A Nonlinear Model Predictive Control Algorithm with Proven Robustness and Resolvability

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***Abstract*— A robustly stabilizing MPC (model predictive control) algorithm with guaranteed resolvability is developed for uncertain nonlinear systems. With resolvability, initial feasibility of the finite-horizon optimal control problem implies future feasibility in a receding-horizon framework. The control consists of two components; (i) feedforward, and (ii) feedback. Feedforward control and the associated nominal trajectory are obtained by online solution of a finite-horizon optimal control problem for the nominal system dynamics. The feedback control policy is designed off-line, based on a bound on the model uncertainty. The entire controller is shown to be robustly stabilizing with a region of attraction composed of initial states for which the finite-horizon optimal control problem is feasible. The controller design for this algorithm is demonstrated on a class of systems with uncertain nonlinear terms that have norm-bounded derivatives and derivatives in polytopes. An illustrative numerical example is also provided.**

I. Introduction

Substantial research over the past two decades [1]-[5][1] established MPC as a powerful control method to solve optimal feedback control problems with state and control constraints. In MPC the control input is calculated by solving online a finite-horizon optimal control problem subject to state and control constraints and with the current state of the system as the initial state. The control input signal is obtained and applied to the system over a specified time interval until the next re-computation provides an updated control input, which is then applied to the system and the cycle repeats. Here, resolvability is equivalent to having a feasible solution to the finite-horizon optimal control problem in a future time given a feasible solution at the current time.

Since the computation of the feedforward control input relies on a nominal system model, the robustness of MPC to system uncertainties is non-trivial to establish. One approach to robustness is reducing the re-computation time intervals sufficiently to establish robust stability. Such an approach is taken in [3] in which explicit upper bounds on re-computation time intervals are derived. However, the upper bounds can be very conservative in terms of a real-time implementation. Other approaches proposed in the literature for robustness [see [1] for review of the literature] such as open-loop min-max MPC [6], [7] where the satisfaction of the constraints for all possible system realizations is considered. Another approach is feedback MPC [8], [9] that replaces the design of a control signal with a sequence of control policies, and explicit feedback policies are obtained for MPC of linear systems with state and control constraints in [10], [11]. In this paper we present a robustly stabilizing MPC algorithm for uncertain nonlinear systems that takes advantage of desirable features of both feedforward and feedback-based MPC approaches. For example, see [12] for a coupled open-loop and feedback design for discrete linear-time-invariant systems. The research on guidance and control in the proximity of small celestial bodies, such as asteroids and comets, provided the motivation for this research, which first appeared in [13].

The algorithm presented here generates a control input that has; (i) feedforward (i.e. open-loop control), and (ii) feedback parts. Feedforward control is obtained from the online solution of a finite-horizon optimal control problem for the nominal system dynamics. The feedback control policy is designed off-line based on a bound on the uncertainty in the system model. The cost in the feedforward optimal control problem has an integral part over the time horizon and a cost on the final state. The cost on the final state serves as a control Lyapunov function in the neighborhood of the origin (see [2], [5], [14] for the use of control Lyapunov functions to prove stability for MPC). The addition of feedback ensures that the actual state trajectories do not deviate from the feedforward trajectories beyond a specified bound. This bound is then used to relax the initial state constraint in the finite-horizon optimal control problem to guarantee resolvability without any bound on the re-computation time intervals. This is particularly useful if the online calculation of the feedforward control is computationally intensive.

The MPC controller is shown to be robustly stabilizing within a region of attraction for which the finite-horizon optimal control problem is feasible. Furthermore, the MPC controller has an explicit characterization of its robustness; it can accommodate uncertainties up to the level considered in the feedback policy design. For the theoretical development of the MPC algorithm, it is assumed that there exists such a feedback policy. Therefore, we also present procedures for the design of feedback policies on a class of uncertain nonlinear systems with derivatives in unit balls or polytopes. This class of systems also contains nominally LTI (linear-time-invariant) systems. Nominally LTI systems with convex state and control constraints lead to convex optimization problems for the feedforward control computations that can be solved online with guaranteed convergence [15], [16], [17], [18]. This is a very useful property for applications where autonomous control input generation is needed.

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[1]The MPC literature is very extensive, and we omit many excellent references for brevity.

The organization of the paper is as follows: Section II presents the MPC algorithm and its theoretical development; Section III presents explicit feedback design procedures for systems with bounded derivatives and derivatives in a polytope, as well as a design of feedforward control of nominally LTI systems; and Section IV demonstrates the MPC algorithm on an illustrative example. Proofs of supporting lemmas are relegated to the Appendix.

The following is a partial list of notation used in this paper: $Q = Q^T \geq 0$ implies Q is a positive semi-definite matrix; $Co\{G_1, \ldots, G_N\}$ represents the convex hull of matrices $G_1, \ldots, G_N$; $\mathbb{R}^n$ is the space of n dimensional vectors with real components; $\mathbb{Z}^+$ is the set of positive integers; and, for $V : \mathbb{R}^N \to \mathbb{R}$, $\nabla V = [\, \partial V/\partial x_1 \, \ldots \, \partial V/\partial x_n \,]$.

II. Description of the System and the Model-Predictive-Control Algorithm

Consider an uncertain nonlinear dynamical system with state $x \in \mathbb{R}^n$ and control $u \in \mathbb{R}^m$ as

$$\dot{x} = f(x, u, t). \tag{1}$$

We will refer to dynamics in (1) as the *real dynamics*. Our objective is to obtain a control input $u(\cdot)$ such that the closed loop system for (1) is *asymptotically stable about the origin*, $x = 0$, *with a region of attraction* $\mathcal{R}_a \subseteq \mathbf{X}$ such that

$$x(t) \in \mathbf{X}\,,\; u(t) \in \mathbf{U}\,,\; \forall\, t \geq t_0, \quad \text{when } x(t_0) \in \mathcal{R}_a\,, \tag{2}$$

and where $\mathbf{X} \subseteq \mathbb{R}^n$ and $\mathbf{U} \subseteq \mathbb{R}^m$ are sets defining the state and control constraints, respectively. We propose a model predictive control (MPC) approach where the control input is composed of two components:

- Feedforward control input, u_o,
- Feedback control input, u_f,

such that

$$u(t) = u_o(t) + u_f(t)\,. \tag{3}$$

Suppose there exist sets $\mathbf{X}_o$, $\mathbf{X}_f$ and $\mathbf{U}_o$, $\mathbf{U}_f$, which all contain the origin, such that[2]

$$\begin{aligned} \mathbf{X}_o + \mathbf{X}_f &\subseteq \mathbf{X}\,, \\ \mathbf{U}_o + \mathbf{U}_f &\subseteq \mathbf{U}\,. \end{aligned} \tag{4}$$

The feedforward component of the control input will be determined through online solution of a finite-horizon optimal control problem, and the feedback part will be determined off-line as a control policy.

For feedforward control design, knowledge of the following nominal plant model is assumed:

$$\dot{z} = F(z, u_o, t)\,, \tag{5}$$

where F is an approximate model of f in (1) and $z \in \mathbb{R}^n$ is the state driven by this control through the nominal model dynamics. Model (5) is used to solve the following finite-horizon optimal control problem (FHC):

[2]For sets A and B, $C = A + B$ implies that: If $a \in A$ and $b \in B$ then $a + b \in C$.

FHC

$$\min_{u_o(\cdot)} J(u_o; t_s, T, z(t_s)) \quad \text{where}$$

$$J = \int_{t_s}^{t_s+T} h(z(\tau), u_o(\tau))d\tau + V(z(t_s+T))$$

$$\text{subject to} \begin{cases} \dot{z} = F(z, u_o, t) \\ z(t) \in \mathbf{X}_o\,, \quad \forall\, t \in [t_s,\, t_s+T] \\ u_o(t) \in \mathbf{U}_o\,, \quad \forall\, t \in [t_s,\, t_s+T] \\ z(t_s+T) \in \Omega_o \\ x(t_s) - z(t_s) \in \mathbf{X}_f \end{cases}$$

where $x(t_s)$ is the state of the real system dynamics (1).

The FHC is a typical problem solved in an MPC framework, except for the last constraint given by

$$x(t_s) - z(t_s) \in \mathbf{X}_f\,, \tag{6}$$

which is a relaxation on the initial state of the FHC. This relaxation together with feedback control u_f will provide a theoretical guarantee for the resolvability of the FHC and lead to a robustly stabilizing controller.

The following conditions are instrumental for the stability proof (In Section III we present a class of uncertain systems for which the satisfaction of these conditions is ensured).

Condition 1: Function h in the FHC satisfies

$$h(z,u) \geq a||z||^p + b||u||^r\,, \quad \forall\, z,\, u\,, \tag{7}$$

with $p \geq 1$, $r \geq 0$, a and b both positive constants, and $h(0,0) = 0$. ⋄

Condition 2: Function V in the FHC is positive definite [19] and there exists a feedback control law $u = \mathcal{L}(x)$ and $u_o = \mathcal{L}(z)$ such that V defines a Lyapunov function for (1) and (5) satisfying

$$\nabla V(x) f(x, \mathcal{L}(x), t) + h(x, \mathcal{L}(x)) \leq 0\,,\; \forall\, x \in \Omega_o\,, \tag{8}$$

$$\nabla V(z) F(x, \mathcal{L}(z), t) + h(z, \mathcal{L}(z)) \leq 0\,,\; \forall\, z \in \Omega_o\,, \tag{9}$$

where $\Omega_o \subset \mathbf{X}_o$ contains the origin. Additionally, feedback law $\mathcal{L}$ renders $\Omega_o \subset \mathbb{R}^n$ invariant for dynamics (1) and (5), i.e., if $x(t_0) \in \Omega_o$ ($z(t_0) \in \Omega_o$) for some t_0, then $x(t) \in \Omega_o\,,\; \forall\, t \geq t_0$ ($z(t) \in \Omega_o\,,\; \forall\, t \geq t_0$). It is also assumed that

$$\mathcal{L}(x) \in \mathbf{U}_o\,, \qquad \forall\, x \in \Omega_o\,. \tag{10}$$

⋄

Condition 3: There exist closed balls[3] around the origin $\mathbf{B}_R$ and $\mathbf{B}_r$ in $\mathbb{R}^n$ with radii $R > r$ such that set Ω_o in the FHC satisfies the following

$$\mathbf{X}_f \subseteq \mathbf{B}_r \subset \mathbf{B}_R \subseteq \Omega_o\,. \tag{11}$$

⋄

Condition 4: There exists a feedback control law $u_f = \mathcal{K}(x,z)$ in (3) that renders the set $\mathbf{X}_f$ invariant for $\eta \triangleq x - z$ with dynamics (1) for x and (5) for z, that is, if $\eta(t_0) \in \mathbf{X}_f$ for some $t_0 \geq 0$, then $\eta(t) \in \mathbf{X}_f \;\; \forall\, t \geq t_0$ and for all $u_o(\cdot)$. Additionally, $u_f = \mathcal{K}(x,z) \in \mathbf{U}_f$ if $\eta(t) \in \mathbf{X}_f$. ⋄

The following algorithm describes our MPC approach:

[3]$\mathbf{B}_\rho \triangleq \{v \,:\, \|v\| \leq \rho\}$.

MPC Algorithm

Begin with $k = 0$ and iterate the following steps over computation times t_k for $k \in \mathbb{Z}^+$.

1) Measure the state $x(t_k)$ of the real system (1).
2) Solve the FHC at time $t_s = t_k$ with $T = T_k$ and obtain $u_{o,k}$ with $u_o(t) = u_{o,k}(t)$ on $t \in [t_k, t_k + T_k]$.
3) Apply $u = u_{o,k} + u_f$ to the real system (1) and u_o to nominal system (5) to obtain x and z_k, with $z(t) = z_k(t)$ on $[t_k,\, t_{k+1}]$, where $u_f = \mathcal{K}(x, z)$.
4) If $z(\tilde{t}) \in \Omega_o$ for some $\tilde{t} \geq 0$, then $u_o = \mathcal{L}(z)$, for $t \geq \tilde{t}$.
5) If $x(\bar{t}) \in \Omega_o$ for some $\bar{t} \geq 0$, then $u = \mathcal{L}(x)$, for $t \geq \bar{t}$.

Lemma 1 (Resolvability of the FHC): Suppose that the FHC is feasible at t_0 with T_0, and let t_k for $k \in \mathbb{Z}^+$ be the times that a solution of the FHC is computed. Then, the feasibility of the FHC is guaranteed at t_k with $T_k \geq T_{k-1} - \delta_k$, $\forall k \in \mathbb{Z}^+$, $\delta_k = t_k - t_{k-1}$, $0 \leq \delta_k < T_{k-1}$ provided conditions 2 and 4 hold. $\diamond$

Lemma 2 (Shrinking Optimal Cost with Receding Horizon): Let t_k $(k \in \mathbb{Z}^+)$ be computation times for the FHC satisfying $\inf_k(t_k - t_{k-1}) \geq \epsilon$ for some $\epsilon > 0$. Suppose the FHC is feasible for some t_{k-1} and T_{k-1} with optimal cost J^*_{k-1}, and conditions 1, 2, 3, and 4 hold. Then, the FHC is feasible for t_k with $T_k \in [T_{k-1} - \delta_k,\, T_{k-1}]$, and if $z_{k-1}(t_{k-1}) \notin \Omega_o$ and $z_{k-1}(t_k) \notin \Omega_o$ the optimal cost satisfies

$$J^*_k - J^*_{k-1} \leq -\beta\,, \quad \text{for some} \quad \beta > 0. \tag{12}$$

$\diamond$

Remark 1: The MPC Algorithm can be modified by removing Step 4 and continuing to resolve the FHC even when the nominal trajectory is in Ω_o. In that case, the existence of the local controller in Ω_o still leads to the inequality (39) in the proof of Lemma 2, and we can establish asymptotic stability of the nominal system and resolvability of the FHC by adapting the proof of Theorem 1 in [2]. $\diamond$

The following theorem is the main result of this paper:

Theorem 1: Consider system (1) with a control input described by the MPC Algorithm. Suppose that conditions 1-4 are satisfied. Then, the resulting closed loop system is asymptotically stable with a region of attraction $\mathcal{R}_a$,

$$\mathcal{R}_a = \{\xi \in \mathbb{R}^n : \text{FHC is feasible with } x(t_s) = \xi\}\,. \tag{13}$$

$\diamond$

Proof: Given the MPC Algorithm and $x(t_0)$ such that the FHC is feasible with some T_0, suppose there exists $k \in \mathbb{Z}^+$ such that $z_{k-1}(t_{k-1}) \notin \Omega_o$ and $z_{k-1}(t_k) \notin \Omega_o$. This implies $z_{k-1}(t) \notin \Omega_o$ for $t \in [t_{k-1},\, t_k]$ and equation (12) holds. Consequently, if the nominal trajectory z does not enter Ω_o in finite time, then there exists $k \in \mathbb{Z}^+$ such that $J^*_k < 0$, which is a contradiction. This together with Condition 2 imply the existence of finite time $\tilde{t} \geq 0$ such that $z(t) \in \Omega_o$, $\forall t \geq \tilde{t}$. Further, using Condition 2 once more, the closed loop nominal system (5) converges asymptotically to the origin when $x(t_0) \in \mathcal{R}_a$. Therefore, there exists $\bar{t} \geq \tilde{t} \geq 0$ such that $\|z(t)\| \leq R - r$ for $t \geq \bar{t}$, where $R > r > 0$ are as defined in Condition 3. This leads to

$$\|x(t)\| \leq \|x(t) - z(t)\| + \|z(t)\| \leq r + (R - r) = R\,,\ \forall\, t \geq \bar{t}\,,$$

which implies that

$$x(t) \in \Omega_o\,, \qquad \forall\, t \geq \bar{t}\,.$$

Since we apply Step 5 in MPC Algorithm for $t \geq \bar{t}$, using Condition 2,

$$\lim_{t\to\infty} \|x(t)\| = 0\,.$$

This proves the convergence of $x(t)$ to the origin. The stability of the origin is a direct implication of conditions 2 and 3, which completes the proof. ■

III. MPC for a Class of Systems with Derivatives in Convex Sets

In this section, we will specialize MPC results to the following class of systems,

$$\begin{aligned} \dot{x} &= Ax + Bu + E\phi(t, q) \\ q &= C_q x + D_q u\,, \end{aligned} \tag{14}$$

where $\phi : \mathbb{R} \times \mathbb{R}^{\mathrm{n_q}} \to \mathbb{R}^{\mathrm{n_p}}$ is a continuously differentiable function representing the uncertain nonlinear part of the dynamics, i.e. $f(x, u, t) = Ax + Bu + E\phi(t, q)$ in (1). Since we develop explicit solutions to the existence conditions 1-4 for (14), this is a particularly important class of systems.

The nominal system dynamics is assumed to have the following form,

$$\begin{aligned} \dot{z} &= Az + Bu_o + E\psi(t, q_o) \\ q_o &= C_q z + D_q u_o\,, \end{aligned} \tag{15}$$

where $\psi : \mathbb{R} \times \mathbb{R}^{\mathrm{n_q}} \to \mathbb{R}^{\mathrm{n_p}}$ is an approximation for ϕ in the real system (14), i.e. $F(z, u_o, t) = Az + Bu_o + \psi(t, q_o)$ in (5). Here, the following are assumed:

Condition 5: Functions ϕ and ψ are continuously differentiable and there exists a closed and convex set of matrices $\Theta \subseteq \mathbb{R}^{n_p \times n_q}$ such that

$$\frac{\partial\phi}{\partial q}(t, q) \in \Theta \quad \text{and} \quad \frac{\partial\psi}{\partial q}(t, q) \in \Theta\,, \qquad \forall\, q,\, t\,. \tag{16}$$

$\diamond$

Condition 6: There exists a scalar $\gamma > 0$ such that

$$\|\phi(t, q_o) - \psi(t, q_o)\| \leq \gamma\,, \quad \forall\, t\,,\ z \in \mathbf{X}_o\,,\ u_o \in \mathbf{U}_o\,, \tag{17}$$

where $q_o = C_q z + D_q u_o$ as in 15. $\diamond$

Remark 2: Condition 6 is satisfied when $\mathbf{X}_o$ and $\mathbf{U}_o$ are compact sets, and ϕ is continuous in its arguments and has no dependence on t. For example suppose that there exists a norm bound on the uncertainty, i.e.,

$$\|\phi(t, q_o) - \psi(t, q_o)\| \leq \rho\|q_o\|\,, \quad \forall\, t\,, q_o\,,$$

where $\rho > 0$. Furthermore, there exist positive scalars ρ_1 and ρ_2 such that

$$\|z\| \leq \rho_1 \ \ \forall\, z \in \mathbf{X}_o\,, \quad \text{and} \quad \|u_o\| \leq \rho_2 \ \ \forall\, u_o \in \mathbf{U}_o\,,$$

which satisfies Condition 6 for $\gamma = \rho(\rho_1\|C_q\| + \rho_2\|D_q\|)$. $\diamond$

This class of systems has a subclass that is particularly interesting for MPC applications, namely when $\psi(t,z) = \psi(t)$, which can be viewed as an exogenous input. In this case, the nominal system (15) is an LTI system. Since the dynamics of the nominal system give a set of equality constraints for the finite-horizon optimal control problem, having an LTI nominal system is useful when all other state and control constraints define a convex feasible domain.[4] Then, the optimal control problem becomes a convex optimization problem, which can be numerically solved reliably and autonomously in real-time by using interior point methods [16]. Therefore, analysis of systems of the form (14) with LTI nominal plants is useful for real-time autonomous control.

The dynamics between the real state and the nominal state, $\eta \triangleq x - z$, are called "*error dynamics*" and are given by

$$\dot{\eta} = A\eta + Bu_f + E\left[\phi(t,q) - \psi(t,q_o)\right], \tag{18}$$

where $u_f = u - u_o$. This equation is then rewritten as,

$$\dot{\eta} = A\eta + Bu_f + E\left[\phi(t,q) - \phi(t,q_o)\right] + E\left[\phi(t,q_o) - \psi(t,q_o)\right]. \tag{19}$$

The following lemma (see [20] for a proof), which is a generalization of mean value theorem, is used to obtain a linear differential inclusion (LDI) [21] for the error dynamics in (19).

Lemma 3: Consider a continuously differentiable function $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ with its Jacobian given by $\partial\varphi(q)/\partial q$. Suppose that there exists a closed convex set $\Lambda \in \mathbb{R}^{n\times m}$ such that

$$\frac{\partial\varphi}{\partial q}(q) \in \Lambda, \qquad \forall q.$$

Then, for every q_1 and q_2 there exists $\Delta \in \Lambda$ such that

$$\varphi(q_2) - \varphi(q_1) = \Delta(q_2 - q_1).$$

$\diamond$

Equation (19) can be written as

$$\dot{\eta} = A\eta + Bu_f + E[\pi(t,\eta,u_f) + w(t,z,u_o)], \tag{20}$$

where $\pi(t,\eta,u_f) = \phi(t, C_q x + D_q u) - \phi(t, C_q z + D_q u_o)$, and $w(t,z,u_o) = \phi(t, C_q z + D_q u_o) - \psi(t, C_q z + D_q u_o)$, where w is assumed to satisfy

$$\|w(t,z,u_o)\| \le \gamma, \quad \forall\, z \in \mathbf{x}_o,\ u_o \in \mathbf{U}_o,\ t \ge 0 \tag{21}$$

by Condition 6. Applying Lemma 3 with Condition 5,

$$\pi(t,\eta,u_f) = \theta(t)(C_q\eta + D_q u_f), \quad \text{where} \quad \theta(t) \in \Theta,\ \forall t. \tag{22}$$

This description of the error dynamics is particularly useful to obtain feedback laws that satisfy Condition 4 for a class of uncertain nonlinear systems. Here, we consider two well known classes where Jacobian matrices are either norm bounded or are in polytopes, and we give the corresponding feedback results.

The following condition gives a polytopic description of the state and an ellipsoidal description of the control constraint sets in the FHC. Note that more general convex characterizations of the constraint sets are also possible and can easily be integrated into the design framework.

[4] All equality constraints in a convex optimization problem must be linear equalities.

Condition 7: The following hold for the constraint sets in the FHC,

$$\begin{aligned}
\mathbf{X}_o &= \{x \in \mathbb{R}^n : a_i^T x \le 1,\ i = 1,\ldots,m_o\},\\
\mathbf{X}_f &= \{x \in \mathbb{R}^n : b_i^T x \le 1,\ i = 1,\ldots,m_f\},\\
\mathbf{U}_o &= \{u \in \mathbb{R}^m : u^T \Pi_o u \le 1\},\\
\mathbf{U}_f &= \{u \in \mathbb{R}^m : u^T \Pi_f u \le 1\},
\end{aligned} \tag{23}$$

where Π_o, Π_f are symmetric positive-definite matrices. $\diamond$

The following is a corollary of Theorem 1 that describes a design procedure for systems with norm-bounded derivatives.

Corollary 1: Consider an uncertain nonlinear system (14) with a nominal model given by (15) satisfying conditions 5, 6, and 7 with

$$\Theta = \{\theta \in \mathbb{R}^{n_p \times n_q} : \|\theta\| \le 1\}. \tag{24}$$

Suppose that there exist matrices $S = S^T > 0$, $Q = Q^T > 0$, L, Y and positive scalars λ, β, μ, c_1, and c_2 satisfying the following matrix inequalities,

$$\begin{bmatrix} \begin{array}{c} SA^T + AS + BL + L^T B^T + S/\lambda \\ +(\beta + \lambda\gamma^2)EE^T \end{array} & SC_q^T + L^T D_q^T \\ C_q S + D_q L & -\beta I \end{bmatrix} \le 0 \tag{25}$$

$$\begin{bmatrix} \begin{pmatrix} QA^T + AQ + BY \\ +Y^T B^T + \mu EE^T \end{pmatrix} & QC^T + Y^T D^T & QC_q^T + Y^T D_q^T \\ CQ + DY & -I & 0 \\ C_q Q + D_q Y & 0 & -\mu I \end{bmatrix} \le 0 \tag{26}$$

$$\begin{bmatrix} S & L^T \\ L & \Pi_f^{-1} \end{bmatrix} \ge 0, \quad \begin{bmatrix} Q & Y^T \\ Y & \Pi_o^{-1} \end{bmatrix} \ge 0, \tag{27}$$

$$a_i^T Q a_i \le 1, \quad i = 1,\ldots,m_o, \tag{28}$$

$$b_i^T S b_i \le 1, \quad i = 1,\ldots,m_f, \tag{29}$$

$$Q \ge c_1 I > c_2 I \ge S, \tag{30}$$

where C and D are matrices satisfying

$$C^T D = 0.$$

Then, the ellipsoids $\varepsilon_Q = \{x : x^T Q^{-1} x \le 1\}$ and $\varepsilon_S = \{x : x^T S^{-1} x \le 1\}$ satisfy $\varepsilon_Q \subseteq \mathbf{X}_o$ and $\varepsilon_S \subseteq \mathbf{X}_f$. Then, the MPC Algorithm with

$$\begin{aligned} h(x,u) &= \|Cx\|^2 + \|Du\|^2, \\ V(x) &= x^T Q^{-1} x, \end{aligned} \tag{31}$$

$$\begin{aligned} \mathcal{L}(x) &= Kx, & K &= YQ^{-1}, \\ \mathcal{K}(x,z) &= K_f(x-z), & K_f &= LS^{-1}, \end{aligned} \tag{32}$$

and invariant ellipsoids ε_Q and ε_S replacing Ω_o and $\mathbf{X}_f$, respectively, results in an asymptotically stable closed loop system for (14) with region of attraction $\mathcal{R}_a$ given in (13). $\diamond$

The following corollary establishes the results of Corollary 1 for systems with uncertain nonlinear terms that have derivatives contained in polytopes.

Corollary 2: Consider a uncertain nonlinear system (14) with a nominal model given by (15) satisfying all the assumptions of Corollary 1 with the following modification: For (24), assume that there exists a set of matrices $\Sigma_1, \ldots, \Sigma_N$ such that

$$\Theta = \left\{\theta \in \mathbb{R}^{n_p \times n_q} \ : \ \theta \in Co\left\{\Sigma_1, \ldots, \Sigma_N\right\}\right\} . \qquad (33)$$

Then, all the conclusions of Corollary 1 hold if inequalities (25) and (26) are replaced by the following, for $i = 1, \ldots, N$

$$A_i S + S A_i^T + B_i L + L^T B_i^T + S/\lambda + (\beta + \lambda\gamma^2) E E^T \leq 0, \quad (34)$$

$$\begin{bmatrix} A_i Q + Q A_i^T + B_i Y + Y^T B_i^T & Q C^T + Y^T D^T \\ C Q + D Y & -I \end{bmatrix} \leq 0, \quad (35)$$

where $A_i = A + E\Sigma_i C_q\,,\ B_i = B + E\Sigma_i D_q\,.$ $\diamond$

Remark 3: All the matrix inequalities given in Corollary 1 and 2 are LMIs (linear matrix inequalities) except (25), (26), (34), and (35). But, these are also LMIs for a given $\lambda > 0$. Therefore, a simple line search on λ can be applied to solve the system of matrix inequalities. $\diamond$

Remark 4: The proofs of Corollaries 1 and 2 come from establishing invariant ellipsoids around both the nominal trajectory and the origin by using LMIs [22], [21]. $\diamond$

A. Computation of Feedforward Control for Nominally LTI Systems

In this section, we describe a methodology to calculate the feedforward control for nominally LTI systems. In this case, ψ in (15) is only a function of time t but not q_o. Additionally, $\mathbf{X}_o$ and $\mathbf{U}_o$ are assumed to be convex sets. The feedforward control can be parameterized by a zero-order-hold approach, i.e., u_o is piecewise constant on time intervals of fixed length δt. The resulting sampled-data system can then be written as,

$$z_{j+1} = A_d z_j + B_d u_{o,j} + \xi_j\,, \qquad (36)$$

where

$$A_d = e^{A\delta t}, \ \ B_d = \int_0^{\delta t} e^{A(\delta t - \tau)} B d\tau, \ \ \xi_j = \int_{t_j}^{t_{j+1}} e^{A(\delta t - \tau)} E\psi(t) d\tau.$$

This allows us to approximate the integral part of the cost J in the FHC with a finite sum,

$$\int_{t_s}^{t_s+T} h(z(\tau), u_o(\tau)) d\tau \approx \delta t \sum_{j=0}^{N} e_j\, h(z_j, u_{o,j})\,,$$

where e_j results from the specific numerical integration technique used, and $N = T/\delta t$. Then, the state and control constraints are only imposed at temporal nodes, i.e.

$$z_j \in \mathbf{X}_o, \ u_j \in \mathbf{U}_o, \qquad j = 0, \ldots, N.$$

If $\mathbf{X}_o$ and $\mathbf{U}_o$ are sets that can be described by semidefinite constraints, such as linear, quadratic, or conic inequalities [15], then the FHC is approximated by a finite-dimensional parameter optimization problem. Specifically, it becomes a semidefinite programming problem (SDP), which can be solved in polynomial time. There exist algorithms and software [16], [18], [17] that compute the global optimum with a deterministic stopping criteria, and with a prescribed level of accuracy. Therefore, they are very well-suited for real-time, onboard computations.

IV. An Illustrative Example

In this section, we present an example illustrating the MPC algorithm. In this example, a “standard” approach to MPC, without the relaxation of the initial state constraint in the feedforward problem (6) and without the feedback, is shown to fail. The same problem is then solved successfully by the MPC algorithm. The dynamics of the system are given by,

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ -0.1 \end{bmatrix} \omega \sin^2(C_q x) \quad (37) \\ C_q &= \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad x = \begin{bmatrix} q & \dot{q} \end{bmatrix}^T \\ \omega &\in [0,\ 0.5]\,. \end{aligned}$$

Here ω is an uncertain parameter. We assume that the nominal system is the linear part of (37), that is $\psi(t, q) = 0$ and $\phi(t, q) = \omega \sin^2 q$ in (15) and (14), respectively. This implies that $\|\partial\phi/\partial q\| \leq 1$, and Condition 5 is satisfied with Θ as in (24). The state constraints are given by,

$$-0.25 \leq x_1 \leq 5 \quad \text{and} \quad -1 \leq x_2 \leq 2$$

and the control constraint is $|u| \leq 1.4$. For the MPC algorithm, we partition the control constraint into

$$|u_o| \leq 1.2 \quad \text{and} \quad |u_f| \leq 0.2\,.$$

The integral cost function $h(z, u) = \|Cx\|^2 + \|Du\|^2$ is determined by,

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Other design parameters needed in Corollary 1 are $\gamma = 0.5$, $\Pi_o = (1/1.2^2)I$, $\Pi_f = (1/0.2^2)I$, and

$$\begin{aligned} a_1 &= [1/4.95 \ \ 0]\,, & b_1 &= [20 \ \ 0] \\ a_2 &= [-5 \ \ 0]\,, & b_2 &= [-20 \ \ 0] \\ a_3 &= [0 \ \ 1/1.9]\,, & b_3 &= [0 \ \ 10] \\ a_4 &= [0 \ \ -1/0.9]\,, & b_4 &= [0 \ \ -10]\,. \end{aligned}$$

Then, the values of the solution variables obtained by solving the LMIs (SDPT3 [18] is used to generate the numerical solutions) in Corollary 1 are

$$K_o = \begin{bmatrix} -5.6850 & -2.0990 \end{bmatrix}, \quad K_f = \begin{bmatrix} -3.4747 & -2.6140 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0.0400 & -0.0480 \\ -0.0480 & 0.2935 \end{bmatrix}, \quad S = \begin{bmatrix} 0.0018 & -0.0017 \\ -0.0017 & 0.0059 \end{bmatrix},$$

and the associated invariant ellipsoids ε_Q and ε_S are shown in Figure 1.

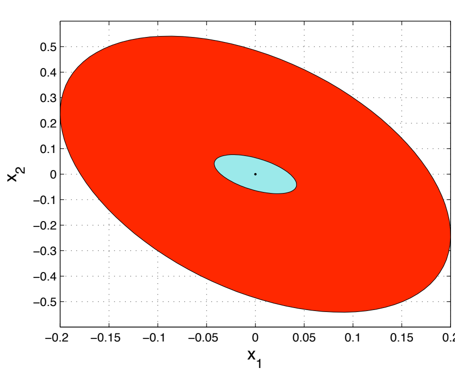

Fig. 1. Invariant Ellipsoids $\varepsilon_S \subset \varepsilon_Q$ for the MPC Algorithm

Figure 2 presents a simulation for a typical MPC implementation, where $\omega = 0.3$. In the simulation, the finite-horizon optimal control problem is solved by applying a zero-order-hold discretization with a time increment $\delta t = 0.5$ seconds, and constraints are guaranteed at the temporal nodes. The feedforward solution is recomputed every 10 seconds with a finite time horizon of 30 seconds. The trajectory enters the invariant set Ω_o (the ellipsoid around the origin in the plot) and asymptotically converges to the origin. However, the state constraint on x_1 is violated, as indicated by the state crossing a solid line at $x_1 = -0.25$.

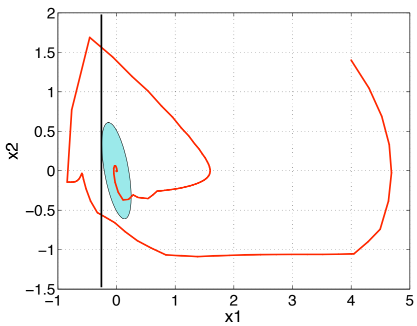

Fig. 2. State Trajectory with Standard MPC

The same simulation is repeated with the MPC algorithm of this paper (figure 3), and the trajectory asymptotically converges to the origin without violating any state or control constraints. Figure 4 depicts the associated feedforward and feedback components of the control, which are within set control constraints. Additionally, figure 5 shows the error between the nominal and real trajectories also goes to zero.

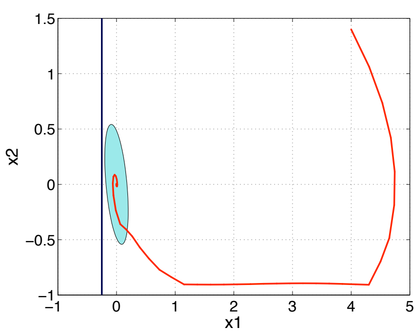

Fig. 3. State Trajectory with the MPC Algorithm

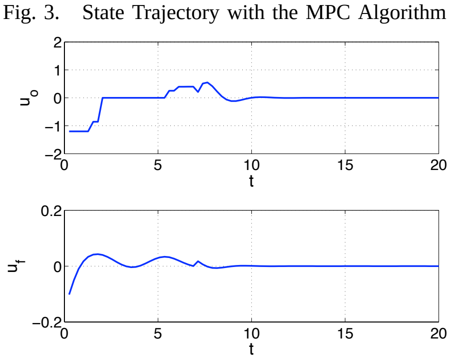

Fig. 4. Feedforward and feedback control as a function of time

The time interval between two computations can be reduced to make "standard" MPC give feasible state trajectories, which is observed by reducing the computation time interval to 5 seconds in this example (results not shown);

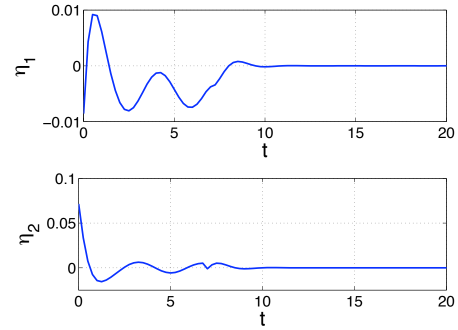

Fig. 5. State error η as a function of time

however, this reduction cannot always be pre-determined, and the required reduction may be computationally too demanding for a real-time implementation for a complex system. Therefore, using the MPC algorithm of this paper can be especially useful when explicit guarantees are needed for feasibility once an initial feasible solution is computed.

V. Conclusions

We presented a general MPC algorithm that differs from earlier counterparts in terms of using separate feedforward and feedback control components, along with a relaxation of the initial state constraint for the solution of the finite-horizon optimal control problem at each computation. The relaxation makes possible the off-line design of the feedback control policy and guarantees resolvability and asymptotic stability once an initial feasible solution is obtained for the finite-horizon optimal control problem. This MPC algorithm is robust to system uncertainties that are accounted for explicitly in the feedback control design; this characterization, which can easily be extended to external disturbances, is particularly desirable for real-time autonomous control applications. Design procedures are also given for a particular class of uncertain nonlinear systems, along with an illustrative example to demonstrate the algorithm.

Appendix

Proof for Lemma 1 (Resolvability of the FHC): Suppose at t_{k-1} the FHC is feasible with T_{k-1} and $u_{o,k-1}(t)$ for $t \in [t_{k-1}, t_{k-1} + T_{k-1}]$, and $z_{k-1}(\cdot)$ is the corresponding state trajectory of (5). Let $t_k = t_{k-1} + \delta_k$ and resolve the FHC. Note, $u_{o,k}(t) = u_{o,k-1}(t)$ for $t \in [t_k,\, t_{k-1} + T_{k-1}]$ is one feasible solution of the FHC with $T_k = T_{k-1} - \delta_k$ since $x(t_k) - z_{k-1}(t_k) \in \mathbf{X}_f$ is invariant due to Condition 4 (i.e. $z_k(t_k) = z_{k-1}(t_k)$ is the initial state of a feasible trajectory).

Now, we show that we can extend this feasible trajectory on $[t_k,\, t_{k-1} + T_{k-1}]$ to $[t_k,\, t_k + T_k]$ for any given $T_k \geq T_{k-1} - \delta_k$ by considering the following control input,

$$u_{o,k}(t) = \begin{cases} u_{o,k-1}(t), & t \in [t_k,\, t_{k-1} + T_{k-1}]; \\ u_{o,k}(t) = \mathcal{L}(z(t)), & t \in [t_{k-1} + T_{k-1},\, t_k + T_k]. \end{cases} \tag{38}$$

which follows from Condition 2 by noting that $z_k(t_{k-1} + T_{k-1}) = z_{k-1}(t_{k-1} + T_{k-1}) \in \Omega_o$. Consequently (38) defines a feasible trajectory on $[t_k,\, t_k + T_k]$, implying the FHC has a feasible solution at t_k with time horizon T_k, once it is feasible at t_{k-1} with T_{k-1}. Now, we can conclude the proof by using induction. ∎

Proof of Lemma 2 (Shrinking Optimal Cost with Receding Horizon): Since the FHC is feasible at t_{k-1} with T_{k-1} and $u_{o,k-1}(\cdot)$ provides the optimal cost J^*_{k-1}, $u_{o,k-1}(\cdot)$ can also be used to provide a feasible solution for the FHC at t_k with $T_k \in [T_{k-1} - \delta_k,\, T_{k-1}]$ by using (38) as in the proof of Lemma 1. So, $z_k(t) = z_{k-1}(t)$ is a feasible trajectory for $t \in [t_k, t_{k-1} + T_{k-1}]$. We will show that (12) is satisfied with $T_k = T_{k-1}$ which will directly imply that (12) is satisfied with $T_k \in [T_{k-1} - \delta_k,\, T_{k-1}]$ by construction of the proof.

The cost with control input (38) and $T_k = T_{k-1}$ is

$$J_k = \int_{t_k}^{t_{k-1}+T_{k-1}} h(z_{k-1}(\tau), u_{o,k-1}(\tau))d\tau + \int_{t_{k-1}+T_{k-1}}^{t_k+T_{k-1}} h(z_k(\tau), u_{o,k}(\tau))d\tau + V(z_k(t_k + T_{k-1}))\,.$$

Since

$$J^*_{k-1} = \int_{t_{k-1}}^{t_k} h(z_{k-1}(\tau), u_{o,k-1}(\tau))d\tau + \int_{t_k}^{t_{k-1}+T_{k-1}} h(z_{k-1}(\tau), u_{o,k-1}(\tau))d\tau + V(z_{k-1}(t_{k-1} + T_{k-1})),$$

we have $J_k - J^*_{k-1} =$

$$\int_{t_{k-1}+T_{k-1}}^{t_k+T_{k-1}} h(z_k(\tau), u_{o,k}(\tau))d\tau - \int_{t_{k-1}}^{t_k} h(z_{k-1}(\tau), u_{o,k-1}(\tau))d\tau + V(z_k(t_k + T_{k-1})) - V(\underbrace{z_{k-1}(t_{k-1} + T_{k-1})}_{=z_k(t_{k-1}+T_{k-1})})\,.$$

Condition 2 implies the following with $u_{o,k}(t) = \mathcal{L}(z_k(t))$ on $t \in [t_{k-1} + T_{k-1},\, t_k + T_{k-1}]$:

$$\int_{t_{k-1}+T_{k-1}}^{t_k+T_{k-1}} \dot{V}(z_k(\tau))d\tau + \int_{t_{k-1}+T_{k-1}}^{t_k+T_{k-1}} h(z_k(\tau), u_{o,k}(\tau))d\tau \le 0, \text{ and}$$

$$V(z_k(t_k + T_{k-1})) - V(z_k(t_{k-1} + T_{k-1})) + \int_{t_{k-1}+T_{k-1}}^{t_k+T_{k-1}} h(z_k(\tau), u_{o,k}(\tau))d\tau \le 0.$$

This implies that

$$J_k - J^*_{k-1} \le -\int_{t_{k-1}}^{t_k} h(z_{k-1}(\tau), u_{o,k-1}(\tau))d\tau\,. \tag{39}$$

Given conditions 1 and 3, if $||z|| \ge R$, then $h(z, u_o) \ge a||z||^p \ge aR^p$ since $p \ge 1$. Since $\delta_k \ge \epsilon > 0$, then for $t \in [t_{k-1}, t_k]$,

$$\int_{t_{k-1}}^{t_k} h(z_{k-1}(\tau), u_{o,k-1}(\tau))d\tau \ge \underbrace{aR^p\epsilon}_{\beta} > 0. \tag{40}$$

Combining inequalities (39) and (40) shows that $J_k - J^*_{k-1} \le -\beta < 0$, and since $J^*_k \le J_k$, then

$$J^*_k - J^*_{k-1} \le -\beta < 0. \qquad \blacksquare$$

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