Multipulse PPM on Memoryless Channels*

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Abstract

We examine several properties of n-pulse Pulse Position Modulation (PPM) [SN89] on memoryless channels. First, we derive the maximum likelihood decision rule and an exact expression for the symbol error rate for $n \geq 1$, avoiding a numerically unstable aspect of the $n = 1$ formula of [GK76] and generalizing the $n = 2$ result of [SV03]. Next, we compare the capacity of multipulse PPM to that of conventional single-pulse PPM when average power, peak power, and bandwidth constraints are simultaneously imposed. On this basis, we demonstrate that multipulse PPM does not produce appreciable gains over conventional PPM except at high average power.

1 Introduction

Pulse position modulation (PPM) is a common signaling scheme for optical communications or whenever a high peak to average power ratio is desirable. PPM is a form of constrained on-off keying (OOK) in which every frame of M slots contains one '1' and $M - 1$ '0's. Information is encoded by letting each group of $\log_2(M)$ bits designate which of the M slots contains the one.

Multipulse PPM is a generalization of PPM that allows more than one pulse per symbol. It was proposed by [SN89] as a technique to improve the bandwidth efficiency of optical signaling; [SN89] also derived an approximate symbol error rate (SER) for multipulse PPM on the quantum-limited (no background noise) channel. [Geo94] extended the analysis to noisy Poisson channels, deriving an explicit Maximum Likelihood (ML) rule, an exact error formula for the quantum-limited channel, and a bound when background noise is present. The exact SER for n-pulse PPM on a Poisson channel was derived in [SV03] for $n = 2$, and shown to involve a triple summation.

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In this paper, we derive the ML detection rule for a slightly more general class of channels and provide an exact and tractible formula for the SER for $n \geq 1$. Next, we compare the capacity of multipulse PPM to that of conventional single-pulse PPM when average power, peak power, and bandwidth constraints are individually or simultaneously imposed.

2 The Maximum Likelihood Rule

Let $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_{|S|}\}$ be the set of allowable n-pulse M-PPM symbols. Then $|S| = \binom{M}{n}$. Each $\mathbf{x}_i = (x_{i,1}, \ldots, x_{i,M})$ is an M-vector with n '1's and $M-n$ '0's. Let $\mathcal{I}_k$ denote the set of indices for which $\mathbf{x}_k$ has a '1'. Let $\mathbf{Y} = (Y_1, \ldots, Y_M)$ denote the slot values received from the channel. Assuming the underlying channel is memoryless, we let $p_1(y)$ and $p_0(y)$ denote the conditional probability that a received slot has value y given a '1' (pulse) or '0' (no pulse), respectively, is transmitted in the slot. The log-likelihood ratio receiving value y in a slot is denoted by $\Lambda(y) = \log\left[\frac{p_1(y)}{p_0(y)}\right]$, which for algebraic convenience we assume to be finite for all y. We also assume that $\Lambda(y)$ is monotonic in y_i, as is the case for many channels of practical interest (e.g., Poission, Gaussian, and Webb-McIntyre-Conradi). Let $P_1(y)$ ($P_0(y)$) denote the cumulative distributions, i.e., the probability that a received signal (nonsignal) slot has value less than or equal to y.

The likelihood of receiving $\mathbf{Y} = \mathbf{y}$, given $\mathbf{X} = \mathbf{x}_k$, is

$$P(\mathbf{Y} = \mathbf{y} | \mathbf{X} = \mathbf{x}_k) = \left(\prod_{i \in \mathcal{I}_k} p_1(y_i)\right)\left(\prod_{j \notin \mathcal{I}_k} p_0(y_j)\right) = \left(\prod_{i \in \mathcal{I}_k} \frac{p_1(y_i)}{p_0(y_i)}\right)\left(\prod_{j=1}^{M} p_0(y_j)\right)$$

The ML decision rule is

$$\hat{\mathbf{X}} = \max_k \log[P(\mathbf{Y} = \mathbf{y} | \mathbf{X} = \mathbf{x}_k)] = \max_k \sum_{i \in \mathcal{I}_k} \Lambda(y_i) = \max_k \sum_{i \in \mathcal{I}_k} y_i, \tag{1}$$

where in the last equation we made use of the monotonicity of the log-likelihood ratio. Thus, the ML multipulse PPM symbol decision is the one corresponding to the largest n slot values observed. This generalizes the Poisson channel result of [Geo94] to the class of channels having a monotonic log-likelihood ratio.

3 The Symbol Error Rate

Since the channel is memoryless, the probability of symbol error is independent of the location of the n pulses. For the purposes of computing the SER, we shall assume the first n slots contain pulses, and the remaining $M - n$ do not. From above, an ML decision can only be correct if $\min(Y_1, \ldots, Y_n) \geq \max(Y_{n+1}, \ldots, Y_M)$. If the inequality holds with strictly inequality, the correct decision is made. On discrete channels, it may hold with equality with positive probability. If it holds with equality, then there are multiple decisions that are ML decisions, only one of which is the correct one. To count these, let $y_{\min} = \min(Y_1, \ldots, Y_n)$ denote the smallest signal slot value, let $l = |\{Y_i : i > n, Y_i = y_{\min}\}|$ denote the number of nonsignal slots that attain that value, and let $m = |\{Y_i : i \leq n, Y_i = y_{\min}\}|$ denote the number of signal slots that attain that value. That is, there are m signal slots with a value of $y_{\min}$, $n - m$ signal slots with higher values, l nonsignal slots with a value of $y_{\min}$, and $M - n - l$ nonsignal slots with lower values. For a given l and m, there are $\binom{l+m}{m}$ distinct ML decisions, and so the probability that the correct ML decision is made is $I(l, m) = 1/\binom{l+m}{m}$. The probability of correct symbol detection is given by the expected value of $I(l, m)$ over $\mathbf{Y}$, and the SER is one minus that quantity. Thus, on a discrete channel having integer outputs,

$$SER = 1 - \sum_{y_{\min}=0}^{\infty} \sum_{l=0}^{M-n} \sum_{m=1}^{n} I(l,m) \underbrace{\binom{M-n}{l} p_0(y_{\min})^l P_0(y_{\min} - 1)^{M-n-l}}_{\substack{\text{probability exactly } l \text{ of } M - \\ n \text{ nonsignal slots have value} \\ y_{\min}\text{, all others smaller}}} \underbrace{\binom{n}{m} p_1(y_{\min})^m (1 - P_1(y_{\min}))^{n-m}}_{\substack{\text{probability exactly } m \text{ of } n \text{ sig-} \\ \text{nal slots have value } y_{\min}\text{, all} \\ \text{others larger}}} \tag{2}$$

In (2), we use the notational convenience that $0^0 = 1$, which can occur with the $P_0(\cdot)$ and $1 - P_1(\cdot)$ terms.

As an example of applying (2), we computed the performance of n-pulse PPM on a Poisson channel. The probability mass functions are $p_0(k) = \frac{K_b^k}{k!} e^{-K_b}$ and $p_1(k) = \frac{(K_s+K_b)^k}{k!} e^{-(K_s+K_b)}$, where K_b represents the average received value of a nonsignal slot, and $K_s + K_b$ represents the average received value of a signal slot (signal plus noise). We computed the SER of n-pulse 16-ary PPM for $n \in \{2, 3, 4, 8\}$ and $K_b \in \{0, 0.1, 0.5\}$, as shown in Fig. 1. All terms of the infinite sum that the computer could distinguish from zero were included in the computation, which generally was fewer than 500 terms. A desktop computer took about a minute to compute the 2000 points on the plot shown in Fig. 1, an average of 0.03 seconds per point.

For $n = 1$, (2) reduces to a double summation that is equivalent to the formula of [GK76], which is a single infinite sum. Unlike the [GK76] formula, (2) does not involve the difference of two nearly equal quantities, a problem which leads to computational difficulties for low SER rates (e.g., below 10^{-11}).

For $n \geq 1$, (2) is a triple summation. This is a generalization of [SV03], which handled the case of $n = 2$ for the Poisson channel. That derivation involved identifying and computing the probabilities of six distinct events, and a straightforward extension of that analysis to higher

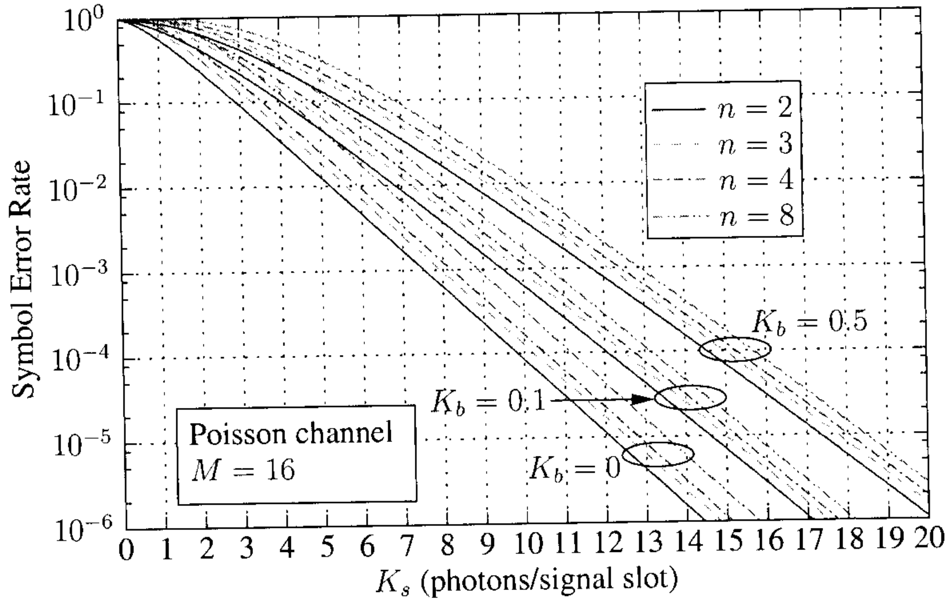

Figure 1: The performance of n-pulse 16-PPM, for $n \in \{2, 3, 4, 8\}$.

values of n results in a number of events that is exponential in n and a formula involving an $(n+1)$-fold infinite summation.

4 Capacity of bandwidth and power constrained channels

Multipulse PPM was designed to improve throughput for a given bandwidth. Afterall, for a fixed bandwidth (slot width), 2-pulse M-PPM nearly doubles the number of bits per symbol with only a small penalty in symbol error rate (see Fig. 1), and a similar comparison with n-pulse PPM is even more stark. However, there are a number of problems with this type of comparison: (a) all other things equal (bandwidth, peak power), the energy per 2-pulse PPM symbol is also twice that of a conventional PPM symbol, (b) uncoded SER alone does not adequately predict the achievable data rates for the channel, and (c) no attempt is made to separately optimize the PPM order for the multipulse and conventional cases.

A better question to ask is, for a given available bandwidth, average power, peak power, and statistical channel characterization, does the class of multipulse PPM achieve higher channel capacity (throughput in bits/sec) than the class of conventional PPM?

The capacity of M-PPM on a soft output memoryless channel is [MH03]

$$C(M) = \frac{1}{M} E_{Y_1,\ldots,Y_M} \log_2 \left[\frac{M L(Y_1)}{\sum_{j=1}^{M} L(Y_j)} \right] \text{ bits/slot,} \tag{3}$$

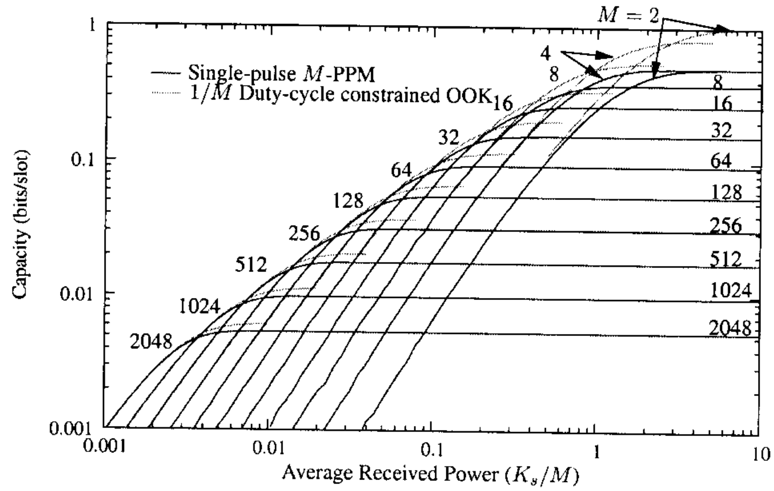

Figure 2: On a Poisson channel with $K_b = 1$, the capacity of M-PPM and the capacity of $1/M$ duty-cycle constrained OOK are nearly equal, $1 \leq M \leq 2048$.

where Y_1 has distribution $p_1(\cdot)$ and Y_i has distribution $p_0(\cdot)$ for all $i > 1$. This is shown for the Poisson channel in Fig. 2. Note that for a given average power, there is an optimal PPM order M^*, meaning that a good design will operate on the upper shell of the curves shown in the figure.

For a given average and peak-to-average power, the class of n-pulse M-PPM can be optmized over both n and M. As this can result in large n and M, it is helpful to consider a modulation which has a certain average duty cycle (fraction of '1's) but is otherwise unconstrained OOK. For example, if the duty cycle is required to be 1/16, this could be obtained with n-pulse M-PPM with $n = 1, M = 16$, or $n = 2, M = 32$, or $n = 4, M = 64$, and so on. Since multipulse PPM is a special case of the more general duty-cycle constrained modulation, its capacity can be no higher than the duty-cycle constrained modulation. The capacity of a memoryless channel with input restricted to duty cycle $1/M$ is

$$C = \frac{1}{M} E_Y \log \frac{p_0(Y)}{f_Y(Y)} + \frac{1}{1 - 1/M} E_Y \log \frac{p_1(Y)}{f_Y(Y)}$$

where $f_Y(y) = \frac{1}{M} p_0(y) + \frac{M-1}{M} p_1(y)$ is the probability mass function for a randomly chosen slot.

Fig. 2 indicates that over a broad range of average powers, the duty-cycle constrained OOK (and thus, the class of multipulse PPM as well) offers negligible capacity advantage over conventional PPM. That is, for a given bandwidth (fixed slot width), the class of multipulse PPM can achieve viritually no higher throughput (in bits/sec) than the class of conventional PPM.

For high average powers, multipulse PPM can offer up to twice the capacity of conventional single-pulse PPM. This manifests itself when the optimal PPM order satisfies $M^* \leq 8$.

Our approach has a number of advantages over, e.g., [SV03]:

- It can be used to identify capacity when any of average power, peak-to-average power, or bandwidth constraints are present, either alone or simultaneously: an average power constraint is a vertical slice of Fig. 2, a peak-to-average power constraint is a bound on the range of M, and all modulations are compared on an equal bandwidth basis because capacity is expressed in bits/slot. In [SV03], only one constraint (e.g., peak power) is considered at a time, letting the others (e.g., average power) vary broadly, making it difficult to make appropriate conclusions for practical systems that have multiple simultaneous constraints.
- The comparison is done on the basis of capacity, not simply SER. SER per se is an inadequate measure of the quality of a modulation scheme, because a modulation scheme may have an inferior SER but combine with an error-correcting codes of a certain rate that produce a superior end-to-end system on the basis of throughput, bandwidth, average power, and peak power.
- The comparison between PPM and multipulse PPM is done on independently optimized orders. It may not be appropriate to compare 16-PPM to 2-pulse 16-PPM, for example, because they have different average powers and/or peak-to-average power ratios. Instead, in Fig. 2 the upper shell for PPM should be compared to the upper shell for multipulse PPM.

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