & G^{\mathrm{T}} \\ G & 0 \end{bmatrix} \begin{bmatrix} a \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}$$

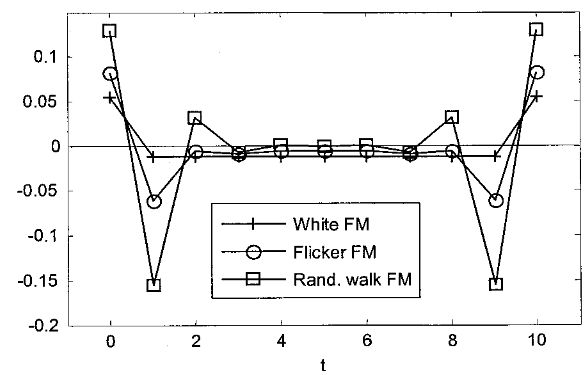
Solve for a, θ.

Mean square error: $\mathrm{E}\left[c_2 - \hat{c}_2\right]^2 = -g^{\mathrm{T}}\theta = -2\theta_2$

Flicker FM: prediction RMSE/τ



Drift estimation coefficients



Conclusions

- Solved two problems of estimation from finite phase data
  - Prediction
  - Trend (slope, drift, aging) estimation
- Clock noise model contained in GACV
- Similar linear-equation formalism for both problems
- Uses
  - Finding good suboptimal estimators based on fewer data
  - Checking other optimal estimation methods (e.g., Kalman filtering)