Guaranteed Spatial Initialization of Distributed Spacecraft Formations

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In a previous paper the authors developed a formation initialization (FI) algorithm for a deep space, N-spacecraft formation. It was demonstrated analytically that this FI algorithm guaranteed the initialization of a formation restricted to a plane. The main contribution of this paper is to extend this planar guarantee to deep space formations with arbitrary initial conditions. As part of the guarantee of initialization, a bound on the time-to-initialize is obtained. The guaranteed FI algorithm is then demonstrated for a two-spacecraft formation with realistic deep space mission constraints (e.g. limited field-of-view relative sensors and attitude constraints). The two-spacecraft scenario is challenging in that it has the least relative sensor field-of-view overlap. Finally, for this scenario, the distribution of time-to-initialize is characterized through a 150,000-case Monte Carlo analysis.

I. Introduction

A set of more than one spacecraft is *formation flying* if their dynamic states are coupled through an automatic control law such that a chain of couplings can be followed, irrespective of directions, between every pair. This coupling enables the set of spacecraft to function collaboratively as a single instrument. For example, hundred-meter class synthetic apertures composed of formation flying spacecraft are being considered for imaging black hole event horizons[1] and detecting Earth-sized planets circling other stars.[2]

For these missions, *precision* formation flying is required; relative spacecraft positions and/or attitudes must be controlled to at least the level of centimeters and arcminutes, respectively. However, for precision formations located beyond high-accuracy terrestrial navigation aids (e.g., the Global Positioning System cannot be used at Earth-Moon libration points), spacecraft only know their positions with respect to the Earth to no better than a few kilometers. As a result, formation spacecraft must rely on onboard *relative* sensors to obtain the precise relative dynamic state information (i.e., relative positions and velocities) necessary for collaboration.

Furthermore, formation spacecraft must generally communicate to function as a single instrument. The process of both establishing inter-spacecraft communication and using onboard sensors to obtain relative dynamic state information is called *Formation Initialization* (FI).

Depending on the communication (comm) and relative sensing suites of a formation, FI can range from a trivial to a complex process. For example, assuming the use of omni-directional comm antennas and relative sensors, FI consists simply of turning these systems on; communication and relative state measurements immediately follow. At the other end of the spectrum, however, formation spacecraft may have only a single limited field-of-view (FOV) comm antenna and a single limited-FOV relative sensor. This latter case may arise due either to failures of an omni-directional system, mission mass or power constraints, for example, that require a reduction in hardware capability, or lack of scalability (e.g., an omni-directional sensor that functions for a three spacecraft formation may not function for a forty spacecraft formation due to excessive spacecraft occultations, multi-path and cross-linking).

In previous papers,[3,4] the authors developed the first FI algorithm for an arbitrarily-sized deep space formation. The goal of that development was an FI algorithm that (*i*) guarantees initialization in a boundable time period, (*ii*) is scalable, (*iii*) functions over a wide range of relative sensor/comm suites, and (*iv*) admits attitude constraints (e.g., a pointing requirement on a sun shield).

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To achieve these goals, the following representative relative sensor/comm suite was selected for FI algorithm development: one limited-FOV relative sensor and omni-directional communication.[3,4] Furthermore, since the relative bearing information available from a formation comm system varies depending on its specific configuration, a priori comm-based relative bearing information is *not* used. For example, consider a formation in which each spacecraft is equipped with two comm antennas with hemispherical fields-of-view. For one hardware design, a spacecraft may be able to determine which comm antenna is receiving, and hence locate the transmitting spacecraft to within a hemisphere of sky. However, comm hardware can also be implemented so that the receiving antenna is not distinguishable. Moreover, the formation comm topology may significantly limit the information obtainable. If inter-spacecraft comm is routed through specific spacecraft, then in general very coarse bearing information can only be determined to subsets of the formation.

Since the FI algorithm of Refs. 3 and 4 does not utilize any comm-based relative bearing information, it can be sub-optimal. However, performance has been traded for robustness to different hardware configurations: the algorithm can function over a wide range of relative sensor and comm configurations. By discarding possible comm-based information, an FI algorithm was obtained that: (*i*) is independent of the size of the formation, (*ii*) is not computationally intensive, (*iii*) requires only limited inter-spacecraft communication and coordination, and (*iv*) can be analytically guaranteed to initialize a formation within a known bounded time.

In Ref. 4, the basic FI algorithm was shown to guarantee formation initialization for deep space formations restricted to a plane. The primary contribution of this paper is to extend this planar guarantee to deep space formations with arbitrary translational initial conditions. To do so, however, it is necessary to make some minor modifications to the basic FI algorithm of Ref. 3 and 4.

In this paper, we first briefly review the basic FI algorithm presented in Ref. 3 and 4. As part of this review, the assumptions used throughout this paper are discussed. Following the review, we present the modified FI algorithm. Next, various lemmas needed for the main result of this paper are proven. Then we prove the main result, namely, that the modified FI algorithm guarantees initialization for a deep space formation with arbitrary initial conditions. After proving the main result, a Monte Carlo analysis is presented demonstrating the performance and efficacy of the modified FI algorithm. Finally, we present some conclusions.

II. Basic FI Algorithm

Before reviewing the basic FI algorithm, we summarize the assumptions upon which the results in this paper are based.

A. Summary of Assumptions

First, recall that each spacecraft is equipped with one limited-FOV relative position sensor and an omni-directional communication capability. In particular, the relative sensor is based upon the functionality of the Autonomous Formation Flying (AFF) sensor.[5] The AFF consists of a set of three receivers and a transmitter on each spacecraft. All four antennas on each spacecraft are co-sighted with identical, conical fields-of-view. The half-angle of the conical FOV is denoted by θ_{FOV}. A typical value for θ_{FOV} is 70°.

A critical aspect of the AFF is that it is a distributed sensor; it requires a transmitter on one spacecraft and three receivers on another. Therefore, for the AFF sensor to function two spacecraft must fall within the transmission/reception patterns (i.e., the FOV) of each other *simultaneously*. This configuration is referred to as a "front-to-front lock" or F/F lock.

This simultaneity condition is illustrated in Figure 1, in which three spacecraft (SC) are shown with cones that represent the fields-of-view of their relative sensors. SC_1 and SC_2 meet the simultaneity requirement, and hence there is an F/F lock between them. Note also that SC_3 falls within the FOV of SC_1. However, SC_1 is *not* in the FOV of SC_3, and so there is no F/F lock between them.

Each spacecraft is also assumed to have a sensitive payload that cannot be exposed to direct sunlight (e.g., the infra-red telescopes of the Terrestrial Planet Finder mission[2]). Figure 2 shows a schematic of a generic formation spacecraft with a payload and a body-fixed reference frame affixed to the spacecraft center of mass. Note the sun-shield and the relative sensor with its conical FOV. We assume further that the body x-axis is aligned with the sun-shade normal and that the body z-axis is aligned with the relative sensor boresight.

To adequately protect the payload, it is assumed that body x-axis must remain within a specified angle of the spacecraft-to-sun line (or simply sun-line). This *sun-angle constraint* limits the allowable attitudes that each spacecraft can achieve. For example, unlimited rotations about the body y- and z-axes are not permitted. The maximum angle between the body x-axis and the sun-line is specified as θ_s, with typical values ranging from 25° to 45°.

Each spacecraft is also equipped with a star tracker that provides accurate attitude information. As a result, even though spacecraft do not initially know the relative positions of one another, they are assumed to know their own attitudes.

Further, the formation is assumed to be in deep space where disturbances are negligible over the duration of FI, which is typically a few hours. In particular, it is assumed that the relative translational dynamics of the spacecraft are well approximated by a double integrator model,[6] in which the only forces on the spacecraft are due to its own thrusters. Therefore, thrust-free spacecraft trajectories are straight lines.

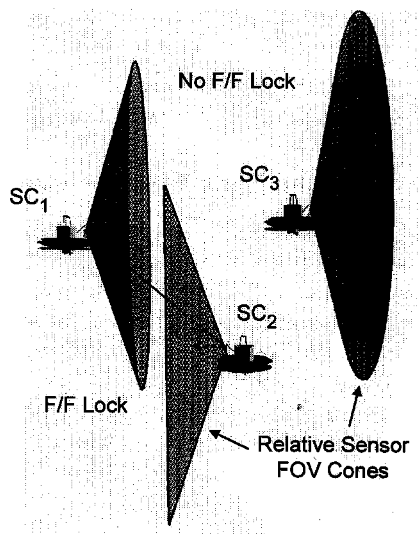

Figure 1. Illustration of Relative Sensor Simultaneity Requirement

In addition to the assumptions above, the lemmas and theorems herein rely on a common set of assumptions, which are presented below. Consider two spacecraft denoted as SC_i and SC_j.

The common assumptions are:

A1. SC_i and SC_j do not have a priori knowledge of their relative range and bearing.

A2. No collisions occur during FI.

A3. The relative sensor of SC_i is located at the spacecraft's center-of-mass.[a]

A4. The formation is located in deep space, where the only force on a spacecraft is due to its own thrusters.

A5. $\theta_R \leq \theta_s$, where θ_R is a characteristic angle of the FI algorithm (discussed in the next section) that depends on θ_{FOV}. A formula for θ_R is given in equation (1).

A6. $\theta_{FOV} \geq 45°$

In our FI algorithm translational control forces (i.e., thrusters) are not applied until after an F/F lock. Therefore, a non-rotating frame attached to the center-of-mass of SC_i is a valid inertial frame of reference by assumption A4. Without loss of generality, SC_i is assumed fixed at a point O and SC_j moves with constant velocity $\vec{v} = \vec{v}_j - \vec{v}_i$ relative to SC_i, where $\vec{v}_i$ and $\vec{v}_j$ denote their constant, absolute velocities. An inertial frame of reference, denoted $\mathcal{F}_I = \{\vec{n}_x, \vec{n}_y, \vec{n}_z\}$, is affixed to SC_i at point O. The axes are oriented as discussed below.

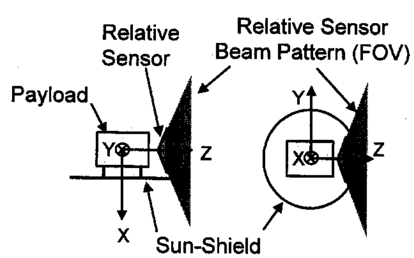

Figure 2. Formation Spacecraft Schematic

Two additional ramifications of A4 are that (i) the direction from a spacecraft to the sun does not change over the duration of FI,[b] and (ii) the sun direction is identical for all formation spacecraft (i.e., the extent of the formation is negligible compared to the distance to the sun). Therefore, without loss of generality, $-\vec{n}_z$ is defined as the sun direction.

We make two further assumptions.

[a]This assumption is a valid approximation when the spacecraft separation is significantly larger than the size of the spacecraft.

[b]It is shown in the sequel that initialization will complete in less than 3 hours for typical spacecraft parameters. For a formation located in a 1 AU orbit, the direction to the sun will change 0.1° in this time.

A7. The initial sensor boresight direction of SC_i is aligned with $\vec{n}_x$, and its body x-axis is aligned with $-\vec{n}_z$.

A8. The attitude of SC_j is initialized so that its relative sensor boresight is anti-parallel to that of SC_i (i.e., in the $-\vec{n}_x$ direction), and its body x-axis is aligned with $-\vec{n}_z$.

Regarding A7 and A8, recall that accurate absolute attitude information is available to the spacecraft. These assumptions are consistent with the initial conditions for the basic FI algorithm presented next.

B. Summary of the Basic FI Algorithm

Given the assumptions A1-A8, the major technical challenge of the posed FI problem is ensuring F/F lock (i.e., simultaneous relative sensor lock) under the sun-angle constraint. In particular, to achieve an F/F lock it is necessary but not sufficient for each spacecraft to search the entire sky with its relative sensor FOV.

We address the simultaneity condition for an F/F lock by arbitrarily assigning the spacecraft to one of two groups, G_A and G_B. Using known attitude information, the relative sensor boresights in one group are set anti-parallel to the boresights in the other group. Then the two groups perform a three-phase, synchronized full sky search such that the relative sensor boresights remain anti-parallel. Since the boresights are always anti-parallel between the two groups, when a spacecraft from one group is in the relative sensor FOV of a spacecraft from the other group, the reverse is also true, thereby ensuring F/F lock.

The three-phase full sky search consists of an In-Plane Search (IPS), an Out-of-Plane Search (OPS), and a Near-Field Search (NFS).

The IPS begins with all spacecraft pointing their body x-axes at the sun, the spacecraft in G_A pointing their z-axes in the same arbitrary direction ($\vec{n}_x$), and the spacecraft in G_B pointing their z-axes opposite those of the spacecraft in G_A. Then the spacecraft all rotate 1.5 revolutions about their x-axes in the same direction at the same angular speed. This IPS maneuver is illustrated in Figure 3. Given $\theta_{FOV} = 70°$, IPS searches 94% of the sky. One and a half revolutions are used as a result of Lemma 2, which is discussed subsequently.

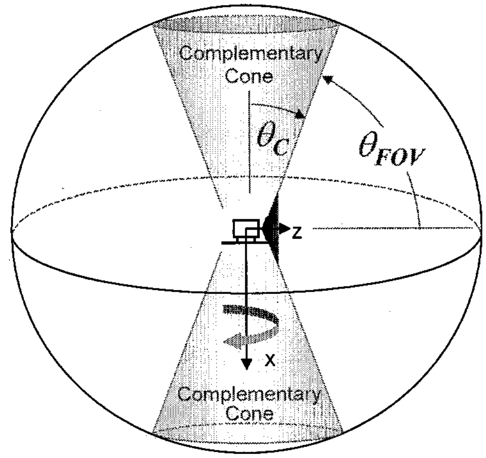

Figure 3. Maneuver for IPS

After IPS, there is a conical region above and below each spacecraft that has not been searched. These two areas are referred to as *complementary cones*, and they are characterized by an angle $\theta_C \triangleq \pi/2 - \theta_{FOV}$. See Figure 3. The Out of Plane Search (OPS) phase searches these complementary cones. However, due to the sun constraint angle θ_s, a complementary cone cannot be searched by a simple spacecraft rotation. As a result, a spacecraft searches a complementary cone by first rotating an angle θ_R about the body y-axis, and then rotating about the sun line. The angle θ_R is consistent with the sun constraint angle (see assumption A7) and its import is discussed in the next section. Note also that two complementary cones must be searched, and that the OPS differs slightly between the two groups of spacecraft. This difference keeps the relative sensor boresights anti-parallel.

The sequence of maneuvers for OPS for both groups is shown in Figure 4. Figure 4 also includes the maneuvers for a modified OPS (mOPS) introduced subsequently. Note that there is an additional final maneuver for OPS to realign the body x-axis with the sun-line. This maneuver is not shown in Figure 4. For spacecraft in G_A (resp. G_B), this extra maneuver consists of a $-\theta_R$ (resp. θ_R) rotation about the body y-axis.

Even after OPS, however, the entire sky has not been searched. There remains an unsearched *near field* about each spacecraft that results from the sun-angle constraint and the offset of the relative sensor from the spacecraft center of mass (to avoid clipping by the sun shield, see Figure 2). Essentially, there are two small volumes adjacent to each spacecraft (located along the sun line) that the relative sensor cannot search.

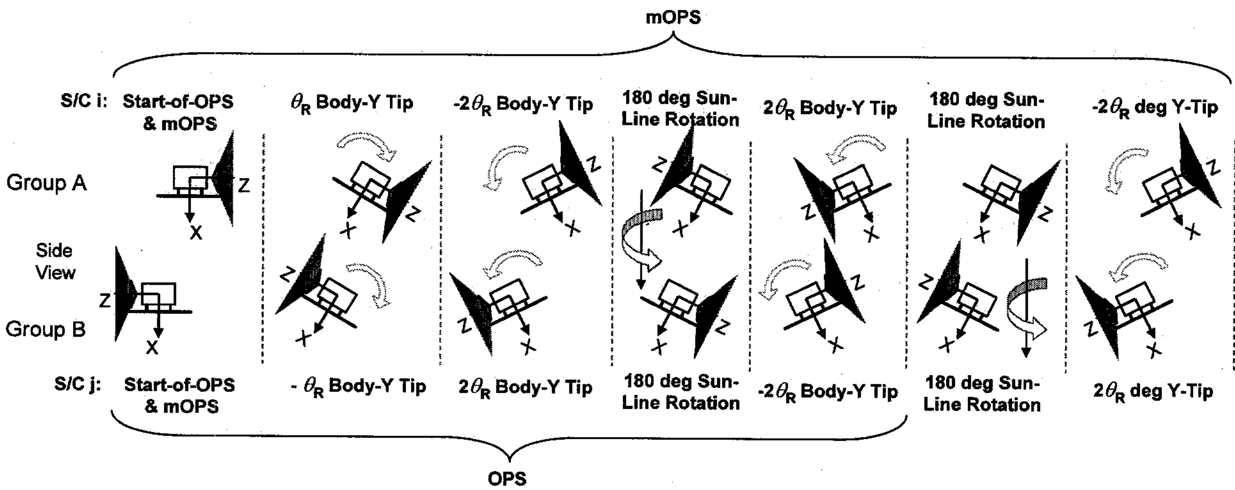

Figure 4. Maneuvers for OPS and Modified-OPS (mOPS)

If a spacecraft is located in the near field of another, it must maneuver out of the near field or be measured by a third spacecraft.

At the end of IPS and OPS spacecraft that have not achieved an F/F lock assume that they are in the near field of another spacecraft. These "lost" spacecraft perform predetermined translational maneuvers to move themselves out of the near-fields. The NFS consists of these predetermined maneuvers. The NFS is analyzed in Ref. 4. It is not pertinent to the initialization guarantee by assumption A3, and so it is not discussed further.

In addition to the synchronized three-phase sky search just discussed, the basic FI algorithm of Ref. 3 and 4 includes logic for merging *sub-formations*. A sub-formation is a set of more than one spacecraft that have obtained relative state information. For example, assume spacecraft A and B have achieved F/F lock, as have spacecraft C and D. While each pair of spacecraft know their relative positions, neither pair knows the relative positions of the other pair: each pair is a sub-formation.

To complete the initialization of a formation, these sub-formations must be merged. However, while Ref. 4 includes the algorithm for merging sub-formations, this merging algorithm is not important for the results of this paper, and it is not discussed further. We note only that the primary use of the assumed omni-directional communication capability is to coordinate the merging of sub-formations.

III. Modified FI Algorithm

In Ref. 4, the basic FI algorithm discussed in the previous section was shown to guarantee initialization for a planar, deep space formation. To extend this guarantee to deep space formations with arbitrary initial conditions, two modifications are necessary. The first modification is to add two additional rotations to OPS. This modified OPS is referred to as mOPS, and it is discussed next. The second modification to the basic FI algorithm is to repeat the three-phase sky search. That is, instead of performing IPS, then OPS and finally NFS, the modified algorithm consists of the following sequence: IPS, mOPS, IPS, mOPS, and finally NFS.

This extended search sequence addresses the primary challenge in extending the planar guarantee to formations with arbitrary initial conditions, namely, the interaction of IPS and OPS. For example, consider the following two-spacecraft scenario. During IPS, each spacecraft is located in a complementary cone of the other. Since IPS does not search the complementary cones, the spacecraft do not obtain an F/F lock. As OPS begins, which would search the complementary cones, the spacecraft exit the complementary cones due to their relative motion. However, the portion of the sky external to the complementary cones has already been searched, and will not be completely searched again. Hence, initialization can fail.

The modified *five-phase* search sequence (i.e., IPS, mOPS, IPS, mOPS, NFS) is shown to guarantee initialization in Section IV. This guarantee is proven by combining the five-phase sequence with two facts stated as lemmas in the following: (*i*) if a spacecraft remains in the IPS region of another during the entirety of IPS, then F/F lock is guaranteed, and (*ii*) if a spacecraft remains in the complementary cones of another during the entirety of mOPS, then F/F lock is also guaranteed. In the next section, we introduce the mOPS and prove fact (*ii*) above. Following the mOPS section, the fact (*i*) is proven.

A. Modified Out-of-Plane Search: mOPS

Modified-OPS consists of OPS, followed by an additional two attitude maneuvers. The maneuvers are (*i*) a 180° sun-line rotation followed by (*ii*) a rotation of $2\theta_R$ about the body y-axes ($-2\theta_R$ for G_A and $2\theta_R$ for G_B). The full set of maneuvers for mOPS is shown in Figure 4. Note that there is an additional final maneuver for mOPS to realign the body x-axis with the sun-line that is not shown in Figure 4. It consists of a θ_R rotation about the body y-axis for spacecraft in G_A.

The rationale underlying mOPS is made clear in the proof of the following lemma. As part of the proof, a formula for θ_R is given.

Lemma 1 *Consider SC_i and SC_j satisfying assumptions $A1-A8$. In addition, if the following assumption holds*

A9. SC_j remains within or on the boundary of the complementary cones of SC_i during the entirety of a modified out-of-plane search (mOPS)

then SC_i and SC_j obtain an F/F lock during mOPS.

Proof: The proof proceeds by first dividing each spacecraft's complementary cones into four regions, each of which is searched entirely during a single rotation of mOPS. Then a correspondence between the regions of two spacecraft is determined such that the following property holds. Assume SC_j is in SC_i's Region "A." If SC_i searches its Region "A" while SC_j searches its *corresponding* Region "B," an F/F lock occurs. Region "B" is said to correspond to Region "A."

To guarantee an F/F lock during mOPS (in which it is not known that SC_j is in Region "A"), SC_i searches all its regions, while SC_j is synchronized to search its corresponding regions. This approach is first complicated by the fact that the spacecraft are moving with respect to one another. As a result, the spacecraft can change regions. The proof concludes by accounting for this relative motion. The key observation is that SC_j travels on a line with respect to SC_i, hence the spacecraft can only change regions at most once.

A second, technical complication is that the initial attitudes of SC_i and SC_j differ from one mOPS to the next. This difference is due to the 1.5 revolutions of IPS. In order to address both modified Out-of-Plane Searches, a second inertial frame $\mathcal{F}_N$ is defined based on $\mathcal{F}_I$ and the attitude of SC_i at the start of a particular mOPS. Note that while $\mathcal{F}_N$ differs from one mOPS to the next, is a valid inertial frame for analyzing a specific mOPS.

Beginning the proof proper, the four complementary cone regions for a spacecraft are shown in Figure 5. The two cones in the figure are the complementary cones of a spacecraft located at the origin O'. In the sequel, SC_i is located at O'. The regions are defined with respect to a second inertial frame $\mathcal{F}_N = \{\vec{n}_1, \vec{n}_2, \vec{n}_3\}$. $\mathcal{F}_N$ itself is defined so that $\vec{n}_3$ is aligned with $\vec{n}_z$, and $\vec{n}_1$ is aligned with the boresight of SC_i at the beginning of mOPS. For the first mOPS $\vec{n}_1 = -\vec{n}_x$, and for the second mOPS $\vec{n}_1 = \vec{n}_x$.

The four complementary cone regions are delineated by the plane containing the $\vec{n}_2$ and $\vec{n}_3$ axes. Each region includes its boundary plane. For example, Region I is the closed right half of the lower complementary cone. Figure 5 also introduces further notation. In particular, $\vec{e}_{ij}$ is a unit vector from SC_i to SC_j.

Having defined the four regions of the complementary cones, the spacecraft maneuver necessary to search a specific region is now determined. As an example consider Region III. This region is searched by a spacecraft obtaining an attitude that can be constructed as follows. First, the relative sensor boresight (i.e., the body z-axis) is aligned with $-\vec{n}_1$. Then the spacecraft rotates θ_R about its body y-axis ($\vec{n}_2$ in this case). The attitudes needed to search the other regions are constructed in a similar manner. Specifically, the body z-axis is first aligned with either $\vec{n}_1$ or $-\vec{n}_1$. Then the spacecraft rotates θ_R or $-\theta_R$ about the body y-axis.

Analytic geometry can be used to show that the angle θ_R that a spacecraft must rotate about its body y-axis to completely search a region is given by

$$\theta_R \triangleq \tan^{-1}\left(\frac{\cos\theta_{FOV}}{\sqrt{1-2\cos^2\theta_{FOV}}}\right). \quad (1)$$

This result requires $\theta_{FOV} \geq 45°$ (see A6). Recall that A5 requires $\theta_R \leq \theta_s$. For a sun constraint angle of 25°, A7 requires $\theta_{FOV} \geq 67°$.

Next, the correspondence between the regions of SC_i and SC_j is derived. Recall, the regional correspondence is such that if SC_i searches Region "A" while SC_j searches the corresponding Region "B" F/F lock can occur. First note that if SC_j is in a complementary cone of SC_i (see A9), then SC_i is also in a complementary cone of SC_j. This fact can be shown by noting that A9 implies $|\vec{n}_3 \cdot \vec{e}_{ij}| \geq \cos\theta_C$. This equation in turn implies $|\vec{n}_3 \cdot \vec{e}_{ji}| \geq \cos\theta_C$, since $\vec{e}_{ji} = -\vec{e}_{ij}$.

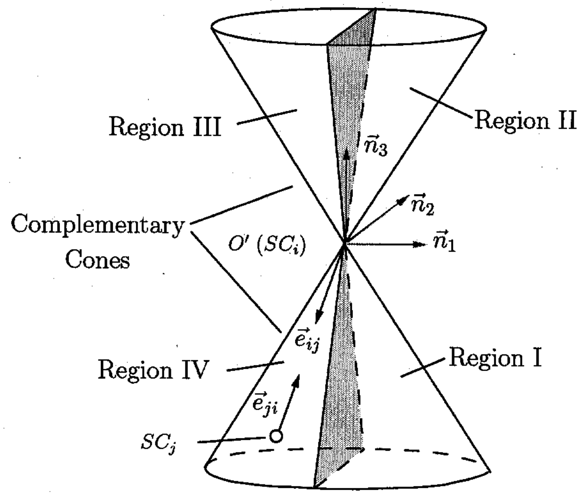

Figure 5. Four Complementary Cone Regions

To determine specific correspondences, first assume SC_j is in SC_i's Region IV, which is denoted Region IV_i. In this case, $\vec{n}_3 \cdot \vec{e}_{ij} \leq 0$ and $\vec{n}_1 \cdot \vec{e}_{ij} \leq 0$. Again substituting $-\vec{e}_{ji}$ for $\vec{e}_{ij}$, we obtain $\vec{n}_3 \cdot \vec{e}_{ji} \geq 0$ and $\vec{n}_1 \cdot \vec{e}_{ji} \geq 0$. With reference to Figure 5, these two conditions together with the fact that SC_i is in SC_j's complementary cones imply that SC_i is in Region II_j. This analysis can be repeated for the remaining regions of SC_i. The correspondence obtained is shown in Table 1.

Location of SC_j	Location of SC_i
Region I_i	Region III_j
Region II_i	Region IV_j
Region III_i	Region I_j
Region IV_i	Region II_j

Table 1. Complementary Cone Region Correspondence

Now consider the mOPS maneuvers shown in Figure 4. Note that a spacecraft in G_A, say SC_i, performs rotation so that its regions are searched in the following sequence: I_i, II_i, III_i, IV_i, I_i, and II_i. Also, a spacecraft in G_B, say SC_j, performs rotations so that it searches the regions corresponding to SC_i's, namely: III_j, IV_j, I_j, II_j, III_j, and IV_j. Therefore, when SC_i searches the region in which SC_j is located, SC_j is searching the region SC_i is located in, and an F/F lock is achieved.

We must now show that there exists a time when SC_i searches a region in which SC_j is located. Recall both spacecraft are assumed to be in each other's complementary cones for all of mOPS. As a result, SC_j can only change cones by passing through the apex where SC_i is located. Assumption A2 prohibits this change, since a collision would result. Also, by A4 the motion of the spacecraft can be represented as the motion of SC_j along a line with respect to SC_i. Since a line and a plane intersect once, not at all, or along the entirety of the line, SC_j can change regions at most once. As a result, SC_j can only change between Regions II_i and III_i, or between Regions I_i and IV_i.

If the line of SC_j's trajectory intersects the plane delineating the four regions either not at all or along the entirety of the line, then SC_j does not change regions. Hence, F/F lock is achieved when the region in which SC_j is located is searched.

If the line of SC_j's trajectory intersects the plane once, then SC_j changes regions at most once (SC_j only travels along a ray coinciding with the line). If the region in which SC_j is initially located is searched before SC_j changes regions, then F/F lock is achieved during the first search of Regions I_i through IV_i.

Now assume SC_j leaves its initial region before that region is searched by SC_i. Two cases are possible. First, if SC_j transitions from II_i to III_i or from I_i to IV_i, then SC_j is found during the first search of either

Region III_i or IV_i. Second, if SC_j transitions from IV_i to I_i or from III_i to II_i, then SC_j is found at the latest during the second search of I_i and II_i.

All cases have been considered, and in each case an F/F lock was achieved. □

B. Restricted Initialization Guarantee for IPS

The previous section proved that if two spacecraft remain in each other's complementary cones during the entirety of mOPS, an F/F lock is achieved. This section proves the second key lemma used used in the guarantee of initialization for the modified FI algorithm. The second lemma is similar to Lemma 1, but it applies to IPS.

Lemma 2 *Consider SC_i and SC_j that satisfy assumptions $A1-A8$. If the following assumption also holds*

A10. SC_j remains outside the complementary cones of SC_i during the entirety of In-Plane Search (IPS)

then SC_i and SC_j achieve an F/F lock during IPS.

Proof: The proof reduces the three dimensional relative motion of the spacecraft to two dimensions by projecting SC_j's position and relative sensor FOV onto the plane spanned by $\vec{n}_x$ and $\vec{n}_y$. Then the theorem from Ref. 4 that guarantees F/F lock for a planar formation is applied. This planar theorem is restated in the sequel.

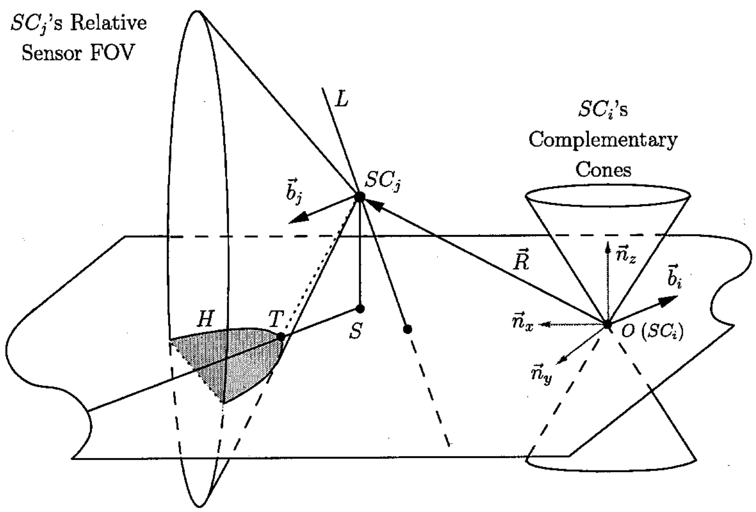

Figure 6. Geometry and Notation for IPS Lemma 2

Figure 6 introduces the notation used in this proof. From A4, SC_j is traveling on a line L with constant velocity $\vec{v}$ with respect to SC_i. SC_i is located at the origin O of the inertial reference frame $\mathcal{F}_I$. The position of SC_j with respect to SC_i is given by $\vec{R} = \vec{R}_o + \vec{v}t$, where $\vec{R}_o$ is the initial position of SC_j with respect to SC_i. The complementary cones of SC_i are shown for context (SC_j's complementary cones are omitted for clarity). A cone is also shown that represents the relative sensor FOV of SC_j. Again for clarity, a cone representing the FOV of SC_i's relative sensor is omitted. However, the boresight directions of both spacecraft's relative sensors are indicated by the unit vectors $\vec{b}_i$ and $\vec{b}_j$. The intersection of SC_j's relative sensor FOV with the $\vec{n}_x, \vec{n}_y$-plane is the hyperbola H. The points S and T are, respectively, the projection of SC_j's location onto the $\vec{n}_x, \vec{n}_y$-plane and the vertex of the hyperbola H.

First, a virtual spacecraft SC_j^p is defined such that if SC_i and SC_j^p achieve F/F lock, then SC_i and SC_j also achieve F/F lock (this fact is proved subsequently). SC_j^p is located at point S. The attitude of SC_j^p is to identical to the attitude of SC_j at all times.

Now it is shown that SC_j^p moves with constant velocity in the $\vec{n}_x, \vec{n}_y$-plane, which is a condition for the planar theorem that is going to be applied. Projecting $\vec{R}$ onto the $\vec{n}_x, \vec{n}_y$ plane yields $\vec{R}^p = (\vec{R}_o - \vec{n}_3 \cdot \vec{R}_o) + (\vec{v} - \vec{n}_3 \cdot \vec{v})t$.

Again with reference to Figure 6, the relative sensor FOV of SC_j^p is defined to be the ray $\vec{ST}$. As a result, SC_j^p has a θ_{FOV} of 0°. In this case, an F/F lock occurs only if both spacecraft's sensor boresights are colinear.

The theorem from Ref. 4 that guarantees an F/F lock for a planar formation is now restated.

Theorem 1 *Two spacecraft confined to a plane that satisfy assumptions A1-A4 and A7 and A8, and that perform an In-Plane Search (IPS) achieve F/F lock for any $\theta_{FOV} \geq 0$.*

Summarizing, SC_i and SC_j^p are confined to plane and the necessary assumptions are satisfied. By Theorem 1, SC_i and SC_j^p achieve an F/F lock.

It is now shown that an F/F lock between SC_i and SC_j^p implies an F/F lock between SC_i and SC_j. Following a similar argument as used in the proof of Lemma 1, since SC_j remains outside the complementary cones of SC_i by A10, SC_i also remains outside the complementary cones of SC_j.

We now show that SC_i can never be located on the line segment $\vec{ST}$ (see Figure 6). Proceeding by contradiction, assume that at some time SC_i belongs to $\vec{ST}$. It can be shown that this assumption implies SC_i is in a complementary cone of SC_j, which contradicts the result of the previous paragraph.

Since SC_i is never part of $\vec{ST}$, when SC_i and SC_j^p obtain an F/F lock, SC_i must be located on the ray $\vec{ST}$ beyond point T. By construction, this portion of the ray $\vec{ST}$ (and hence SC_i) is in the relative sensor FOV of SC_j. Finally, since SC_i is in SC_j's sensor FOV and the relative sensor boresights are anti-parallel, SC_j is also in SC_i's relative sensor FOV. F/F lock is achieved. □

Remark: Lemma 2 only guarantees that an F/F lock occurs sometime during the 1.5 revolutions of IPS. The results in Ref. 7 can be used to solve for the exact time of F/F lock.

IV. Guaranteed Initialization for Arbitrary Formation Initial Conditions

The main result of this paper is proved in this section. The two lemmas from the previous section are used to prove that the modified FI algorithm guarantees the initialization of a deep space formation with arbitrary initial conditions. The proof is based upon enumerating the possible scenarios resulting from the relative motion of SC_j (e.g., SC_j starts in the IPS region of SC_i, but enters a complementary cone before IPS completes), and showing in each case that an F/F lock is eventually achieved.

Before presenting the main result, one additional lemma is needed. The lemma delimits the scenarios that must be considered in the proof of guaranteed initialization.

Lemma 3 *A line and a double cone may intersect: once, twice, along the entirety of the line, or not at all.*

Proof: With reference to Figure 6, the line L may be parameterized as $\vec{R} = \vec{R}_o + \vec{v}t$ for all $t \in \mathbb{R}$. The line L intersects a double cone (i.e., the two complementary cones) at times t when

$$\vec{n}_z \cdot \vec{R} = \pm|\vec{R}| \cos\theta_C. \tag{2}$$

Squaring and rearranging leads to

$$(\vec{R} \cdot \vec{n}_z)(\vec{n}_z \cdot \vec{R}) = \cos^2\theta_C \vec{R} \cdot \vec{R}. \tag{3}$$

Let R, v and n_z be representations of the vectors $\vec{R}$, $\vec{v}$ and $\vec{n}_z$. Then (3) becomes

$$R^T Q R = 0 \tag{4}$$

where $Q = n_z n_z^T - \cos^2\theta_C I$, and I is the 3×3 identity matrix. It can be shown that (4) is a general condition for the point specified by the vector R to be part of the double cone with apexes at the origin, axis of symmetry n_z and half-cone angle θ_C.

Let $R = R_o + vt$, where R_o is a representation of the vector $\vec{R}_o$. Substituting into (4) and rearranging, the line L intersects a double cone at the times t that are real solutions to

$$at^2 + bt + c = 0 \tag{5}$$

where $a = v^T Q v$, $b = 2v^T Q R_o$ and $c = R_o^T Q R_o$.

Consider $a \neq 0$. Depending on the discriminant $\sqrt{b^2 - 4ac}$, there are no intersections, two different intersections or one intersection (i.e., if the discriminant equals zero, then there is a repeated real root). Now consider $a = 0$. If $b \neq 0$, then there is one solution. If $b = 0$, then $c = 0$, otherwise there is a contradiction. In this case, all times t satisfy the equation $0 = 0$, and so the line L must belong to the double cone.

Summarizing, there are either no, one, two or infinite real solutions to (5). In addition, these solutions represent actual intersections (as opposed to extraneous roots) since the right-hand side of (2) is constant and includes both signs. □

The main result of this paper is the following theorem. It guarantees the initialization of any two-spacecraft deep space formation following the modified FI algorithm. The proof immediately extends to an N-spacecraft formation by considering a pair of spacecraft (in differing groups) at a time.

Theorem 2 *Two spacecraft that meet assumptions A1-A8 and follow the modified FI algorithm will obtain an F/F lock.*

Proof: Two volumes of space are defined with respect to SC_i. The complementary cone volume (CCV) consists of SC_i's complementary cones (interior and boundary). The in-plane volume (IPV) is everything else.

The proof proceeds by enumerating the possible initialization scenarios on three levels. The first level of enumeration is based on the number of times SC_j changes volumes. Recall that SC_j is traveling on a line (A4). From Lemma 3, SC_j can intersect the complementary cone boundary zero, one, two or an infinity of times. In the case of an infinity of intersections, SC_j is traveling on the boundary of the complementary cones, and hence it is in the CCV. As a result, only zero, one or two volume changes are possible (e.g. IPV to CCV to IPV is two volume changes).

For a given number of volume changes, the second level of enumeration is based on the volume SC_j starts in.

The third and final level of enumeration is the most complex. Proceeding through the scenario dictated by the first two levels (e.g. two volume changes, with SC_j starting in the CCV), it is alternately assumed that an F/F lock is either achieved or not before SC_j's next volume change. If an F/F lock occurs by assumption, nothing more need be done. If an F/F lock does not occur, then the next volume change is considered. Eventually, an F/F lock is achieved or a contradiction results.

We proceed with the enumeration.

1. No Volume Change

   Further enumeration is not necessary since SC_j must remain in its starting volume. If SC_j is in the CCV, then by Lemma 1 an F/F lock is achieved during the first mOPS. If SC_j is in the IPV, then by Lemma 2 an F/F lock is achieved during the first IPS.

2. One Volume Change

   2.a) SC_j starts in the IPV.

   Since there is one volume change, SC_j must enter the CCV and remain in it.

   2.a.i) SC_j enters the CCV after an F/F lock has occurred.

   F/F lock occurs by assumption, and during the first IPS.

   2.a.ii) SC_j enters the CCV before an F/F lock has occurred.

   SC_j must have entered the CCV before the completion of the first IPS. Otherwise SC_j would have been in the IPV during the entirety of the first IPS and by Lemma 2 an F/F lock would have occurred. Since SC_j must remain in the CCV and it entered before the start of the first mOPS, by Lemma 1 an F/F lock occurs during the first mOPS.

   2.b) SC_j starts in the CCV.

   Since there is one volume change, SC_j must enter the IPV and remain in it.

   2.b.i) SC_j enters the IPV after an F/F lock has occurred.

   F/F lock occurs by assumption, and during the first mOPS.

   2.b.ii) SC_j enters the IPV before an F/F lock has occurred.

   SC_j must have entered the IPV before the end of the first mOPS, otherwise by Lemma 1 an F/F lock would have occurred. Since SC_j must now remain in the IPV and it entered the IPV before the start of the second IPS, by Lemma 2 an F/F lock will occur during the second IPS at the latest.

3. Two Volume Changes

3.a) SC_j starts in the IPV.

Since there are two volume changes, SC_j must enter and then exit the CCV.

3.a.i) SC_j enters the CCV after an F/F lock has occurred.

F/F lock occurs by assumption, and during the first IPS.

3.a.ii) SC_j enters the CCV before an F/F lock has occurred.

SC_j must have entered the CCV before the completion of the first IPS, otherwise by Lemma 2 an F/F lock would have occurred. As a result, SC_j is in the CCV prior to the start of the first mOPS, no F/F lock has occurred, and SC_j must eventually enter the IPV and remain in it thereafter. This case is identical to Case 2b, and an F/F lock occurs during the second IPS at the latest.

3.b) SC_j starts in the CCV.

Since there are two volume changes, SC_j must enter and then exit the IPV.

3.b.i) SC_j enters the IPV after an F/F lock has occurred.

F/F lock occurs by assumption, and during the first mOPS.

3.b.ii) SC_j enters the IPV before an F/F lock has occurred.

SC_j must have entered the IPV before the completion of the first mOPS. Otherwise, SC_j would have been the CCV during the entirety of the first mOPS, and by Lemma 1 an F/F lock would have occurred. However, there are now two sub-cases.

A. SC_j reenters the CCV after an F/F lock has occurred.

F/F lock occurs by assumption, and by the end of the second IPS at the latest.

B. SC_j reenters the CCV before an F/F lock has occurred.

SC_j originally entered the IPV before the completion of the first mOPS (i.e., before the start of the second IPS). If SC_j reentered the CCV after the completion of the second IPS, then it would have been in the IPV for the entirety of the second IPS, and by Lemma 2 an F/F lock would have occurred. Therefore, SC_j must reenter the CCV before the completion of the second IPS. As a result, SC_j is in the CCV and must remain there for the entirety of the second mOPS. By Lemma 1, an F/F lock occurs during the second mOPS.

In all cases, an F/F lock occurs. □

V. FI Monte Carlo Analysis

A Monte Carlo analysis was used to demonstrate the FI algorithm and characterize the distribution of the time required to initialize. We considered a two spacecraft formation in deep space. The translational initial conditions of the two spacecraft had positions uniformly distributed within a cube 1 km on a side and component velocities uniformly distributed over ± 20 cm/s. The modified FI algorithm was run for 150,000 different initial conditions. A θ_{FOV} of 70° was used. Hence, $\theta_R = 21.3°$. All spacecraft rotations are performed at 0.25°/s, which is consistent with typical star tracker rate limitations.

To be in agreement with assumption A3, the relative sensor on each spacecraft is located at its center-of-mass. In this case, there is no near field, and the Near Field Search (NFS) was not included in the Monte Carlo analysis. At a rotation speed of 0.25°/s, the abbreviated four-phase sky search of the modified FI algorithm (i.e., IPS, mOPS, IPS, and mOPS) takes approximately 2.4 hours.

Note that when considering only the sky-search portion of the modified FI algorithm (as opposed to the merging of sub-formations), a two spacecraft formation is the most challenging. For more than two spacecraft one spacecraft's FOV often encompasses a large portion of the complementary cones and near fields of other spacecraft. As a result, formation initialization can complete during the first IPS even if two spacecraft are in each other's complementary cones.

The results of the Monte Carlo analysis are shown in the histogram of Figure 7. The abscissa is the time in seconds that was needed to achieve F/F lock, divided into 60 second-wide bins. The ordinate shows the number of initial conditions that led to F/F lock in the corresponding 60 second window. Also, of the 150,000 cases run, approximately 50,000 were such that the spacecraft started in F/F lock. These cases are not shown in Figure 7.

As can be seen from the figure, in most cases F/F lock was achieved during the first IPS (97.33%, including the 33.37% that started in F/F lock). Of the remaining 2.67% that did not achieve F/F lock during the first IPS, 2.65% achieved F/F lock during the first mOPS, and 0.02% achieved F/F lock during the second IPS. In all 150,000 cases F/F lock was achieved before the second mOPS. However, based on the proof of Theorem 2, we calculated and verified initial conditions that require the second mOPS to obtain F/F lock. In $\mathcal{F}_I$, the initial conditions of SC_2 are (SC_1 is motionless at the origin):

$$r_2 = [2.08 \;\; 25.2 \;\; -305.6]\, m$$
$$v_2 = [0 \;\; 0 \;\; 0.0831]\, m/s$$

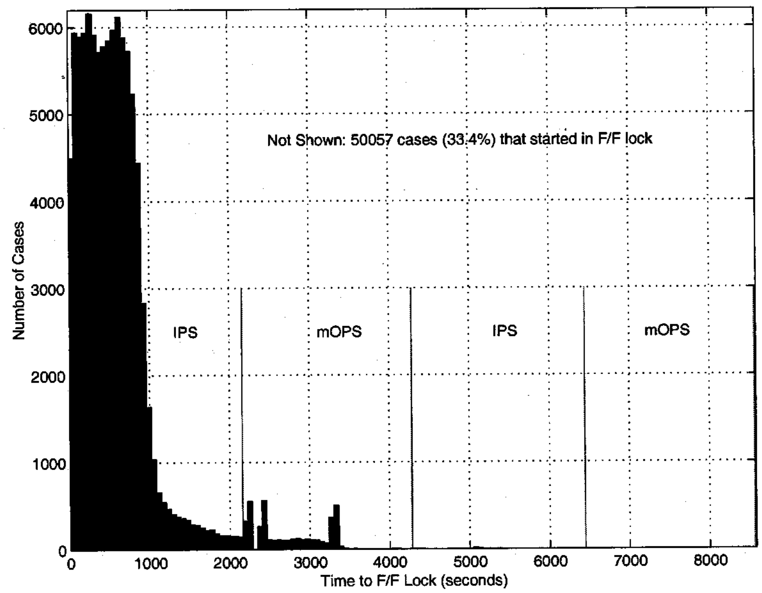

Figure 7. Modified FI Algorithm Monte Carlo Analysis. Note: cases that achieved F/F lock immediately are not shown.

Though not proven, simulation evidence suggests that the set of initial conditions requiring the second mOPS is not of zero measure.

As guaranteed by Theorem 2, all Monte Carlo cases initialized successfully. As a last observation, note the spikes in Figure 7 during the first mOPS. These spikes correspond to the spacecraft completing their rotations of θ_R or 180° to completely search a specific complementary cone region.

VI. Conclusions

In this paper we have first developed a modified version of the formation initialization (FI) algorithm originally presented in Refs. 3 and 4. Then we showed analytically that this modified algorithm guarantees the initialization of an N-spacecraft deep space formation with arbitrary initial conditions. The modified algorithm included additional rotations during the modified Out-of-Plane Search (mOPS) and the repetition of IPS and mOPS.

The FI algorithm developed takes into account realistic mission constraints such as limited relative sensor field-of-view (FOV) and sun-angle restrictions. Further, in ignoring possible a priori information available from the inter-spacecraft communication system, an algorithm was developed that applies to a wide range of formation relative sensor and communication suites, requires minimal coordination, is computationally simple and, most importantly, guarantees formation initialization in a boundable time period, typically 2.5 hours.

Finally, the modified FI algorithm was subjected to a Monte Carlo analysis in which 150,000 different initial conditions were simulated. The majority (97%) completed during the first IPS, with only 37 of the 150,000 cases requiring the second IPS to achieve F/F lock. In summary, a practical and broadly applicable formation initialization algorithm has been developed that has an analytic guarantee of success and a bound on the initialization time.

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References

[1]Gendreau, K., White, N., Owens, S., Cash, W., Shipley, A., and Joy, M., "The MAXIM X-ray Interferometry Mission Concept Study," *36th Liege Int. Astrophysics Colloquium*, Leige, Belguim, 2001.

[2]Lawson, P., "The Terrestrial Planet Finder," *IEEE Aerospace Conference*, 2001, pp. 2005–2011.

[3]Hadaegh, F., Scharf, D., and Ploen, S., "Initialization of Distributed Spacecraft for Precision Formation Flying," *IEEE Conference on Control Applications*, 2003, pp. 1463–1468.

[4]Scharf, D., Ploen, S., Hadaegh, F., Keim, J., and Phan, L., "Guaranteed Initialization of Distributed Spacecraft Formations," *AIAA Guidance, Navigation,&Control Conference*, 2003.

[5]Wu, S.-C. and Kuang, D., "Positioning with Autonomous Formation Flyer (AFF) on Space-Technology 3," *ION GPS*, Nashville, TN, 1999.

[6]Scharf, D., Hadaegh, F., and Kang, B., "On the Validity of the Double Integrator Approximation in Deep Space Formation Flying," *Int. Symp. Formation Flying Missions & Technologies*, Toulouse, France, 2002.

[7]Bikdash, M., Hadaegh, F., Scharf, D., and Ploen, S., "Three-Dimensional Analysis of Basic Acquisition Algorithms in Deep Space," *American Control Conference*, 2004.