Second-order multiple-scattering theory associated with backscattering enhancement for a millimeter wavelength weather radar with a finite beam width

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[1] Effects of multiple scattering on reflectivity are studied for millimeter wavelength weather radars. A time-independent vector theory, including up to second-order scattering, is derived for a single layer of hydrometeors of a uniform density and a uniform diameter. In this theory, spherical waves with a Gaussian antenna pattern are used to calculate ladder and cross terms in the analytical scattering theory. The former terms represent the conventional multiple scattering, while the latter terms cause backscattering enhancement in both the copolarized and cross-polarized components. As the optical thickness of the hydrometeor layer increases, the differences from the conventional plane wave theory become more significant, and essentially, the reflectivity of multiple scattering depends on the ratio of mean free path to radar footprint radius. These results must be taken into account when analyzing radar reflectivity for use in remote sensing.

1. Introduction

[2] Millimeter wavelength weather radars have been extensively used to increase accuracy of measuring hydrometeor characteristics (e.g., raindrops, liquid cloud particles). In this frequency regime, multiple-scattering effects become important so as to be taken into account when using radar reflective intensity in retrieval algorithms of hydrometeor density. The occurrence of multiple scatterings was confirmed in 35 GHz radar measurements by the presence of depolarized signals reflected from spherical raindrops [Ito et al., 1995; Iguchi et al., 1992]. Ito et al. [1995] proposed an approximate analytical formula to calculate copolarized and depolarized intensities caused by second-order scattering in a single layer of hydrometeors for plane wave incidence. Since this formula is derived from the expansion of generalized spherical harmonics based on the time-dependent radiative transfer theory [Oguchi, 1980], it is particularly useful to the analysis with a short pulsed weather radar in remote sensing.

[3] From the early 1970s to the early 1990s, multiple scattering in randomly distributed particles was intensively studied through the analytical method of electromagnetic wave [de Wolf, 1971; Golubentsev, 1984; Tsang and Ishimaru, 1985; Kravtsov and Saichev, 1982]. In the course of study, two main contributions of multiple scattering to reflective intensity were revealed; one is the conventional multiple scattering called ladder term, and the other called cross term, is the contribution from interference of two ray paths mutually satisfying the condition of time reversal paths, as will be addressed in this paper. The contribution of a cross term is negligible to that of the corresponding ladder term unless the transmitting and receiving directions satisfy the condition of nearly backscattering. Once this condition is satisfied, as is always the case of ground- based monostatic radars, the cross term becomes comparable to the ladder term, resulting in backscattering enhancement. However, for the case of spaceborne radars, we must not always expect the backscattering enhancement because the platform motion at a high speed introduces a significant displacement, breaking the right backscattering condition. If the scattering angle is far from backscattering, it is sufficient to consider only the ladder term. As Barbanenko et al. [1991] and Ishimaru [1991] reviewed, the backscattering enhancement is a universal wave phenomenon observed whenever the multiple scattering is substantial. Therefore it is not appropriate to use the second-order scattering formula of Ito et al. [1995]
for monostatic radars without evaluating the effect of backscattering enhancement.

[4] An application of backscattering enhancement to weather radars was studied for turbulent media by Lure et al. [1989]. However, for millimeter wavelength radars, de Wolf et al. [2000] and Kobayashi [2002] showed that the incoherent scattering from cloud particles carried by the turbulent air cannot be observed except for extreme case. Hence in this paper, only the scattering from each particle, namely delta function–like singular incoherent component will be considered. For this scattering component, backscattering enhancement was studied as a scalar theory [Ishimaru and Tsang, 1988; Akkermans et al., 1986; Barbanenkov and Ozrin, 1988; Golubentsev, 1984; Tsang and Ishimaru, 1984, 1985] and as a vector theory [Mandt et al., 1990; Mandt and Tsang, 1992; Akkermans et al., 1988; Stephen and Cwilich, 1986; Kuzmin et al., 1992; Mishchenko, 1991, 1992]. Numerical simulations for the vector theory were found in references [Oguchi and Ibara, 2005; van Albada and Lagendijk, 1987]. Among these vector theories, the perturbation theories [Mandt et al., 1990; Mandt and Tsang, 1992; Kuzmin et al., 1992; Mishchenko, 1991, 1992] can be considered more appropriate for hydrometeors because of its small volume fraction of scatterers of order of $10^{-5}$, comparing with the diffusion theories [Akkermans et al., 1988; Stephen and Cwilich, 1986]. Especially, two formalisms of Mishchenko [1991, 1992] and Mandt et al. [1990] are advantageous so as to have the explicit forms of scattering amplitude matrix in position space representation. The former formalism includes the contributions of ladder and cross terms of all orders, and seems to suit for numerical simulation, but not for analytical expression. The latter, on the other hand, includes at most the second-order terms, but it can give an analytically simple form for a system of a finite layer thickness. Furthermore the second-order theory can be considered to be sufficient for a dilute system such as hydrometeors as mentioned in references [Ito et al., 1995; Iguchi et al., 1992].

[5] In all the previous theoretical works, a plane wave is incident to a layer of randomly distributed particles, and the reflected wave is collected by a receiver at infinite range. On the other hand, in remote sensing, a spherical wave with a finite beam width, usually approximated as a Gaussian antenna pattern within the antenna mainlobe, is incident, and the reflected wave is received by an antenna at a finite range. For the single scattering, the plane wave theory can be applied to the spherical wave of a finite beam width with a slight correction on range and gain, as will be also shown in this paper. On the other hand, for multiple scattering, it is not appropriate to adopt the plane wave theory as it is, because a finite footprint radius can be considered to give a smaller reflectivity than the plane wave theory predicts, especially when the footprint radius is smaller than the mean free path of an illuminated body. The mean free path in a layer of hydrometeors can reach 1000–2000 m for millimeter wavelength wave, while a typical footprint radius is on order of hundreds meters for airborne applications, and a few kilometers for spaceborne cases. In this study, a time-independent multiple-scattering theory is formulated for a spherical wave along with a Gaussian antenna pattern, based on the plane wave theory of Mandt et al. [1990]. It means that the formalism of this paper can be applied to the stationary process such as CW radars (not FM-CW radars), but not to pulsed radars except for the special case described in section A3. To estimate amounts of the multiple scattering for pulsed radars in the general case, it is necessary to develop time-dependent algorithms as a future work, which are briefly discussed in section 4. Our analysis considers only a single layer of spherical water particles of uniform diameter and uniform number density. For simplicity of theoretical derivation, particles are assumed to be stationary in air throughout this paper. The limit of this last assumption will be discussed in section 3.

[6] The main purpose of this paper is to illustrate that reflectivity in multiple scattering for a finite beam width is smaller than the value predicted by the plane wave theory. The result will form the basis of a future retrieval algorithm of hydrometeors that takes a drop size distribution into consideration.

2. Formalism

[7] Ishimaru and Tsang [1988] derived a second-order multiple-scattering theory for anisotropic scatterers through the analytic method of electromagnetic wave approximated in scalar. A plane wave with infinite duration was used to lead a time-independent theory in which the thickness “d” of a layer is an adjustable parameter. Mandt et al. [1990] later expanded this scalar theory to a vector counterpart by introducing the three-dimensional characteristics of electromagnetic waves. In this section, the formalism of Ishimaru-Tsang and Mandt et al. is expanded to a spherical wave with a finite beam width represented by a Gaussian function. To deal with complication introduced by the finite beam width, further simplifications of integrals will be performed for the second-order ladder and cross terms as will be shown in this section and section A1.

2.1. First-Order Ladder Term

[8] As briefly mentioned in section 1, all the ladder terms are incorporated into radiative transfer theory. The first-order ladder term, corresponding to the conventional single scattering intensity, is depicted schematically in Figures 1a and 1b for a radar with finite beam width. To
illustrate the backscattering enhancement, at first a bistatic radar is considered by assuming a small value of scattering angle \( q_s \ll 1^\circ \) as shown in Figures 1a and b, and then the formula for monostatic radar is derived by taking the limit of \( q_s \rightarrow 0 \). In Figure 1a, the point O is the center of the footprint at the top boundary of the random medium layer. The unit vector \( \hat{k}_i \) denotes the direction of the incident wave along the transmitting antenna axis TO, and the transmitting range \( r_i \) is the length of TO. When the incident spherical wave is scattered by a point scatterer at point \( r' \) in the medium, the field at point \( r' \) can be represented for a large range \( r_r \) and a small 3 dB aperture angle \( q_d \ll 1^\circ \) in the form of

\[
\psi(r') \approx \sqrt{P_t G_0} \frac{1}{4\pi r_i^{-1}} \exp[ikr_i] \cdot \exp[i\kappa_r r'] \exp[-r_{r,\perp}^2/4\sigma_r^2] \psi_0
\]

In equation (1), \( P_t, G_0 \) and \( k \) denote the total transmitting power, the center gain of the antenna, and the wave number in air, respectively. \( r_r \) denotes the transverse length of the point \( r_0 \) as shown in Figure 1. The 3 dB footprint radius \( \sigma_r \) can be defined as

\[
\sigma_r^2 = r_i^2 \frac{\theta_d^2}{2^3 \ln 2}
\]

The vector \( \psi_0 \) in equation (1) represents an initial polarization state. The effective incident wave number \( k_i \) in the medium can be represented, based on work by Ishimaru and Tsang [1988]:

\[
k_i \approx k \hat{k}_i - i\kappa_{iz} \hat{z}
\]

The incident direction \( \hat{k}_i \) can be defined with polar coordinates as

\[
\hat{k}_i = \sin \theta_i \cos \varphi_i \hat{x} + \sin \theta_i \sin \varphi_i \hat{y} + \cos \theta_i \hat{z}
\]

in which the base vector \( \hat{z} \) in the Cartesian system is taken in the zenith. The directions of \( \hat{x} \) and \( \hat{y} \) are arbitrary in the orthogonal plane to \( \hat{z} \). In equation (3), the imaginary part \( \kappa_{iz} \) is represented in the form of

\[
\kappa_{iz} = -\kappa_e/2 \cos \theta_i
\]

Note that the negative signs in equations (3) and (5) have been chosen because of \( \theta_i \approx \pi \). The extinction rate \( \kappa_e \) in equation (5) is defined by the Foldy-Twersky-Oguchi formula [Oguchi, 1973; Tsang and Kong, 2001]:

\[
\kappa_e = \text{Im} \left[ 4\pi N_0 k^{-1} F(k_i, \hat{k}_i) \right]
\]

in which \( N_0 \) is the number density of hydrometeors. Since \( F(k_s, \hat{k}_i) \) denotes the scattering amplitude matrix scattered from the directions \( k_s \) to \( \hat{k}_i \), \( F(k_s, \hat{k}_i) \) in equation (6) means the forward scattering amplitude matrix.
negligible so that the above approximation is satisfied. However, the difference in gain is generally considerable and its magnitude can be approximated by the bistatic radar configuration depicted in Figure 1b, because the antenna moves by a large amount during the round trip time of pulse. The corresponding angle shift $\theta_s$ is approximately equal to 0.0025°. In rigor, we have to note that, for the spaceborne radar, the antenna gain at the receiving position changes from that at the transmitting position. However, this difference in gain is generally negligible, so that the above approximation is satisfied. For a small scattering angle $\theta_s \ll 1°$ along with the condition for the receiving range $r_s(\gg |r'|)$, the Green function of the signal received by the antenna R, located at point $r$, can be approximated in the form of

$$G_{01}(r, r') \approx \sqrt{\pi G_0/k^2 r_s^{-1} \exp[ikr_s]}$$

$$\cdot \exp[-i\kappa_s r'] \exp[-r_s^2/4\sigma^2] \bar{I}_2$$

(7)

in which the two-dimensional identical operator/dyad is defined with bra-ket notations in quantum mechanics as

$$\bar{I}_2 = |\hat{\theta}| \langle \hat{\theta} | + |\hat{\varphi}| \langle \hat{\varphi}|$$

(8)

In equation (8), the unit vectors in the polar and azimuth directions are represented by $\hat{\theta}$ and $\hat{\varphi}$, respectively. The effective scattered wave number $\kappa_s$ in the medium can be defined in a similar manner to equations (3) and (5):

$$\kappa_s \approx k \hat{k}_s + i\kappa_{sz}^s \hat{z}$$

(9)

with

$$\hat{k}_s = \sin \theta_s \cos \varphi_s \hat{x} + \sin \theta_s \sin \varphi_s \hat{y} + \cos \theta \hat{z}$$

(10)

$$\kappa_{sz}^s = \kappa_{e}^s / 2\cos \theta_s$$

(11)

The scattering process in Figures 1a and 1b is represented by the Dyson's diagram in Figure 1c. For completely random distributions of a uniform density $N_0$ (i.e., no particle-particle correlation), the first-order ladder term intensity can be represented via the diagram of Figure 1c in the integral form:

$$I_L^{(1)} = (4\pi)^2 N_0 \int_{-d}^{0} dz \int d|r'| \psi^*(r') F(\hat{k}_s, \hat{k}_i)$$

$$\cdot \bar{G}_{01}(r', r') \bar{G}_{01}(r, r') F(\hat{k}_i, \hat{k}_i) \psi(r')$$

(12)

in which the superscript $\dagger$ indicates the complex conjugate of a dyad or a vector.

Figure 2. Second-order ladder term. An incident field $E_A$ and its identical conjugate field $E_B$ emit from antenna T and scatter at points b and a successively in the random medium, being received by antenna R. (a) Diagram in which $G_{11}$ is a Green function in the medium. (b) Geometrical configuration.

[9] Substitutions of equations (1) and (7) into equation (12) yield for nearly backscattering condition,

$$I_L^{(1)} \approx P_s G_0^2 \lambda^2 \theta_0^2 (2^7 \pi \ln 2 r_s^2)^{-1} N_0 \left\{ 2\left( \kappa_{iz}^s + \kappa_{sz}^s \right) \right\}^{-1}$$

$$\cdot \sum_{\alpha} \left[ \left| \langle \hat{\alpha} | F(-\hat{k}_i, \hat{k}_i) | \psi_0 \rangle \right|^2$$

$$\cdot \left\{ 1 - \exp \left[ -2\left( \kappa_{iz}^s + \kappa_{sz}^s \right) d \right] \right\}$$

(13)

where the approximation of $r_s \approx r_i$ and $\hat{k}_s \approx -\hat{k}_i$ have been used. In equation (13), the summation over unit vector $\hat{\alpha}$ is taken over a complete set of polarizations such as the $h$ and $v$ directions. When the initial polarization is in the $h$ direction and the absorption is negligible, equation (13) projected onto the $v$-polarized component can be written in the form of

$$I_L^{(1)} = P_s G_0^2 \lambda^2 \theta_0^2 (2^7 \pi^2 \ln 2 r_s^2)^{-1} \xi_{\text{back}} d$$

(14)
has been proven to be equivalent to the radar equation as expected.

### 2.2. Second-Order Ladder Term

The second-order ladder term for bistatic radars can be represented by the diagram in Figure 2a, and the corresponding geometry is represented by Figure 2b. The field $E_A$ and its identical conjugate field $E_B$ are transmitted from the antenna T. These fields first scatter at point b and, then at point a, eventually returning to the receiving antenna R. Under the condition that the mean free path of the medium is much larger than the wavelength, which is almost always satisfied for hydrometeors, a Green function in the medium can be approximated from a complete form of van Bladel [1961]

$$G_{11}(r, r') \approx (4\pi)^{-1}|r - r'|^{-1} \exp[i/k|r - r'|] I_2$$

Substituting equations (1), (7), and (15) into the integral form corresponding to the diagram in Figure 2a, the second-order ladder term for the finite beam width can be derived as shown in Appendix A1:

$$I_{2}^{(2)} = P_i G_0^2 \lambda^2 \theta_0^2 \left(2\pi / \eta \right)^{-1} N_0^2 \left\{ 2 \left( \kappa_{iz}^n + \kappa_{sz}^n \right) \right\}^{-1}$$

$$\cdot \int_0^\infty d\eta \int_0^{2\pi} d\varphi \int_0^d d\zeta \frac{\eta}{1 + \eta^2}$$

$$\cdot \exp \left[ -\kappa_{iz} \sqrt{1 + \eta^2} \right] \exp \left[ -\kappa_{sz}^2 / 4\sigma_r^2 \right]$$

$$\cdot \sum_{\alpha} \left\{ \left| \hat{\alpha} \right| F(k_s, \hat{r}) F(-\hat{r}, \hat{k}_s) \psi_0 \right|^2$$

$$\cdot \left[ \exp \left[ -2\kappa_{iz}^n \right] - \exp \left[ 2\kappa_{iz}^n - 2\left( \kappa_{iz}^n + \kappa_{sz}^n \right) d \right] \right]$$

$$+ \left| \hat{\alpha} \right| F(k_s, -\hat{r}) F(-\hat{r}, \hat{k}_s) \psi_0 \right|^2$$

$$\cdot \left[ \exp \left[ -2\kappa_{sz}^n \right] - \exp \left[ 2\kappa_{sz}^n - 2\kappa_{sz}^n d \right] \right]$$

$$= \int_0^\infty d\eta \int_0^{2\pi} d\varphi \int_0^d d\zeta \frac{\eta}{1 + \eta^2}$$

$$\cdot \exp \left[ -\kappa_{iz} \sqrt{1 + \eta^2} \right] \exp \left[ -\kappa_{sz}^2 / 4\sigma_r^2 \right]$$

$$\cdot \sum_{\alpha} \left\{ \left| \hat{\alpha} \right| F(k_s, \hat{r}) F(-\hat{r}, \hat{k}_s) \psi_0 \right|^2$$

$$\cdot \left[ \exp \left[ -2\kappa_{iz}^n \right] - \exp \left[ 2\kappa_{iz}^n - 2\left( \kappa_{iz}^n + \kappa_{sz}^n \right) d \right] \right]$$

$$+ \left| \hat{\alpha} \right| F(k_s, -\hat{r}) F(-\hat{r}, \hat{k}_s) \psi_0 \right|^2$$

$$\cdot \left[ \exp \left[ -2\kappa_{sz}^n \right] - \exp \left[ 2\kappa_{sz}^n - 2\kappa_{sz}^n d \right] \right]$$

(16)

Figure 3. Second-order cross term. An incident field $E_A$ takes the same path as in Figure 2b, while the conjugate field $E_B$ emits from antenna T and scatters at points a and b successively, being received by antenna R. (a) Diagram. (b) General scattering $\theta_s \neq 0$. (c) Backscattering $\theta_s = 0$, where $E_B$ takes the time reversal path of $E_A$.

where the volumetric backscattering reflectivity $\xi_{zhk} = 4\pi N_0 |F_{iz}(-k_s, k_j)|^2$ has been introduced. If the layer thickness $d$ is replaced with the longitudinal resolution length $c/2 \tau$, we can retrieve the radar equation for volume scattering with a pulse duration of $\tau$. Hence the first-order ladder term for a small scattering angle $\theta_s \ll 1^\circ$ has been proven to be equivalent to the radar equation as expected.

### 2.3. Second-Order Cross Term

The diagram of the second-order cross term is depicted in Figure 3a. The direction of the conjugate field is arbitrary. In this paper, the notation of the direction of the fields will follow that of Kravtsov and Soloviev [1982]. In the geometry of Figure 3b, the field...
$E_4$ travels in the same path as that in Figure 2b, while the conjugate field $E_B$, starting from the transmitting antenna T, scatters first at point a and, then at point b, eventually returning to the receiving antenna R. Since the fields $E_A$ and $E_B$ have different path lengths, random distributions of points a and b will cause strong decorrelation, generally giving negligible contribution to the measured intensity. However for the backscattering measurement ($\theta_s = 0$) depicted in Figure 3c, the fields $E_A$ and $E_B$ have the same path lengths, giving finite correlation that leads to backscattering enhancement. The ray path of $E_B$ in Figure 3c is referred to as the time reversal path of $E_A$. As the transmitting and receiving gains are assumed to be symmetric in this study, the received fields $E_A$ and $E_B$ have also the same magnitudes. Although this assumption is not indispensable to obtain the backscattering enhancement, it is assumed for the sake of simplicity in calculation.

The second-order cross term for the finite beamwidth can be derived in a similar manner to the second-order ladder term. The result is

$$I_c^{(2)} = P_G\delta^2 \Delta^2 (2\pi \ln 2 \gamma_0^2)^{-1} N_0^2 \left(\kappa_{ie} + \kappa_{ic} \right)^{-1}$$

$$\cdot \int_0^\infty d\eta \int_0^{2\pi} d\varphi \int_0^d d\zeta \frac{\eta}{1 + \eta^2}$$

$$\cdot \exp\left\{-\frac{\kappa_{ie} \sqrt{1 + \eta^2} + \kappa_{ic}}{\kappa_{ic}} \right\}$$

$$\cdot \exp\left\{-\frac{\zeta^2 \eta^2}{4\sigma_0^2} \right\} \left\{1 - \exp\left[-2 \left(\kappa_{ie} + \kappa_{ic} \right) \right]\right\}$$

$$\cdot \left(d - \zeta \right) \right\} \Re \left\{ \sum \left\langle \hat{\alpha} | F\left(\hat{k}_s, \hat{r} \right) F\left(\hat{r}, \hat{k}_i \right) | \psi_0 \right\rangle^* \right\}$$

$$\cdot \left(\hat{\alpha} | F\left(\hat{k}_s, -\hat{r} \right) F\left(-\hat{r}, \hat{k}_i \right) | \psi_0 \right\rangle \exp\left[i(k_{dz} + t)\zeta \right]\}$$

$$\cdot \left(d - \zeta \right) \right\} \Re \left\{ \sum \left\langle \hat{\alpha} | F\left(\hat{k}_s, \hat{r} \right) F\left(\hat{r}, \hat{k}_i \right) | \psi_0 \right\rangle^* \right\}$$

in which the deviation vector $\mathbf{k}_d$ from the backscattering direction and the new variable $t$ have been introduced on the basis of work by Ishimaru and Tsang [1988]:

$$\mathbf{k}_d = (\hat{k}_s + \hat{k}_i) \equiv k_{dz} \hat{x} + k_{dy} \hat{y} + k_{dx} \hat{z}$$

$$t = k_{dz} \eta \cos \varphi + k_{dy} \eta \sin \varphi$$

Note that the second-order ladder term (equation (16)) and cross term (equation (17)) can be reduced to the forms of Mandt et al. [1990] in the limit of $\sigma_r \to \infty$. However an advantage of the form of equation (17) is that the decorrelation caused by increase in the value of $\mathbf{k}_d$ is explicitly represented through the term $\exp\left[i(k_{dz} + t)\zeta \right]$. Another advantage is that equation (17) is represented in real valued form.

Since Mandt et al. [1990] did not elaborate on the relation between the ladder and cross terms in the second order for the case of backscattering $\theta_s = 0$, we shall derive it. For the backscattering, the following conditions are satisfied for equations (16) and (17):

$$\hat{k}_s = -\hat{k}_i$$

$$\kappa_{ie} = \kappa_{ic}$$

$$t = k_{dz} = 0$$

In general, a scattering amplitude matrix $\mathbf{F}$ satisfies the Saxon’s reciprocal relation [Mishchenko, 1991; Saxon, 1955]:

$$F(-\hat{n}, -\hat{n}_t) = QF^t(\hat{n}_t, \hat{n})Q$$

where the superscript $t$ denotes matrix transpose, and the matrix $Q$ is defined as

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Without losing generality, we can assume the initial polarization:

$$\psi_0 \equiv \hat{\psi} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Using equations (20) and (23) and the fact of $\langle \mathbf{a}|\mathbf{A}|\mathbf{b} \rangle = \langle \mathbf{b}|\mathbf{A}^t|\mathbf{a} \rangle^*$, we can show

$$\langle \hat{\psi} | F\left(\hat{k}_s, -\hat{r} \right) F\left(-\hat{r}, \hat{k}_i \right) | \hat{\psi} \rangle = \langle \hat{\psi} | F\left(\hat{k}_s, \hat{r} \right) F\left(\hat{r}, \hat{k}_i \right) | \hat{\psi} \rangle$$

Inserting equations (21), (22), and (26) into equations (16) and (17), we see for the copolarized component that the cross term is equal to the ladder term:

$$I_c^{(2)}(CO) = I_c^{(2)}(CO)$$

where $CO$ stands for the copolarized component. On the other hand, for the cross polarized component, it is impossible to deduce a simple relation from equations (16) and (17) unless additional symmetries exist in a system. These results are in agreement with general formulas of Mishchenko [1991, 1992].

For the Mie scattering of spherical particles, the matrix $F(\hat{k}_s, \hat{r}) F(\hat{r}, \hat{k}_i)$ can be shown to have an
ical particles, the first-order ladder term of radar reflectivity. When hydrometeors consist of spherical particles: 

\[ F\left(\hat{k}_i, \hat{r}\right) F\left(\hat{r},\hat{k}_i\right) = \begin{bmatrix} a \cos 2\varphi + b & a \sin 2\varphi \\ -a \sin 2\varphi & a \cos 2\varphi - b \end{bmatrix} \] (28)

in which

\[ a \equiv \left[f_{i1}(\theta - \theta_s)f_{i1}(\theta_i - \theta) + f_{i2}(\theta - \theta_s)f_{i2}(\theta_i - \theta)\right]/2 \] (29)

\[ b \equiv \left[f_{i1}(\theta - \theta_s)f_{i1}(\theta_i - \theta) - f_{i2}(\theta - \theta_s)f_{i2}(\theta_i - \theta)\right]/2 \] (30)

and \( f_{i1}(\theta) \) and \( f_{i2}(\theta) \) are the components of Mie scattering amplitude matrix. Alternatively, we can write equation (28) as

\[ \langle \hat{h} | F\left(\hat{k}_i, \hat{r}\right) F\left(\hat{r},\hat{k}_i\right) | \hat{v} \rangle = -\langle \hat{v} | F\left(\hat{k}_i, \hat{r}\right) F\left(\hat{r},\hat{k}_i\right) | \hat{h} \rangle \] (31)

Using equations (20), (23), and (31), we can show for the cross polarized component scattered from \( \hat{v} \) to \( \hat{h} \):

\[ \langle \hat{h} | F\left(\hat{k}_i, \hat{r}\right) F\left(\hat{r},\hat{k}_i\right) | \hat{v} \rangle = \langle \hat{h} | F\left(\hat{k}_i, -\hat{r}\right) F\left(-\hat{r},\hat{k}_i\right) | \hat{v} \rangle \] (32)

Inserting equations (21), (22), and (32) into equations (16) and (17), we can derive the equality between the ladder and cross terms in cross polarization for spherical particles:

\[ I^{(2)}_c(CX) = I^{(2)}_l(CX) \] (33)

where \( CX \) stands for the cross polarized component. A recent numerical simulation by Oguchi and Ihara [2005] further reported that for cross polarization, the equality \( C^{cx} = L^{cx} \) is satisfied only to the second order.

3. Results and Discussion

Multiple scatterings including backscattering enhancement can be considered to effectively increase radar reflectivity. When hydrometeors consist of spherical particles, the first-order ladder term \( I^{(1)}_l \) has only the copolarized component, that is, \( I^{(1)}_l = I^{(1)}_L(CO) \). Furthermore \( I^{(1)}_l \) is almost constant in the vicinity of \( \theta_s = 0 \) (\( \theta_s < 0.3^\circ \)), within which the backscattering enhancement occurs. For this reason, the intensity of a multiple-scattering term will be normalized by \( I^{(1)}_L \) to be converted to an effective reflectivity in the rest of this paper. For instance, the second-order ladder reflectivity in copolarized component \( L^{(2)}_c \) will mean \( L^{(2)}_L(CO)/L^{(1)}_L \).

[17] The reflectivities of the second-order terms for a finite beam width can be numerically calculated through equations (16) and (17). In Figure 4, the reflectivities of the ladder term \( L^{(2)}_c \) and the cross term \( C^{cx}_2 \) in copolarization are represented by solid lines, and those of the ladder term \( L^{cx}_2 \) and the cross term \( C^{cx}_2 \) in cross polarization are represented by dashed lines. Spherical water particles of diameter \( D = 1 \) mm, particle number density \( N_0 = 5 \times 10^4 \) m\(^{-3}\), and layer thickness \( d = 100 \) m are used, which give the mean free path \( l_0 = 77.2 \) m, the optical thickness \( \tau_d = 1.30 \), and the normalized footprint radius \( \sigma_r/l_0 = 0.288 \).

![Figure 4. Reflectivities normalized by the single-scattering intensity versus the scattering angle \( \theta_s \) taken in the plane parallel to the initial polarization. The reflectivities of the ladder term \( L^{(2)}_c \) and the cross term \( C^{cx}_2 \) in copolarization are represented by solid lines, and those of the ladder term \( L^{cx}_2 \) and the cross term \( C^{cx}_2 \) in cross polarization are represented by dashed lines. Spherical water particles of diameter \( D = 1 \) mm, particle number density \( N_0 = 5 \times 10^4 \) m\(^{-3}\), and layer thickness \( d = 100 \) m are used, which give the mean free path \( l_0 = 77.2 \) m, the optical thickness \( \tau_d = 1.30 \), and the normalized footprint radius \( \sigma_r/l_0 = 0.288 \).]
\[ \kappa_e^{-1} = 77.2 \, \text{m}, \ \text{the optical thickness} \ \tau_d \equiv \kappa_e d = 1.30, \ \text{and} \ \text{the normalized footprint radius} \ \sigma_r/l_0 = 0.288. \]

On the basis of a scalar theory including all the ladder and cross terms \cite{Ishimaru and Tsang, 1988; Tsang and Kong, 2001}, truncation at the second order can be considered valid in this range of albedo and optical thickness. Unless specified otherwise, these parameters will be used in the remainder of this section.

Figure 4 shows that the ladder terms \( L_{2}^\alpha \) (solid) and \( L_{2}^\beta \) (dashed) are practically constant for \( \theta_s < 0.1^\circ \) for the above parameters. The cross terms \( C_2 \) are equal to the ladder terms \( L_2 \) at \( \theta_s = 0 \) in both polarizations. These results are theoretically expected from equations (27) and (33). As \( \theta_s \) increases, the cross terms rapidly decrease because of the term \( \exp[i(\kappa_\perp + \theta_\parallel)] \) in equation (17), as described in section 2.3. This rapid reduction in reflectivity is more evident for the cross polarization than for the copolarization. It is noted that the constancy of \( L_2 \) and the decaying behavior of \( C_2 \) in both polarizations in Figure 4 will hold even when changing either the beam width \( \theta_\parallel \) or the range \( r_s \), correspondingly changing the footprint radius \( \sigma_r \). Behaviors of \( L_2 \) and \( C_2 \) as functions of \( \sigma_r \), including the relation to the plane wave theory, will be studied for fixed values of \( \theta_s \) in later figures.

From an operational perspective, it is useful to compare the copolarized return intensity including the cross term, \( 1 + L_{2}^{\alpha} + C_{2}^{\alpha} \), to the conventional return intensity \( 1 + L_{2}^{\beta} \). These values were calculated in Figure 5 for two conditions referred to as parallel and orthogonal angular displacements respectively. Suppose that the scattering plane is parallel to the \( \vec{v} \) direction. In the parallel angular displacement, the initial polarization \( \psi_0 \) is chosen in the \( \vec{v} \) direction, while in the orthogonal one, \( \psi_0 \) is in the direction \( \vec{h} \). In Figure 5a, the values of \( 1 + L_{2}^{\beta} \) (dotted line) for these two conditions coincide, while the values of \( 1 + L_{2}^{\alpha} + C_{2}^{\alpha} \) (solid lines) for the two conditions coincide only at \( \theta_s = 0 \). As \( \theta_s \) increases, the values of the parallel angular displacement show slower decay than those of the orthogonal angular displacement, indicating that the \( C_{2}^{\alpha} \) term is anisotropic in copolarization. Since there is no return of the first-order scattering in cross polarization, that is, \( I_1^{\alpha}(CX) = 0 \), the total cross polarized return is represented by \( L_{2}^{\beta} \) or \( L_{2}^{\alpha} + C_{2}^{\alpha} \), which is equal to linear depolarization ratio (LDR). These values are plotted in Figure 5b, in which not only the term \( L_{2}^{\beta} \) but also the term \( L_{2}^{\alpha} + C_{2}^{\alpha} \) coincide to each other between the parallel and vertical angular displacements, providing no spatial anisotropy. The results of the spatial anisotropy near the backscattering direction, shown in Figures 5a and 5b, are in agreement with the works of \( \text{van Albada and Legendijk} \ [1987] \) and \( \text{van Albada et al.} \ [1987] \), who first attributed the anisotropy to the vector property of electromagnetic wave through computer simulation without theoretical confirmation.
The features of this spatial anisotropy, including its origin, are derived in Appendix A2. Regardless of the angular displacements, the backscattering enhancements appear with angular widths of the order of $2\pi\kappa/k \approx 0.002^\circ$, which is in agreement with the previous plane wave theories [Barbanenkov et al., 1991; de Wolf, 1971; Ishimaru, 1991; Kuzmin et al., 1992; Tsang and Kong, 2001; van Albada and Lagendijk, 1987] and plane wave experiments [Akkermans et al., 1986; Kuga and Ishimaru, 1984; van Albada et al., 1987; Wolf and Maret, 1985]. The spatial anisotropy of reflectivity in remote sensing can thus be summarized as follows. As long as spherical particles are observed with ground-based monostatic radars, the anisotropy of reflectivity in copolarization does not appear because of its scattering angle $\theta_s = 0$. For bistatic radars, including spaceborne monostatic radars with $\theta_s \approx 0.0025^\circ$, we must consider this effect. For simplicity of discussion in the rest of paper, only the parallel angular displacement will be considered.

The dependence of multiple-scattering reflectivities on footprint radius $\sigma_r$ is illustrated in Figure 6. The terms $1 + L_2^{co} + C_2^{co}$, $1 + L_2^{co}$ in copolarization (solid lines), and the terms $L_2^{cx} + C_2^{cx}$ and $L_2^{cx}$ in cross polarizations (dashed lines) are plotted as a function of $\sigma_r/l_0$ in Figure 6a for the backscattering angle, that is, $\theta_s = 0$, and in Figure 6b for $\theta_s = 0.0025^\circ$. The value $\sigma_r/l_0$ means that the footprint radius $\sigma_r$ is normalized by the mean free path $l_0 = 77.2$ m. Note that neither the 3 dB aperture angle $\theta_d$ nor the range $r_s$ are defined explicitly, since they are interrelated to $\sigma_r$ in equation (2). In Figure 6, calculation is terminated at $\sigma_r/l_0 = 0.064$, that is, $\sigma_r = 5$ m, because in such a small footprint radius, the numerical integral in equation (17) does not converge with good accuracy for $\theta_s = 0.0025^\circ$. Above $\sigma_r/l_0 \approx 2.5$, all the reflectivities in Figures 6a and 6b can be regarded as nearly constant, asymptotically approaching to the values that the plane wave theory [Mandt et al., 1990] predicts. The return intensity with only the ladder terms, $1 + L_2^{co} + C_2^{co}$ (bottom solid line) and $L_2^{co}$ (bottom dashed line) in Figure 6a, are identical to those in Figure 6a, while the return intensity including the cross terms, $1 + L_2^{co} + C_2^{co}$ (top solid line) and $L_2^{cx} + C_2^{cx}$ (top dashed line) in Figure 6b, come to lower values than those of Figure 6a because of reduction of backscattering enhancement. For instance, choosing the same value of $\sigma_r/l_0 = 0.288$ as taken for Figure 5, the readings of $1 + L_2^{co} + C_2^{co}$ and $L_2^{co} + C_2^{co}$ in Figure 6a are 1.34 and $-15.6$ dB, corresponding to the values at $\theta_s = 0^\circ$ in Figures 5a and 5b, respectively. Those readings at $\sigma_r/l_0 = 0.288$ in Figure 6b reduce to 1.16 and $-17.1$ dB, corresponding the values at $\theta_s = 0.0025^\circ$ in Figures 5a and 5b, respectively.

Since the normalized reflectivity of the first-order term is ever defined as unity for any $\sigma_r/l_0$, working as a constant bias, it is reasonable to study the effect of only the second-order terms. Figure 7 show the calculated values of $L_2^{co} + C_2^{co}$ and $L_2^{cx} + C_2^{cx}$ with a solid and a dashed line respectively. The monostatic radar is our main concern so that only the backscattering $\theta_s = 0$ will

![Figure 6](image-url)
be considered hereinafter. In Figure 7, both polarizations show rapid reductions in the region \( \sigma_s/l_0 < 1 \), while they asymptotically approach to the values of the plane wave theory as \( \sigma_s/l_0 \) increases, as already seen in Figure 6. It is noted that for \( \theta_s = 0 \), the values of equations (13), (16), and (17) can be shown to be independent of the particle number density \( N_0 \), when the layer thickness \( d \) and the footprint radius \( \sigma_s \) are normalized to the mean free path \( l_0 \). In this sense, we can alternatively interpret the hydrometeor parameters in Figure 7 as the two parameters defined by the particle diameter \( D = 1 \) mm and the optical thickness \( \tau_d = 1.30 \) regardless of the values of \( N_0 \) and \( d \). For ease of readers’ reference, the mean free path \( l_0 \) is plotted for \( D = 1 \) mm as a function of \( N_0 \) in Figure 8.

[22] In Figure 9, the terms \( L_2^{\sigma_0} + C_2^{\sigma_0} \) (solid lines) and \( L_2^{\sigma_0} + C_2^{\sigma_0} \) (dashed lines) are plotted as a function of optical thickness \( \tau_d = d/l_0 \) for several values of normalized footprint radius \( \sigma_s/l_0 \). Here only the frequency of 95 GHz and the particle diameter \( D = 1 \) mm are fixed. Since the mean free path \( l_0 \) is uniquely defined for a given particle density \( N_0 \), the value of \( \tau_d \) should be considered to change by varying the physical layer thickness \( d \). In both polarizations, the reflectivities become less sensitive to increments in \( \tau_d \) as the parameter \( \sigma_s/l_0 \) decreases. This is conspicuous for the cross polarization, in which the multiple-scattering amplitude factors such as \( F(k_s, \vec{r}) F(\vec{r}, \vec{k}_0) \) with \( \vec{r} = (\theta, \varphi) \) have high values around \( \theta = 90^\circ \) for nearly Rayleigh scattering regime. Hence the second-order scattering in the cross polarization is not so sensitive to increment in \( \tau_d \) that is along the direction of \( \theta = 0 \) and \( 180^\circ \), but sensitive to increment in \( \sigma_s/l_0 \) that is along the direction of \( \theta = 90^\circ \). In fact, Figures 7 and 9 show that for a fixed \( \tau_d \), as \( \sigma_s/l_0 \) increases, the corresponding increment in the reflectivities of the cross polarization is larger than that of the copolarization.

[23] Using Figure 9, we can determine the reduction factor from the plane wave theory (i.e., \( \sigma_s/l_0 = \infty \)) for the particle diameter \( D = 1 \) mm and given a footprint radius \( \sigma_s \), a layer thickness \( d \), and a particle number density \( N_0 \). Figure 9 also indicates that the plane wave theory can be applied to a smaller value of \( \sigma_s/l_0 \), as the optical thickness \( \tau_d \) decreases. For example, at a large optical thickness \( \tau_d = 4.0 \), the values of \( L_2^{\sigma_0} + C_2^{\sigma_0} \) (Solid lines) are \(-1.89 \) dB and \(-2.39 \) dB for \( \sigma_s/l_0 = \infty \) and \( \sigma_s/l_0 = 1 \) respectively. These values reduce to \(-11.49 \) dB and \(-11.58 \) dB respectively at a small optical thickness \( \tau_d = 0.05 \). Hence the difference in \( L_2^{\sigma_0} + C_2^{\sigma_0} \) between \( \sigma_s/l_0 = \infty \) and \( \sigma_s/l_0 = 1 \) is 0.50 dB at \( \tau_d = 4 \), while it becomes only 0.09 dB at \( \tau_d = 0.05 \). It means that the plane wave theory can be approximately applied for the footprint radius \( \sigma_s/l_0 = 1 \) at the small optical thickness \( \tau_d = 0.05 \).

![Figure 7](image_url)  
Figure 7. Reflectivity \( L_2^{\sigma_0} + C_2^{\sigma_0} \) in copolarization (solid line) and the reflectivity \( L_2^{\sigma_0} + C_2^{\sigma_0} \) in cross polarization (dashed line) as functions of the normalized footprint radius \( \sigma_s/l_0 \) for the backscattering \( \theta_s = 0 \). The same parameters as in Figure 4 are used. Note that only for \( \theta_s = 0 \), we can alternatively interpret the hydrometeor parameters as the two parameters defined by particle diameter \( D = 1 \) mm and the optical thickness \( \tau_d = 1.30 \) regardless of particle number density \( N_0 \) and layer thickness \( d \). In \( \sigma_s/l_0 < 1 \), the reflectivities reduce to be far from the values predicted by the plane wave theory represented by \( \sigma_s/l_0 \gg 1 \).

![Figure 8](image_url)  
Figure 8. Mean free path \( l_0 \) versus particle number density \( N_0 \) for spherical water of diameter \( D = 1 \) mm.
Throughout the paper, hydrometeor particles have been assumed to be stationary in air. If the particles move around, the phase coherence between the two time reversal paths in the cross term of Figure 3 will break down, resulting in decorrelation. This problem was studied by Akkermans et al. [1988] and Golubentsev [1984] using time-dependent diffusion approximations in scalar. Although consistent derivation requires a time-dependent Green function, we can roughly estimate the dependence on the particle density $N_0$ does not appear explicitly in Figure 9, it is implicitly included in the mean free path $l_0$ as shown in Figure 8.

Since the theory in this paper is derived as a time-independent theory, it can be applied only to the stationary process such as CW radars (not FM-CW radars). To apply for pulsed radars, the theory must be extended to a time-dependent theory. For a single layer of hydrometeors, an analytical solution of the time-dependent radiative transfer theory was derived for the plane wave incidence by Ito et al. [1995], based on generalized spherical harmonics expansions [Oguchi, 1980; Ito and Oguchi, 1987, 1989]. However their approach does not include the effects of the second-order cross term (backscattering enhancement) nor the finite beam width. A promising approach is therefore to combine the formalism of Ito et al. [1995] with that of the present paper. To develop this algorithm, raindrops are more tractable than ice particles. Since the shape of raindrops is close to sphere, large LDR values, such as measured in the Ka band range by Ito et al. [1995] and Iguchi et al. [1992], can be attributed to multiple-scattering effect rather than to nonspherical particle effect. This issue will be studied in the future. On the other hand, when the vertical range resolution $l_{\text{res}} = cT/2$ is larger than the mean free path $l_{\text{free}}$, that is, $l_{\text{res}} > l_{\text{free}}$, the theory of this paper can be used to estimate the amount of second-order scattering near the top surface of hydrometeors as discussed in section A3.

In addition to developing a time-dependent algorithm, it is also necessary to include effects of a generic drop size distribution (DSD), and nonspherical particles. Since the former effect breaks the high symmetrical forms of the scattering amplitude matrices in equations (16) and (17) because of the lack of symmetry for particle exchange,
a new formalism must be developed. The latter effect is also important especially for ice particles in clouds, which often are represented as needles (columns) and plates. The nonsphericity of these particles will introduce spatial anisotropy both in the scattering amplitude matrix and in the propagation Green function. Once these two effects are introduced into the formalism, the second-order scatterings for ice particles can be easily calculated, which is of main interest for W band weather radars.

### Appendix A: Details of Derivation

#### A1. Derivation of the Second-Order Ladder Term

[28] The second-order ladder term can be derived by substituting equations (1), (7), and (15) into the diagram of Figure 2a. In the course of calculation, the center mass substituting equations (1), (7), and (15) into the diagram Ishimaru and Tsang in the following form along with the transformations of za

In equation (A3), the function J(z_a, z_b) can be represented in the following form along with the transformations of

\[
x_\perp = (z_a - z_b) \tan \theta \cos \varphi \\
y_\perp = (z_a - z_b) \tan \theta \sin \varphi
\]

For \( z_a > z_b \)

\[
J(z_a, z_b) = \int_{-d}^{0} d\varphi \int_{-d}^{0} d\theta \left[ \exp \left[ -2\kappa_{z}^\ast z_b + 2\kappa_{z}^\ast z_a \right] \exp \left[ -2\kappa_{z}^\ast z_b + 2\kappa_{z}^\ast z_a \right] \exp \left[ -2\kappa_{z}^\ast z_b + 2\kappa_{z}^\ast z_a \right] \right. \\
\left. \cdot \exp \left[ -2\kappa_{z}^\ast z_b + 2\kappa_{z}^\ast z_a \right] \right]
\]

while for \( z_a < z_b \)

\[
J(z_a, z_b) = \int_{0}^{\pi/2} d\theta \tan \theta \int_{0}^{2\pi} d\varphi \\
\cdot \exp \left[ -(z_a - z_b)^2 \tan^2 \theta / 4\sigma_r^2 \right] \\
\cdot \exp \left[ -\kappa_{z}(z_a - z_b) \sec \theta \right] \\
\cdot \sum_\alpha \left| \left\langle \hat{\alpha} \right| F(\hat{k}_s, -\hat{r}) F(\hat{r}, \hat{k}_t) \right| \psi_0 \right|^2
\]

Up to this point, there is no substantial difference from Mandt et al. [1990]. Since we cannot calculate analytically the integrals over \( z_a \) and \( z_b \) for the case of finite beam width as was done for the plane wave case, further simplification is to be introduced through the following transformations:

\[
\zeta = z_a - z_b \quad \zeta' = (z_a + z_b) / 2
\]

Noting the relations satisfied for a general function \( g(z_a, z_b) \):

\[
\begin{align*}
\int_{-d}^{0} dz_a \int_{-d}^{0} dz_b g(z_a, z_b) &= \int_{-d}^{0} dz_0 \int_{-d}^{0} dz_0 g(\zeta, \zeta') \quad \text{for } z_a > z_b \\
\int_{-d}^{0} dz_a \int_{-d}^{0} dz_b g(z_a, z_b) &= \int_{-d}^{0} dz_0 \int_{-d}^{0} dz_0 g(-\zeta, \zeta') \quad \text{for } z_a < z_b
\end{align*}
\]

we obtain the form of

\[
I_L^{(2)} = \pi P_i G_0^2 \lambda^2 \theta_d^2 (2\pi \ln 2 \sigma_s^2)^{-1} N_0^2 \left\{ 2 \left( \kappa_{z}^\ast + \kappa_{z}^\ast \right) \right\}^{-1} \\
\cdot \int_{0}^{\pi/2} d\theta \tan \theta \int_{0}^{2\pi} d\varphi \\
\cdot \exp \left[ -2\kappa_{z}^\ast \zeta \right] \\
\cdot \sum_\alpha \left| \left\langle \hat{\alpha} \right| F(\hat{k}_s, -\hat{r}) F(\hat{r}, \hat{k}_t) \right| \psi_0 \right|^2
\]

Finally conversion of the integral variable \( \theta \) to \( \eta = \tan \theta \) leads to equation (16).

#### A2. Spatial Anisotropy of Ladder and Cross Terms

[29] Suppose that a scattering plane is parallel to the polarization \( \hat{v} \). When the initial polarization \( \psi_0 \) is parallel(orthogonal) to \( \hat{v} \), this condition is referred to as parallel(orthogonal) angular displacement. Difference
between the parallel and orthogonal angular displacements appear only through the terms $\langle \hat{\chi}|F(\hat{k}_s, +\hat{r}) F(\pm\hat{r}, \hat{k}_i)|\psi_0 \rangle$ in equations (16) and (17). Throughout this section, it is noted that equation (28) is approximately satisfied with high accuracy for $|\theta_0| \leq 0.1^\circ$.

**A2.1. Proof of $L_2^{ca}(\hat{v}\hat{v}) = L_2^{ca}(\hat{h}\hat{h})$**

[30] Since equation (28) is well satisfied for $|\theta_0| \leq 0.1^\circ$,

$$\left| \langle \hat{v}|F(\hat{k}_s, +\hat{r}) F(\hat{r}, \hat{k}_i)|\hat{v} \rangle \right|^2 = 2^{-1}|a|^2 + |b|^2$$

$$+ 2^{-1}|a|^2 \cos 4\varphi + (ab^* + ba^*) \cos 2\varphi$$

(A12)

$$\left| \langle \hat{h}|F(\hat{k}_s, +\hat{r}) F(\hat{r}, \hat{k}_i)|\hat{h} \rangle \right|^2 = 2^{-1}|a|^2 + |b|^2$$

$$+ 2^{-1}|a|^2 \cos 4\varphi - (ab^* + ba^*) \cos 2\varphi$$

(A13)

Using equations (A12) and (A13), we can show

$$\int d\varphi \left( \langle \hat{v}|F(\hat{k}_s, +\hat{r}) F(\hat{r}, \hat{k}_i)|\hat{v} \rangle \right)^2$$

$$= \int d\varphi \left( \langle \hat{h}|F(\hat{k}_s, +\hat{r}) F(\hat{r}, \hat{k}_i)|\hat{h} \rangle \right)^2$$

(A14)

which is also satisfied for $-\hat{r}$. Noting that the integral over $\varphi$ in equation (16) concerns only to the terms $\langle \hat{v}|F(\hat{k}_s, +\hat{r}) F(\pm\hat{r}, \hat{k}_i)|\psi_0 \rangle$, the substitution of equation (A14) and its counterpart for $-\hat{r}$ into equation (16) yields

$$I_L^{(2)}(\hat{v} \hat{v}) = I_L^{(2)}(\hat{h} \hat{h})$$

(A15)

For the nearly backscattering $|\theta_0| \leq 0.1^\circ$, the following relation is approximately satisfied for the spherical particle with high accuracy:

$$I_L^{(1)} = I_L^{(1)}(\hat{v} \hat{v}) \approx I_L^{(1)}(\hat{h} \hat{h})$$

(A16)

We divide equation (A15) by equation (A16) to obtain

$$L_2^{ca}(\hat{v} \hat{v}) = L_2^{ca}(\hat{h} \hat{h})$$

(A17)

Equation (A17) means that the reflectivity of the second-order ladder term in copolarization from $\hat{v}$ to $\hat{v}$ is equal to that from $\hat{h}$ to $\hat{h}$ as shown by the dash-dotted line in Figure 5a.

**A2.2. Spatial Anisotropy of the Cross Term in Copolarized Return**

[31] Since equation (28) is well satisfied for $|\theta_0| \leq 0.1^\circ$, we derive

$$\langle \hat{v}|F(\hat{k}_s, +\hat{r}) F(\hat{r}, \hat{k}_i)|\hat{v} \rangle * \langle \hat{v}|F(\hat{k}_s, -\hat{r}) F(-\hat{r}, \hat{k}_i)|\hat{v} \rangle$$

$$= 2^{-1}a^*a' + b*b' + 2^{-1}a*a' \cos 4\varphi$$

$$+ (a*b' + b*a') \cos 2\varphi$$

(A18)

in which $a'$ and $b'$ correspond to the values $a$ and $b$ in equations (29) and (30) by replacing $\theta$ with $\pi - \theta$. In the same manner,

$$\langle \hat{h}|F(\hat{k}_s, +\hat{r}) F(\hat{r}, \hat{k}_i)|\hat{h} \rangle * \langle \hat{h}|F(\hat{k}_s, -\hat{r}) F(-\hat{r}, \hat{k}_i)|\hat{h} \rangle$$

$$= 2^{-1}a^*a' + b*b' + 2^{-1}a*a' \cos 4\varphi$$

$$+ (a*b' + b*a') \cos 2\varphi$$

(A19)

Substituting equation (A18) and (A19) along with equation (22) into the integral over $\varphi$ in equation (17), we obtain for the $\theta_0 = 0$:

$$\int d\varphi \text{Re}\left\{ \langle \hat{v}|F(\hat{k}_s, +\hat{r}) F(\hat{r}, \hat{k}_i)|\hat{v} \rangle * \langle \hat{v}|F(\hat{k}_s, -\hat{r}) F(-\hat{r}, \hat{k}_i)|\hat{v} \rangle \exp[i(k_{de} + t)\zeta] \right\}$$

$$= \int d\varphi \text{Re}\left\{ \langle \hat{h}|F(\hat{k}_s, +\hat{r}) F(\hat{r}, \hat{k}_i)|\hat{h} \rangle * \langle \hat{h}|F(\hat{k}_s, -\hat{r}) F(-\hat{r}, \hat{k}_i)|\hat{h} \rangle \exp[i(k_{de} + t)\zeta] \right\}$$

(A20)

which leads to

$$I_C^{(2)}(\hat{v}\hat{v}) = I_C^{(2)}(\hat{h}\hat{h})$$

(A21)

Dividing both sides by equation (A16), we obtain

$$C_2^{ca}(\hat{v}\hat{v}) = C_2^{ca}(\hat{h}\hat{h})$$

(A22)

Equation (A22) is satisfied only for the backscattering direction $\theta_0 = 0$ because of the lack of the oscillation term $\exp[i(k_{de} + t)\zeta]$. For $\theta_0 \neq 0$, the term $\exp[i(k_{de} + t)\zeta]$ gives another dependence on the integral variable $\varphi$, and equation (A22) is no longer satisfied. This is the origin of the spatial anisotropy of the backscattering enhancement in copolarization as shown by the two solid lines in Figure 5a.

**A2.3. Proof of $L_2^c(\hat{v}\hat{v}) = L_2^c(\hat{h}\hat{h})$ and $C_2^c(\hat{v}\hat{v}) = C_2^c(\hat{h}\hat{h})$**

[32] These properties can be easily derived for the nearly backscattering $|\theta_0| \leq 0.1^\circ$ by substituting equation (31) and its counterpart for $-\hat{r}$ into equation (17), followed by dividing by equation (A16). The equalities/isotropies $L_2^c(\hat{v}\hat{v}) = L_2^c(\hat{h}\hat{h})$, and $C_2^c(\hat{v}\hat{v}) = C_2^c(\hat{h}\hat{h})$ are represented in Figure 5b.

**A3. Special Condition to Apply the Time-Independent Theory to Pulsed Radars**

[33] Although the returned signals due to the second-order scattering represented by equations (16) and (17) have been derived as time independent process such as CW radars (not FM-CW radars), it is worth comparing the scheme represented by these equations with that of pulsed radars. Figure A1 is a bounce diagram [Bringi
and Chandrasekar, 2001; Freeman, 1996]. Since this diagram is a projection of four dimensional spaces into two dimensional spaces of z coordinate (range) and time, we cannot fully represent information on multiple scattering, but it is still useful. In the diagram, the speed of light is set at unity, so that the time and z coordinate are plotted in the same scale. Suppose that a uniform random medium exists between the ranges \(d_u\) and \(d_b\), and that the leading and trailing edges of a pulse are transmitted at time 0 and T respectively. Then the signal representing the range \(d_3\) is received at time \(R_3\), including the first-order scattering contribution from the line \(Q_3U_3\). When considering the second-order scattering process, a ray scattered first at any point in the dotted region has possibility to be secondly scattered at a certain point on the line \(Q_3S_3\), eventually being received at time \(R_3\). For instance, if an incident ray transmitted at time \(t_a\) is scattered at point “a,” a second-time scattering can occur at a certain point “A” on the line \(aS_3\), as long as the three dimensional distance between points “a” and “A” is properly chosen. It is therefore seen that even for pulsed radars, multiple-scattering effect contaminates into the signal of a given range. However, the contribution of second-order scattering comes only from the dotted region. On the other hand, the time-independent theory, referred to as CW radars, includes the contribution of the second-order scattering from both the solid gray and dotted regions, and at the same time, the first-order scattering comes from all the points on the line \(B_3S_3\). Hence it is challenging to generalize the comparison of the effects of second order scattering between these two schemes. However in the case of \(l_{res} > l_{free}\), where \(l_{res}\) and \(l_{free}\) are the range resolution (\(cT/2\)) and the mean free path of medium respectively, the amounts of second order scattering in both the pulsed and CW radars can be considered roughly equal near the top surface of hydrometeors. This will be explained by using the bounce diagram, in which the dash-dotted line “eh” is drawn with \(U_3f = Q_3h = l_{free}\). Then the contribution of the second-order scattering for pulsed radars can be considered to come roughly from the region “\(U_3egQ_3\)”, while for CW radars applied only between \(d_2\) and \(d_3\) (i.e., the integral limit “\(d\)” in equations (16) and (17) is replaced with \(l_{res} = d_3 - d_2\), the contribution comes from the region “\(U_3fhQ_3\)” Among these regions, the triangle “\(efU_3\)” for pulsed radars contributes to more amount of second-order scattering onto the line “\(Q_3V_3\)” than the triangle “\(ghQ_3\)” for CW radars does, because the larger range suffers more absorption. A small portion of the shared region “\(U_3fgQ_3\)” may also give slight contribution of second-order scattering to the line “\(U_3V_3\)” for pulsed radars. Furthermore as the range increases, the second-order contributions from regions other than “\(U_3egQ_3\)” for pulsed radars will be no longer negligible. As seen from the bounce diagram, these difference-causing effects will be reduced near the top surface. It is thus seen that the second-order scattering

**Figure A1.** Bounce diagram. The speed of light \(c\) is set at unity. A uniform random medium exists from the range \(d_u\) to \(d_b\). The leading and trailing edges of pulses are transmitted at times 0 and T, respectively.
calculated from the time-independent theory, applied to the ranges $d_2$ and $d_3$, is a good estimation for pulsed radars under the condition of $l_{res} > l_{free}$, only if the range $d_2 - d_3$ is located near the top surface, that is, for the range $d_0 - d_1$.

[34] On the contrary, for $l_{res} < l_{free}$, especially for $2l_{res} < l_{free}$, more contribution of the second-order scattering comes from the dotted region “$U_2Q_2S_2S_3$” for pulsed radars, while for CW radars, from the solid gray region “$Q_0d_2d_3Q_3$.” In addition, for pulsed radars, the contribution from the region “$U_3Q_3S_3$” to the line $U_3S_3$ will increase. It is again difficult to evaluate the amounts of these second-order scattering for pulsed radars through the formalism of this paper, and the time-dependent algorithm is needed.

[35] Summarizing the case of $l_{res} > l_{free}$, the formalism of this paper can roughly estimate the amount of second-order scattering of pulsed radars in ranges near the top surface. Our preliminary calculation showed that the mean free path $l_{free}$ is between 300 and 500 meter for rains of 10 mm/hr, depending on drop size distributions. Since the range resolution of the CloudSat and EarthCare missions is 500 meter, the above condition is weakly satisfied.

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